IA008: Computational Logic

6. Modal Logic

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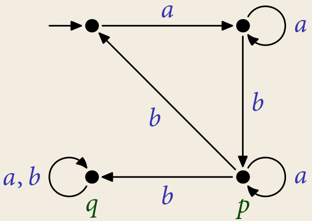
Basic Concepts

Transition Systems

directed graph $\mathfrak{S} = \langle S, (E_a)_{a \in A}, (P_i)_{i \in I}, s_0 \rangle$ with

- **states** S
- **initial state** $s_0 \in S$
- **edge relations** E_a with **edge colours** $a \in A$ (‘**actions**’)
- **unary predicates** P_i with **vertex colours** $i \in I$ (‘**properties**’)



Modal logic

Propositional logic with **modal operators**

- $\langle a\rangle\varphi$ 'there exists an a-successor where φ holds'
- $[a]\varphi$ 'φ holds in every a-successor'

Notation: $\Diamond\varphi, \Box\varphi$ if there are no edge labels

Formal semantics

$$
\begin{array}{lll}
\mathfrak{S}, s \models P & :\text{iff} & s \in P \\
\mathfrak{S}, s \models \varphi \wedge \psi & :\text{iff} & \mathfrak{S}, s \models \varphi \text{ and } \mathfrak{S}, s \models \psi \\
\mathfrak{S}, s \models \varphi \vee \psi & :\text{iff} & \mathfrak{S}, s \models \varphi \text{ or } \mathfrak{S}, s \models \psi \\
\mathfrak{S}, s \models \neg\varphi & :\text{iff} & \mathfrak{S}, s \not\models \varphi \\
\mathfrak{S}, s \models \langle a\rangle\varphi & :\text{iff} & \text{there is } s \to^{a} t \text{ such that } \mathfrak{S}, t \models \varphi \\
\mathfrak{S}, s \models [a]\varphi & :\text{iff} & \text{for all } s \to^{a} t\text{, we have } \mathfrak{S}, t \models \varphi
\end{array}
$$

Examples

$P \wedge \Diamond Q$	'The state is in P and there exists a transition to Q.'
$[a]\bot$	'The state has no outgoing a-transition.'

Interpretations

- **Temporal Logic** talks about time:
  - states: points in time (discrete/continuous)
  - $\Diamond\varphi$ 'sometime in the future φ holds'
  - $\Box\varphi$ 'always in the future φ holds'
- **Epistemic Logic** talks about knowledge:
  - states: possible worlds
  - $\Diamond\varphi$ 'φ might be true'
  - $\Box\varphi$ 'φ is certainly true'

Examples: Temporal Logic

system $\mathfrak{S} = \langle S, \leq, \bar{P} \rangle$

- "P never holds."

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  $\Box(P \to \Diamond Q)$

- "Once P holds, it holds forever."

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- "There are infinitely many P."

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- "There are infinitely many P."

  $\Box \Diamond P$

Translation to first-order logic

Proposition

For every formula φ of propositional modal logic, there exists a formula $\varphi^*(x)$ of first-order logic such that

$$\mathfrak{S}, s \vDash \varphi \quad \text{iff} \quad \mathfrak{S} \vDash \varphi^*(s)\,.$$

Proof

Translation to first-order logic

Proposition

For every formula φ of propositional modal logic, there exists a formula $\varphi^*(x)$ of first-order logic such that

$$\mathfrak{S}, s \models \varphi \quad \text{iff} \quad \mathfrak{S} \models \varphi^*(s)\,.$$

Proof

$$\begin{aligned}
P^* &:= P(x) \\
(\varphi \wedge \psi)^* &:= \varphi^*(x) \wedge \psi^*(x) \\
(\varphi \vee \psi)^* &:= \varphi^*(x) \vee \psi^*(x) \\
(\neg\varphi)^* &:= \neg\varphi^*(x) \\
(\langle a \rangle \varphi)^* &:= \exists y[E_a(x, y) \wedge \varphi^*(y)] \\
([a]\varphi)^* &:= \forall y[E_a(x, y) \rightarrow \varphi^*(y)]
\end{aligned}$$

Bisimulation

$\mathfrak{S}$ and $\mathfrak{T}$ transition systems

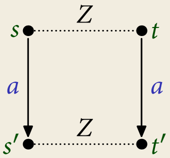
$Z \subseteq S \times T$ is a **bisimulation** if, for all $\langle s, t \rangle \in Z$,

(local) $s \in P \iff t \in P$

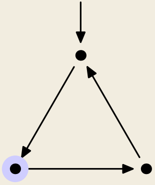
(forth) for every $s \to^a s'$, exists $t \to^a t'$ with $\langle s', t' \rangle \in Z$,

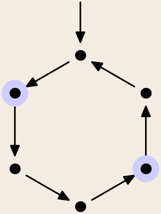
(back) for every $t \to^a t'$, exists $s \to^a s'$ with $\langle s', t' \rangle \in Z$.

$\mathfrak{S}, s$ and $\mathfrak{T}, t$ are **bisimilar** if there is a bisimulation Z with $\langle s, t \rangle \in Z$.

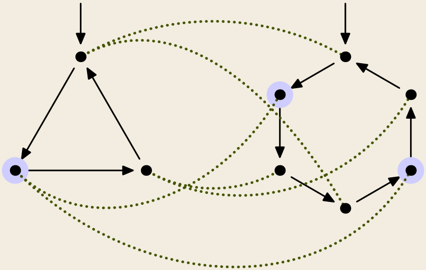


Examples

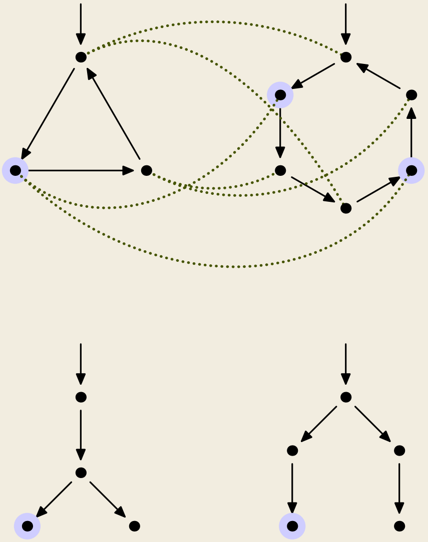




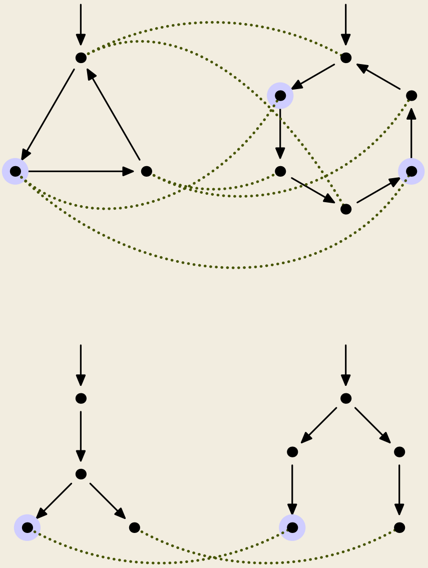
Examples



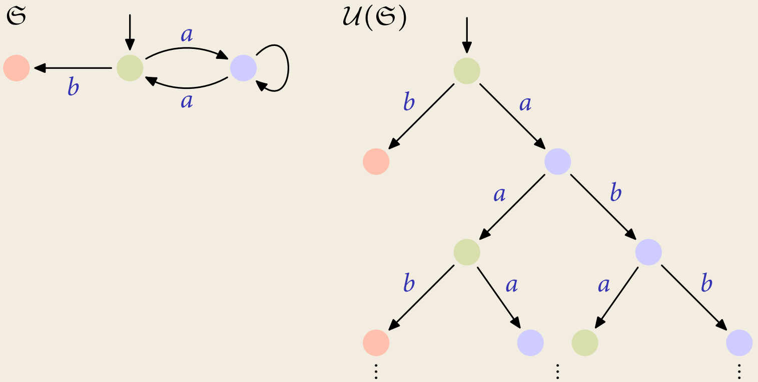
Examples



Examples



Unravelling



Lemma
$\mathfrak{S}$ and $\mathcal{U}(\mathfrak{S})$ are bisimilar.

Bisimulation invariance

Theorem

Two **finite** transition systems $\mathfrak{S}, s$ and $\mathfrak{T}, t$ are **bisimilar** if, and only if,

$$\mathfrak{S}, s \vDash \varphi \quad \Leftrightarrow \quad \mathfrak{T}, t \vDash \varphi\,, \qquad \text{for every modal formula } \varphi\,.$$

Definition

A formula $\varphi(x)$ is **bisimulation invariant** if

$$\mathfrak{S}, s \sim \mathfrak{T}, t \quad \text{implies} \quad \mathfrak{S} \vDash \varphi(s) \Leftrightarrow \mathfrak{T} \vDash \varphi(t)\,.$$

Theorem

A first-order formula φ is equivalent to a **modal formula** if, and only if, it is **bisimulation invariant.**

First-Order Modal Logic

Syntax

first-order logic with modal operators $\langle a \rangle \varphi$ and $[a]\varphi$

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Models

transistion systems where each state s is labelled with a Σ-structure $\mathfrak{A}_s$ such that

$$s \to^a t \quad \text{implies} \quad A_s \subseteq A_t$$

First-Order Modal Logic

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first-order logic with modal operators $\langle a \rangle \varphi$ and $[a]\varphi$

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Examples

- $\Box \forall x \varphi(x) \to \forall x \Box \varphi(x)$ is valid.
- $\forall x \Box \varphi(x) \to \Box \forall x \varphi(x)$ is not valid.

Tableaux

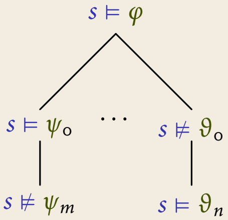
Tableau Proofs

Statements

$s \models \varphi \qquad s \not\models \varphi \qquad s \to^a t$

s, t state labels, φ a modal formula

Rules



Tableaux

Construction

A **tableau** for a formula φ is constructed as follows:

- start with $s_0 \not\models \varphi$
- choose a branch of the tree
- choose a statement $s \models \psi / s \not\models \psi$ on the branch
- choose a rule with head $s \models \psi / s \not\models \psi$
- add it at the bottom of the branch
- repeat until every branch contains both statements $s \models \psi$ and $s \not\models \psi$ for some formula ψ

Tableaux

Construction

A **tableau** for a formula φ is constructed as follows:

- start with $s_0 \nvDash \varphi$
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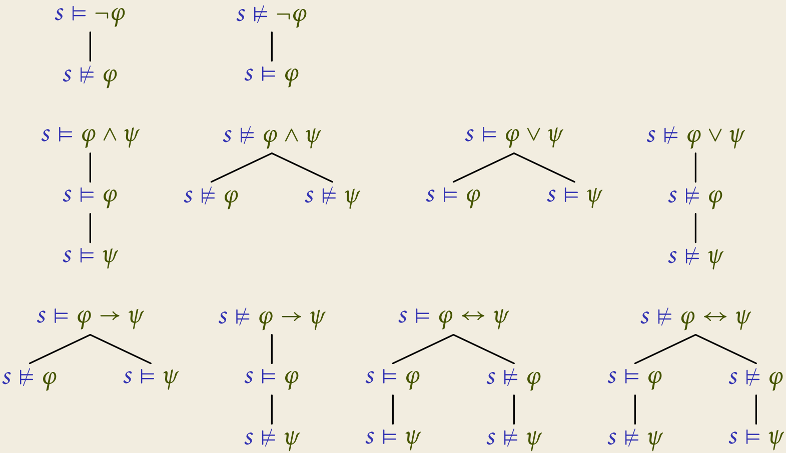
- repeat until every branch contains both statements $s \vDash \psi$ and $s \nvDash \psi$ for some formula ψ

Tableaux with premises Γ

- choose a branch, a state s on the branch, a premise $\psi \in \Gamma$, and add $s \vDash \psi$ to the branch

Rules


s ⊨ ¬φ
s ⊭ φ
s ⊭ ¬φ
s ⊨ φ
s ⊨ φ ∧ ψ
s ⊨ φ
s ⊨ ψ
s ⊭ φ ∧ ψ
s ⊭ φ
s ⊭ ψ
s ⊨ φ ∨ ψ
s ⊨ φ
s ⊨ ψ
s ⊭ φ ∨ ψ
s ⊭ φ
s ⊭ ψ
s ⊨ φ → ψ
s ⊭ φ
s ⊨ ψ
s ⊭ φ → ψ
s ⊨ φ
s ⊭ ψ
s ⊨ φ ↔ ψ
s ⊨ φ
s ⊨ ψ
s ⊭ φ
s ⊭ ψ
s ⊭ φ ↔ ψ
s ⊨ φ
s ⊭ ψ
s ⊭ φ
s ⊨ ψ

Rules

$$\begin{array}{c} s \vDash \langle a \rangle \varphi \\ | \\ s \to^a t \\ | \\ t \vDash \varphi \end{array} \qquad \begin{array}{c} s \nvDash \langle a \rangle \varphi \\ | \\ t' \nvDash \varphi \end{array} \qquad \begin{array}{c} s \vDash [a] \varphi \\ | \\ t' \vDash \varphi \end{array} \qquad \begin{array}{c} s \nvDash [a] \varphi \\ | \\ s \to^a t \\ | \\ t \nvDash \varphi \end{array}$$

$$\begin{array}{c} s \vDash \forall x \varphi \\ | \\ s \vDash \varphi[x \mapsto u] \end{array} \qquad \begin{array}{c} s \nvDash \forall x \varphi \\ | \\ s \nvDash \varphi[x \mapsto c] \end{array} \qquad \begin{array}{c} s \vDash \exists x \varphi \\ | \\ s \vDash \varphi[x \mapsto c] \end{array} \qquad \begin{array}{c} s \nvDash \exists x \varphi \\ | \\ s \nvDash \varphi[x \mapsto u] \end{array}$$

t a new state, t' every state with entry $s \to^a t'$ on the branch,
c a new constant symbol, u an arbitrary term

Example $\varphi \vDash \Box\varphi$

$$s \nvDash \Box\varphi$$

$$s \to t$$

$$t \nvDash \varphi$$

$$t \vDash \varphi$$

Example $\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$$s \not\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

$$s \models \Box(\varphi \to \psi)$$

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$$s \not\models \Box\psi$$

$$s \to t$$

$$t \not\models \psi$$

$$t \models \varphi$$

$$t \models \varphi \to \psi$$

$$t \not\models \varphi \qquad t \models \psi$$

Example $\vDash \Box\forall x\varphi \to \forall x \Box \varphi$

$$s \nvDash \Box\forall x\varphi \to \forall x\Box\varphi$$

$$s \vDash \Box\forall x\varphi$$

$$s \nvDash \forall x\Box\varphi$$

$$s \nvDash \Box\varphi[x \mapsto c]$$

$$s \to t$$

$$t \nvDash \varphi[x \mapsto c]$$

$$t \vDash \forall x\varphi$$

$$t \vDash \varphi[x \mapsto c]$$

Soundness and Completeness

Consequence

ψ is a **consequence** of Γ if, and only if, for all transition systems $\mathfrak{S}$,

$$\mathfrak{S}, s \vDash \varphi\,, \quad \text{for all } s \in S \text{ and } \varphi \in \Gamma\,,$$

implies that

$$\mathfrak{S}, s \vDash \psi\,, \quad \text{for all } s \in S\,.$$

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implies that

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Theorem

A modal formula φ is a consequence of Γ if, and only if, there exists a tableau T for φ with premises Γ where every branch is contradictory.

Complexity

Theorem

Satisfiability for propositional modal logic is in **deterministic linear space.**

Theorem

Satisfiability for first-order modal logic is **undecidable.**

Temporal Logics

Linear Temporal Logic (LTL)

Speaks about **paths.** $P \longrightarrow \bullet \longrightarrow \bullet \longrightarrow P, Q \longrightarrow Q \longrightarrow \bullet \longrightarrow \cdots$

Syntax

- atomic predicates $P, Q, \ldots$
- boolean operations $\wedge, \vee, \neg$
- next $X\varphi$
- until $\varphi U \psi$
- finally $F\varphi := \top U \varphi$
- generally $G\varphi := \neg F \neg \varphi$

Examples

FP	a state in P is reachable
GFP	we can reach infinitely many states in P
$(\neg P) U (P \wedge Q)$	the first reachable state in P is also in Q

Linear Temporal Logic (LTL)

Theorem

Let L be a set of paths. The following statements are equivalent:

- L can be defined in LTL.
- L can be defined in first-order logic.
- L can be defined by a star-free regular expression.

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Translation LTL to FO

$$
\begin{aligned}
P^* &:= P(x) \\
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(\varphi \vee \psi)^* &:= \varphi^*(x) \vee \psi^*(x) \\
(\neg \varphi)^* &:= \neg \varphi^*(x) \\
(X\varphi)^* &:= \exists y[x < y \wedge \neg \exists z(x < z \wedge z < y) \wedge \varphi^*(y)] \\
(\varphi U \psi)^* &:= \exists y[x \leq y \wedge \psi^*(y) \wedge \forall z[x \leq z \wedge z < y \rightarrow \varphi^*(z)]]
\end{aligned}
$$

Linear Temporal Logic (LTL)

Theorem

Satisfiablity of LTL formulae is **PSPACE-complete.**

Theorem

Model checking $\mathfrak{S}, s \vDash \varphi$ for LTL is **PSPACE-complete.** It can be done in

$$\textbf{time } \mathcal{O}\left(|S| \cdot 2^{\mathcal{O}(|\varphi|)}\right) \quad \text{or} \quad \textbf{space } \mathcal{O}\left((|\varphi| + \log |S|)^2\right).$$

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Formula complexity: **PSPACE-complete**

Data complexity: **NLOGSPACE-complete**

Computation Tree Logic (CTL and CTL*)

Applies LTL-formulae to the branches of a tree.

Syntax (of CTL*)

- **state formulae φ**:

$$\varphi ::= P \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid A\psi \mid E\psi$$

- **path formulae ψ**:

$$\psi ::= \varphi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \neg\psi \mid X\psi \mid \psi U \psi \mid F\psi \mid G\psi$$

Examples

EFP	a state in P is reachable
AFP	every branch contains a state in P
$EGFP$	there is a branch with infinitely many P
$EGEFP$	there is a branch such that we can reach P from every of its states

Computation Tree Logic (CTL and CTL*)

Theorem

Satisfiability for CTL is **EXPTIME-complete.**

Model checking $\mathfrak{S}, s \models \varphi$ for CTL is **P-complete.** It can be done in

time $\mathcal{O}\big(|\varphi| \cdot |S|\big)$ or **space** $\mathcal{O}\big(|\varphi| \cdot \log^2 (|\varphi| \cdot |S|)\big)$.

Data complexity: **NLOGSPACE-complete**

Computation Tree Logic (CTL and CTL*)

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Satisfiability for CTL is **EXPTIME-complete.**

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Data complexity: **NLOGSPACE-complete**

Theorem

Satisfiability for CTL* is **2EXPTIME-complete.**

Model checking $\mathfrak{S}, s \models \varphi$ for CTL* is **PSPACE-complete.** It can be done in

$$\textbf{time } \mathcal{O}\big(|S|^2 \cdot 2^{\mathcal{O}(|\varphi|)}\big) \quad \text{or} \quad \textbf{space } \mathcal{O}\big(|\varphi|(|\varphi| + \log|S|)^2\big).$$

Formula complexity: **PSPACE-complete**

Data complexity: **NLOGSPACE-complete**

Fixed points

Theorem (Knaster, Tarski)

Let $\langle A, \leq \rangle$ be a **complete** partial order and $f : A \to A$ **monotone**. Then f has a **least** and a **greatest fixed point** and

$$\mathrm{lfp}(f) = \lim_{\alpha \to \infty} f^{\alpha}(\bot) \quad \text{and} \quad \mathrm{gfp}(f) = \lim_{\alpha \to \infty} f^{\alpha}(\top)$$

Ordinals

$0, 1, 2, 3, \ldots$

Ordinals

$0, 1, 2, 3, \ldots\ \omega$

Ordinals

$$0, 1, 2, 3, \ldots\ \omega,\ \omega + 1,\ \omega + 2, \ldots$$

Ordinals

$$0, 1, 2, 3, \ldots\ \omega,\ \omega + 1,\ \omega + 2, \ldots\ \omega + \omega = \omega 2$$

Ordinals

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Ordinals

$0, 1, 2, 3, \ldots\ \omega,\ \omega + 1,\ \omega + 2, \ldots\ \omega + \omega = \omega 2,\ \omega 2 + 1,\ \omega 2 + 2, \ldots$
$\omega 3, \ldots\ \omega 4, \ldots\ \omega 5, \ldots$

Ordinals

$$0, 1, 2, 3, \ldots\ \omega,\ \omega + 1,\ \omega + 2, \ldots\ \omega + \omega = \omega 2,\ \omega 2 + 1,\ \omega 2 + 2, \ldots$$
$$\omega 3, \ldots\ \omega 4, \ldots\ \omega 5, \ldots\ \omega\omega = \omega^2$$

Ordinals

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$$\omega 3, \ldots\ \omega 4, \ldots\ \omega 5, \ldots\ \omega\omega = \omega^2, \ldots\ \omega^3, \ldots\ \omega^4, \ldots$$

Ordinals

$0, 1, 2, 3, \ldots\ \omega,\ \omega + 1,\ \omega + 2, \ldots\ \omega + \omega = \omega 2,\ \omega 2 + 1,\ \omega 2 + 2, \ldots$
$\omega 3, \ldots\ \omega 4, \ldots\ \omega 5, \ldots\ \omega\omega = \omega^2, \ldots\ \omega^3, \ldots\ \omega^4, \ldots$
ω^ω

Ordinals

$$0, 1, 2, 3, \ldots\ \omega,\ \omega + 1,\ \omega + 2, \ldots\ \omega + \omega = \omega 2,\ \omega 2 + 1,\ \omega 2 + 2, \ldots$$
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$$\omega^\omega, \ldots\ \omega^{\omega^\omega}, \ldots\ \omega^{\omega^{\omega^\omega}}, \ldots$$

Ordinals

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Ordinals

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3 Kinds

- 0

Ordinals

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3 Kinds

- 0
- successor $\alpha + 1$

Ordinals

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$$\omega^\omega, \ldots\ \omega^{\omega^\omega}, \ldots\ \omega^{\omega^{\omega^\omega}}, \ldots\ \varepsilon, \ldots\ \omega_1, \ldots\ \omega_2, \ldots$$

3 Kinds

- 0
- successor $\alpha + 1$
- limit δ

Ordinals

$$0, 1, 2, 3, \ldots\ \omega,\ \omega+1,\ \omega+2, \ldots\ \omega+\omega=\omega 2,\ \omega 2+1,\ \omega 2+2, \ldots$$
$$\omega 3, \ldots\ \omega 4, \ldots\ \omega 5, \ldots\ \omega\omega=\omega^2, \ldots\ \omega^3, \ldots\ \omega^4, \ldots$$
$$\omega^{\omega}, \ldots\ \omega^{\omega^{\omega}}, \ldots\ \omega^{\omega^{\omega^{\omega}}}, \ldots\ \varepsilon, \ldots\ \omega_1, \ldots\ \omega_2, \ldots$$

3 Kinds

- 0
- successor $\alpha + 1$
- limit δ

Proposition

Every non-empty set of ordinals has a least element.

Ordinals

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$\omega^\omega, \dots\ \omega^{\omega^\omega}, \dots\ \omega^{\omega^{\omega^\omega}}, \dots\ \varepsilon, \dots\ \omega_1, \dots\ \omega_2, \dots$

3 Kinds

- 0
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- limit δ

Proposition

Every non-empty set of ordinals has a least element.

Iteration

$$\begin{aligned} f^0(x) &:= x\,, \\ f^{\alpha+1}(x) &:= f(f^\alpha(x))\,, \\ f^\delta(x) &:= \sup_{\alpha<\delta} f^\alpha(x)\,, \quad \text{for limit ordinals } \delta\,. \end{aligned}$$

Proof

Monotonicity $f^{\alpha}(\bot) \leq f^{\beta}(\bot)$ for $\alpha \leq \beta$

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$a = f(a)$ fixed point, $f^{\alpha}(\bot) = f^{\alpha+1}(\bot)$

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The modal μ-calculus (L_μ)

Adds recursion to modal logic.

Syntax

$$\varphi ::= P \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle a \rangle \varphi \mid [a]\varphi \mid \mu X.\varphi(X) \mid \nu X.\varphi(X)$$

(X positive in $\mu X.\varphi(X)$ and $\nu X.\varphi(X)$)

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Semantics

$$F_\varphi(X) := \{\, s \in S \mid \mathfrak{S}, s \vDash \varphi(X) \,\}$$

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Examples

$\mu X(P \vee \Diamond X)$ a state in P is reachable

$\nu X(P \wedge \Diamond X)$ there is a branch with all states in P

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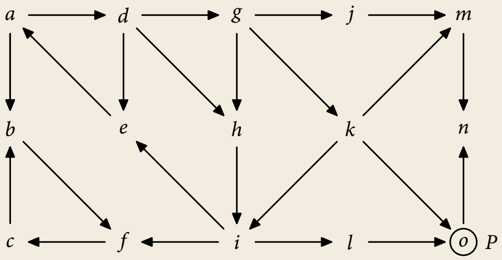

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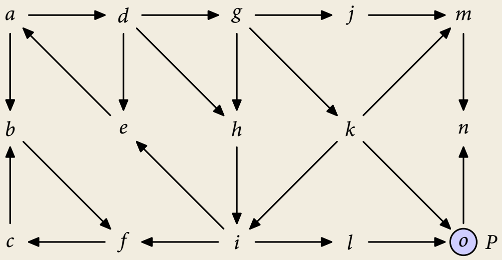

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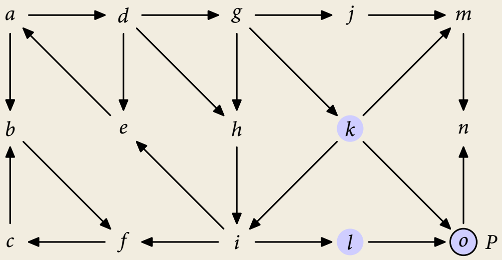

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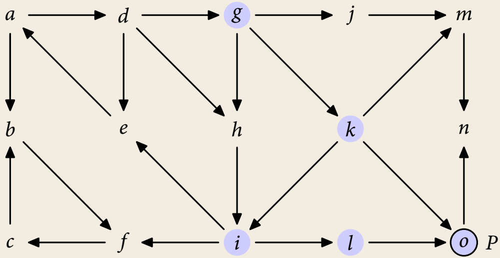

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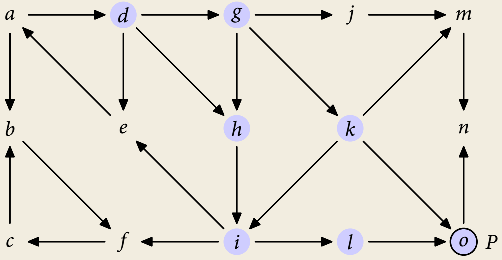

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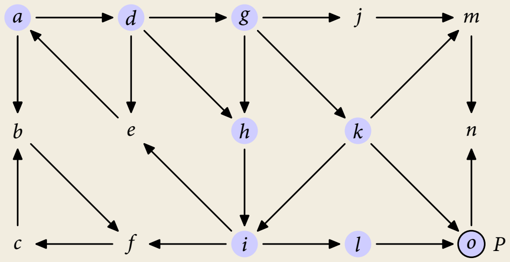

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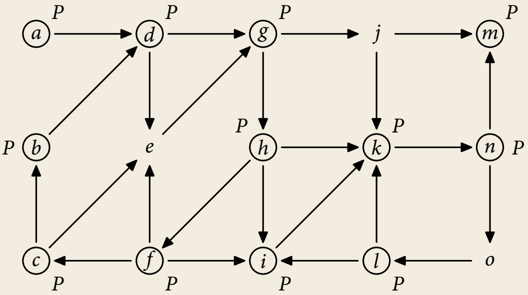

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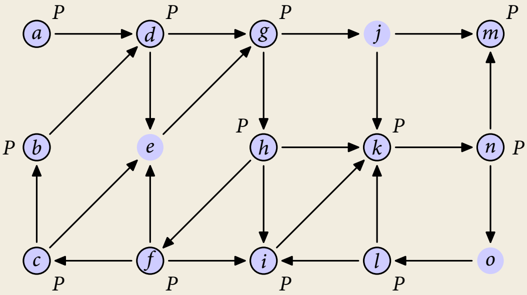

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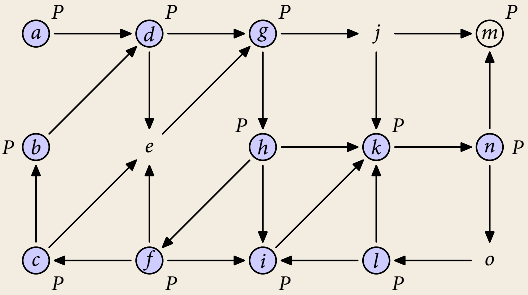

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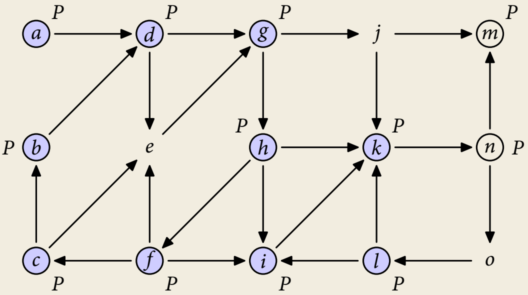

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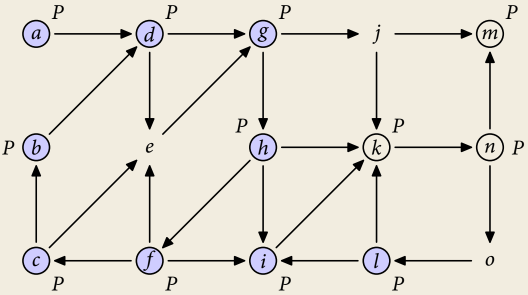

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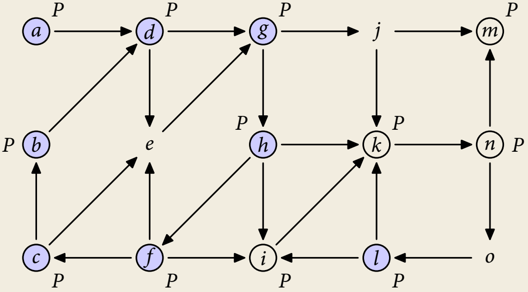

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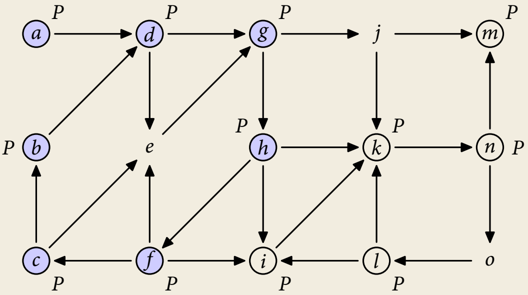

Expressive power

Theorem

For every CTL*-formula φ there exists an equivalent formula φ^* of the modal μ-calculus.

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Proof (for CTL)

$$
\begin{aligned}
P^* &:= P \\
(\varphi \wedge \psi)^* &:= \varphi^* \wedge \psi^* \\
(\varphi \vee \psi)^* &:= \varphi^* \vee \psi^* \\
(\neg\varphi)^* &:= \neg\varphi^* \\
(EX\varphi)^* &:= \Diamond\varphi^* \\
(AX\varphi)^* &:= \Box\varphi^* \\
(E\varphi U\psi)^* &:= \mu X[\psi^* \vee (\varphi^* \wedge \Diamond X)] \\
(A\varphi U\psi)^* &:= \mu X[\psi^* \vee (\varphi^* \wedge \Box X)]
\end{aligned}
$$

The modal μ-calculus (L_μ)

Theorem

A regular tree language can be defined in the **modal μ-calculus** if, and only if, it is **bisimulation invariant.**

Theorem

Satisfiability of μ-calculus formulae is **decidable** and complete for **exponential time.**

Model checking $\mathfrak{S}, s \vDash \varphi$ for the modal μ-calculus can be done in **time** $\mathcal{O}\big((|\varphi| \cdot |S|)^{|\varphi|}\big)$.

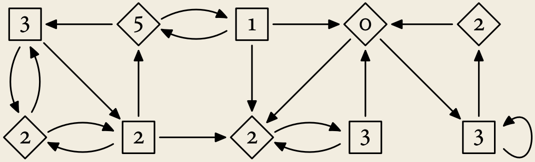
(The satisfiability algorithm uses tree automata and parity games.)

Parity Games

$\mathfrak{G} = \langle V_\Diamond, V_\Box, E, \Omega \rangle \qquad \Omega : V \to \mathbb{N}$

Infinite plays $v_0, v_1, \ldots$ are **won** by Player $\Diamond$ if

$\liminf_{n\to\infty} \Omega(v_n)$ is even.



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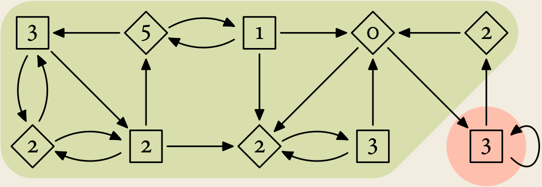

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Parity games are **positionally determined:** from each position some player has a positional/memory-less winning strategy.

Parity Games

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Theorem

Parity games are **positionally determined:** from each position some player has a positional/memory-less winning strategy.

Theorem

Computing the winning region of a parity game with n positions and d priorities can be done in time $n^{\mathcal{O}(\log d)}$.

Model-Checking Games

game for $\mathfrak{S}, s_0 \vDash \varphi$? ($\varphi$ μ-formula in negation normal form)

Model-Checking Games

game for $\mathfrak{S}, s_0 \models \varphi$? ($\varphi$ μ-formula in negation normal form)

Positions

Player $\Diamond$: $\langle s, \psi \rangle$ for $s \in S$ and ψ a subformula

$$\psi = \psi_0 \vee \psi_1 , \qquad \psi = P \text{ and } s \notin P , \qquad \psi = \mu X.\psi_0 ,$$
$$\psi = \langle a \rangle \psi_0 , \qquad \psi = \neg P \text{ and } s \in P , \qquad \psi = \nu X.\psi_0 ,$$
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Player $\Box$: $[s, \psi]$ for $s \in S$ and ψ a subformula

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Model-Checking Games

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Initial position

$\langle s_0, \varphi \rangle$ or $[s_0, \varphi]$

Model-Checking Games

game for $\mathfrak{S}, s_0 \vDash \varphi$?   ($\varphi$ μ-formula in negation normal form)

Edges   ((s, ψ) means either $\langle s, \psi \rangle$ or $[s, \psi]$.)

$$\begin{aligned}
\langle s, \psi_0 \vee \psi_1 \rangle &\to (s, \psi_i)\,, \\
[s, \psi_0 \wedge \psi_1] &\to (s, \psi_i)\,, \\
\langle s, \mu X.\psi \rangle &\to \psi\,, \\
\langle s, \nu X.\psi \rangle &\to \psi\,, \\
\langle s, X \rangle &\to \langle s, \mu X.\psi \rangle \text{ or } \langle s, \nu X.\psi \rangle\,, \\
\langle s, \langle a \rangle \psi \rangle &\to \langle t, \psi \rangle \quad \text{for every } s \to^a t\,, \\
[s, [a]\psi] &\to \langle t, \psi \rangle \quad \text{for every } s \to^a t\,.
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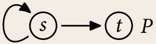
[s, [a]\psi] &\to \langle t, \psi\rangle \quad \text{for every } s \to^a t\,.
\end{aligned}$$

Priorities (all other priorities big)

$$\begin{aligned}
\Omega(\langle s, \mu X.\psi\rangle) &:= 2k+1\,, \qquad \text{if inside of } k \text{ fixed points.} \\
\Omega(\langle s, \nu X.\psi\rangle) &:= 2k\,.
\end{aligned}$$

Model-Checking Games

$\mathfrak{S} =$ 

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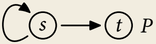
$\mathfrak{S} =$ 

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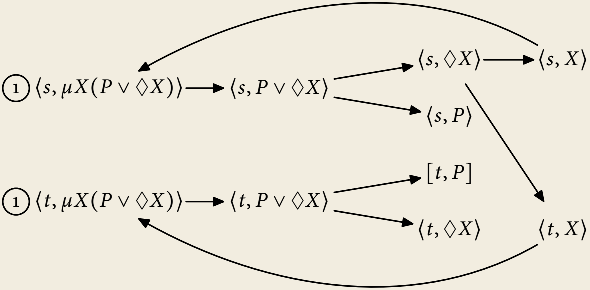

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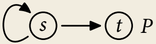
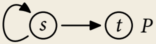
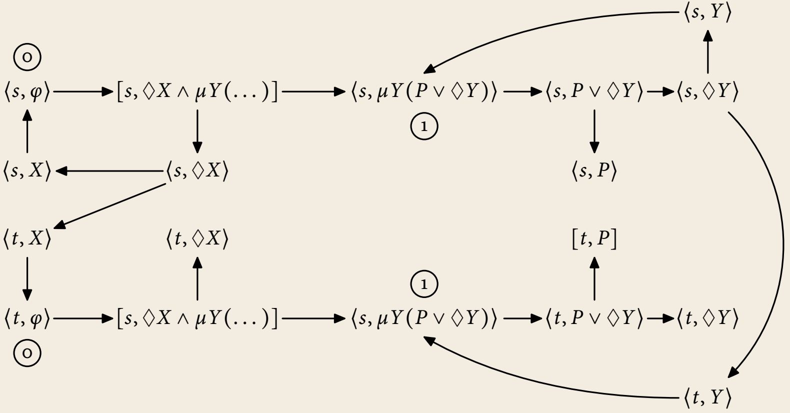
$\mathfrak{S} =$ 

$\varphi = \nu X(\Diamond X \wedge \mu Y(P \vee \Diamond Y))$

Model-Checking Games

$\mathfrak{S} =$ 

$\varphi = \nu X(\Diamond X \land \mu Y(P \lor \Diamond Y))$



Description Logics

Description Logic

General Idea

Extend **modal logic** with operations that are **not bisimulation-invariant.**

Applications

Knowledge representation, deductive databases, system modelling, semantic web

Ingredients

- **individuals:** elements (Anna, John, Paul, Marry,…)
- **concepts:** unary predicates (person, male, female,…)
- **roles:** binary relations (has_child, is_married_to,…)
- **TBox:** terminology definitions
- **ABox:** assertions about the world

Example

TBox

$$\begin{aligned} \text{man} &:= \text{person} \wedge \text{male} \\ \text{woman} &:= \text{person} \wedge \text{female} \\ \text{father} &:= \text{man} \wedge \exists \text{has\_child.person} \\ \text{mother} &:= \text{woman} \wedge \exists \text{has\_child.person} \end{aligned}$$

ABox

man(John)

man(Paul)

woman(Anna)

woman(Marry)

has_child(Anna, Paul)

is_married_to(Anna, John)

Syntax

Concepts

$$\varphi ::= P \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall R\varphi \mid \exists R\varphi \mid (\geq nR) \mid (\leq nR)$$

Terminology axioms

$$\varphi \sqsubseteq \psi \qquad \varphi \equiv \psi$$

TBox Axioms of the form $P \equiv \varphi$.

Assertions

$$\varphi(a) \qquad R(a, b)$$

Extensions

- operations on roles: $R \cap S$, $R \cup S$, $R \circ S$, $\neg R$, R^+, R^*, R^-
- extended number restrictions: $(\geq nR)\varphi$, $(\leq nR)\varphi$

Algorithmic Problems

- **Satisfiability:** Is φ satisfiable?
- **Subsumption:** $\varphi \models \psi$?
- **Equivalence:** $\varphi \equiv \psi$?
- **Disjointness:** $\varphi \wedge \psi$ unsatisfiable?

All problems can be solved with standard methods like **tableaux** or **tree automata.**

Semantic Web: OWL (functional syntax)

```
Ontology(
  Class(pp:man    complete
          intersectionOf(pp:person pp:male))
  Class(pp:woman  complete
          intersectionOf(pp:person pp:female))
  Class(pp:father complete
          intersectionOf(pp:man
            restriction(pp:has_child pp:person)))
  Class(pp:mother complete
          intersectionOf(pp:woman
            restriction(pp:has_child pp:person)))
  Individual(pp:John  type(pp:man))
  Individual(pp:Paul  type(pp:man))
  Individual(pp:Anna  type(pp:woman)
               value(pp:has_child     pp:Paul)
               value(pp:is_married_to pp:John))
  Individual(pp:Marry type(pp:woman))
)
```