Exercise 1 We consider (undirected) graphs as structures of the form $\mathfrak{G} = \langle V, E \rangle$ where E is the binary edge relation. Express the following statements in first-order logic.

(a) All vertices are neighbours.

(b) The graph contains a triangle.

(c) Every vertex has exactly three neighbours.

(d) Every pair of vertices is connected by a path of length at most 3.

Exercise 2 Show that the following formulae are valid using tableau proofs.

(a) $\varphi \wedge \psi \to \psi \wedge \varphi$

(b) $\psi \to ((\varphi \wedge \psi) \vee \psi)$

(c) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

(d) $\varphi \to \neg\neg\varphi$

(e) $((\varphi \wedge \psi) \vee \psi) \to \psi$

(f) $(\varphi \to \psi) \wedge (\varphi \to \vartheta) \to (\varphi \to \psi \wedge \vartheta)$

(g) $(\varphi \to \psi \wedge \vartheta) \to (\varphi \to \psi) \wedge (\varphi \to \vartheta)$

(h) $\neg\neg\varphi \to \varphi$

(i) $\varphi \vee \neg\varphi$

(j) $\neg(\neg\varphi \wedge \neg\psi) \to (\varphi \vee \psi)$

(k) $\varphi \to \exists x\varphi$

(l) $\forall x\varphi \to \varphi$

(m) $\forall xR(x,x) \to \forall x\exists yR(f(x),y)$

(n) $\exists x(\varphi \vee \psi) \to (\exists x\varphi \vee \exists x\psi)$

(o) $(\exists x\varphi \vee \exists x\psi) \to \exists x(\varphi \vee \psi)$

(p) $\forall x\varphi \wedge \forall x\psi \to \forall x(\varphi \wedge \psi)$

(q) $\forall x(\varphi \wedge \psi) \to \forall x\varphi \wedge \forall x\psi$

(r) $\forall x\forall y[\varphi(x) \leftrightarrow \varphi(y)] \wedge \exists x\varphi(x) \to \forall x\varphi(x)$

Exercise 3 Prove that the formulae from Exercise 2 are valid using Natural Deduction.

Exercise 4 Find all consistent sets for the following sets of rules.

(a) $\frac{}{\alpha} \quad \frac{\alpha : \beta}{\delta} \quad \frac{\alpha : \gamma}{\delta}$

(b) $\frac{}{\alpha} \quad \frac{\alpha : \beta\,\gamma}{\beta}$

(c) $\frac{}{\alpha} \quad \frac{\alpha\,\beta}{\gamma} \quad \frac{\alpha : \gamma}{\beta}$

Exercise 5 For each of the following subsets $\Phi \subseteq \wp(\{\alpha, \beta\})$, find a set of rules R such that Φ is the set of all consistent sets for R.

(a) $\{\varnothing, \{\alpha\}, \{\alpha, \beta\}\}$

(b) $\{\{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$

(c) $\{\varnothing, \{\alpha, \beta\}\}$

(d) $\{\{\alpha\}, \{\alpha, \beta\}\}$

Exercise 6 Derive the following additional rules from the basic ones of the Natural Deduction calculus (that is, combine the basic rules to obtain the ones below).

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi} \qquad \frac{\Gamma, \neg\varphi \vdash \neg\psi}{\Gamma \vdash \psi \to \varphi}$$