IA038 Types and Proofs

3. Natural Deduction

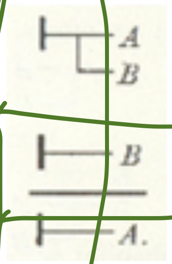
Jiří Zlatuška

March 7, 2023

Notation for proofs: from Frege to Hilbert school

- Frege started with pictorial notation for judgements

Modus Ponens:



B implies A

B is asserted

A follows

- Notation for judgements evolved into

$\vdash A$ as the judgement that A is asserted

- Modus Ponens and axiom schemes:

$$\frac{\vdash B \to A \qquad \vdash B}{\vdash A}$$

$$\vdash A \to A \tag{1}$$

$$\vdash A \to (B \to A) \tag{2}$$

$$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{3}$$

Gentzen's notation and rules for Natural deduction

- Gentzen introduced judgements with assumptions,

$B_1, \ldots, B_n \vdash A$ meaning A proved from assumptions $B_1, \ldots, B_n$

- Modus Ponens in Gentzen's system:

$$\frac{\Gamma \vdash B \rightarrow A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A}$$

here Γ and Δ are lists of propositions, Γ, Δ means union of two lists (here just as finite sets)

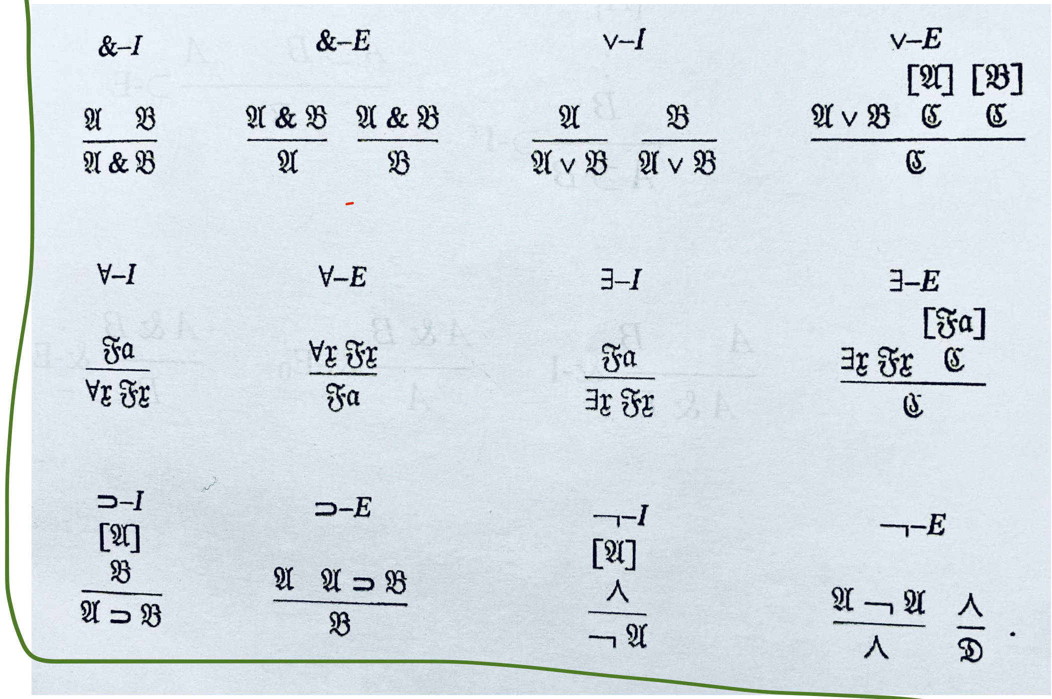
Gentzen’s rules for Natural deduction

$$\frac{}{A \vdash A}\ \text{Id}$$

$$\frac{\Gamma, B \vdash A}{\Gamma \vdash B \to A}\ \to\text{-I} \qquad \frac{\Gamma \vdash B \to A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A}\ \to\text{-E}$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge\text{-I} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-E}_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-E}_2$$

Gentzen’s tree notation:



Example proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{B \wedge A\} \vdash B \wedge A}\,\text{Id}}{\{B \wedge A\} \vdash A}\,\wedge\text{-E}_2 \qquad \dfrac{\dfrac{}{\{B \wedge A\} \vdash B \wedge A}\,\text{Id}}{\{B \wedge A\} \vdash B}\,\wedge\text{-E}_1}{\{B \wedge A\} \vdash A \wedge B}\,\wedge\text{-I}}{\{\} \vdash (B \wedge A) \to (A \wedge B)}\,\to\text{-I} \qquad \dfrac{\dfrac{}{\{B\} \vdash B}\,\text{Id} \qquad \dfrac{}{\{A\} \vdash A}\,\text{Id}}{\{A,\, B\} \vdash B \wedge A}\,\wedge\text{-I}}{\{A,\, B\} \vdash A \wedge B}\,\to\text{-E}$$

Gentzen's rules for Natural deduction

$$\frac{}{A \vdash A}\,\mathrm{Id}$$

$$\frac{\Gamma, B \vdash A}{\Gamma \vdash B \to A}\,\to\text{-I} \qquad \frac{\Gamma \vdash B \to A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A}\,\to\text{-E}$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge\text{-I} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\wedge\text{-E}_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\wedge\text{-E}_2$$

Modern notation (Prawitz, Girard, …):

- *Hypothesis*: A
- *Introductions*:

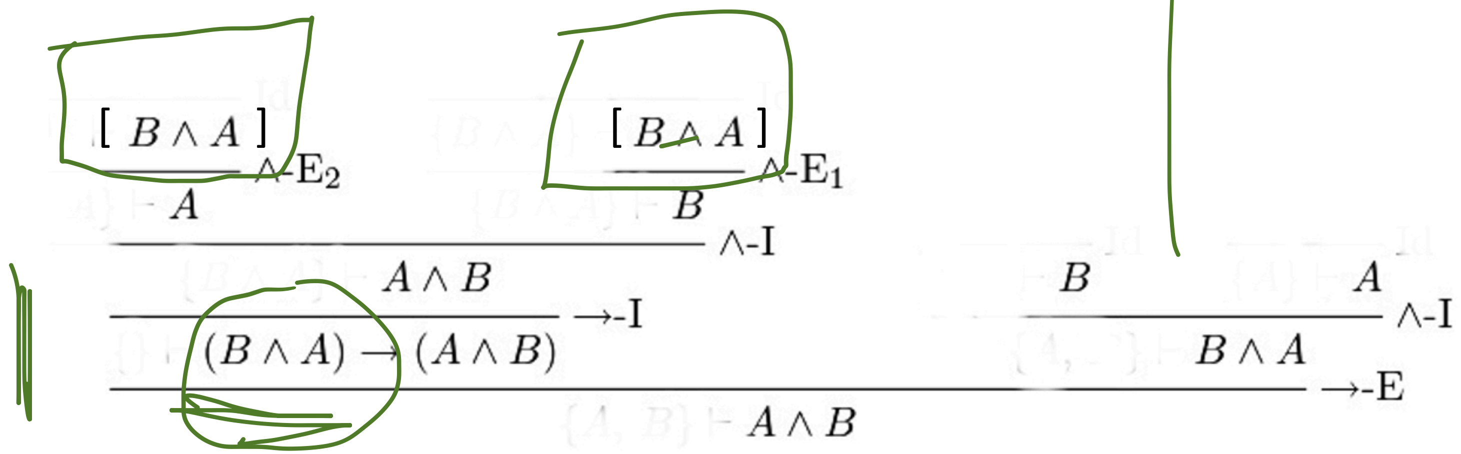
$$\frac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}\,\wedge\mathcal{I} \qquad \frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B}\,\Rightarrow\mathcal{I}$$

- *Eliminations*:

$$\frac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\,\wedge 1\mathcal{E} \qquad \frac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\,\wedge 2\mathcal{E} \qquad \frac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A \Rightarrow B\end{matrix}}{B}\,\Rightarrow\mathcal{E}$$

Example proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{B\wedge A\}\vdash B\wedge A}\,\text{Id}}{\{B\wedge A\}\vdash A}\,\wedge\text{-E}_2 \qquad \dfrac{\dfrac{}{\{B\wedge A\}\vdash B\wedge A}\,\text{Id}}{\{B\wedge A\}\vdash B}\,\wedge\text{-E}_1}{\{B\wedge A\}\vdash A\wedge B}\,\wedge\text{-I}}{\{\}\vdash (B\wedge A)\to(A\wedge B)}\,\to\text{-I} \qquad \dfrac{\dfrac{}{\{B\}\vdash B}\,\text{Id} \quad \dfrac{}{\{A\}\vdash A}\,\text{Id}}{\{A,\ B\}\vdash B\wedge A}\,\wedge\text{-I}}{\{A,\ B\}\vdash A\wedge B}\,\to\text{-E}$$



Proof simplification

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \wedge A]_z}{A}\,\wedge\text{-E}_1 \qquad \dfrac{[B \wedge A]_z}{B}\,\wedge\text{-E}_0}{A \wedge B}\,\wedge\text{-I}}{(B \wedge A) \Rightarrow (A \wedge B)}\,\Rightarrow\text{-I}_z \qquad \dfrac{[B]^y \quad [A]^x}{B \wedge A}\,\wedge\text{-I}}{A \wedge B}\,\Rightarrow\text{E}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{[B]^y \quad [A]^x}{B \wedge A}\,\wedge\text{-I}}{A}\,\wedge\text{-E}_1 \qquad \dfrac{\dfrac{[B]^y \quad [A]^x}{B \wedge A}\,\wedge\text{-I}}{B}\,\wedge\text{-E}_0}{A \,\&\, B}\,\wedge\text{-I}$$

$$\Downarrow$$

$$\dfrac{[A]^x \quad [B]^y}{A \wedge B}\,\wedge\text{-I}$$