

IA038 Types and Proofs

3. Natural Deduction

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Notation for proofs: from Frege to Hilbert school

- Frege started with pictorial notation for judgements

Modus Ponens:

$\vdash \frac{A}{B}$ B implies A

$\vdash B$ B is asserted

$\vdash A$ A follows

- Notation for judgements evolved into $\vdash A$ as the judgement that A is asserted

- Modus Ponens and axiom schemes:

$$\frac{\vdash B \rightarrow A \quad \vdash B}{\vdash A}$$

$$\begin{aligned} & \vdash A \rightarrow A & (1) \\ & \vdash A \rightarrow (B \rightarrow A) & (2) \\ & \vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) & (3) \end{aligned}$$

Gentzen's notation and rules for Natural deduction

- Gentzen introduced judgements with assumptions,

$B_1, \dots, B_n \vdash A$ meaning A proved from assumptions B_1, \dots, B_n

- Modus Ponens in Gentzen's system:

$$\frac{\Gamma \vdash B \rightarrow A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A}$$

here Γ and Δ are lists of propositions, Γ, Δ means union of two lists (here just as finite sets)

Gentzen's rules for Natural deduction

$$\frac{\Gamma, B \vdash A}{\Gamma \vdash B \rightarrow A} \rightarrow\text{-I}$$

$$\frac{}{A \vdash A} \text{Id}$$

$$\frac{\Gamma \vdash B \rightarrow A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A} \rightarrow\text{-E}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge\text{-I}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-E}_1$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-E}_2$$

Gentzen's tree notation:

&-I

$$\frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \& \mathcal{B}}$$

&-E

$$\frac{\mathcal{A} \& \mathcal{B}}{\mathcal{A}} \quad \frac{\mathcal{A} \& \mathcal{B}}{\mathcal{B}}$$

v-I

$$\frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \vee \mathcal{B}} \quad \frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \vee \mathcal{B}}$$

v-E

$$\frac{\mathcal{A} \vee \mathcal{B} \quad \begin{array}{l} [\mathcal{A}] \\ \mathcal{C} \end{array} \quad \begin{array}{l} [\mathcal{B}] \\ \mathcal{C} \end{array}}{\mathcal{C}}$$

v-I

$$\frac{\exists a \mathcal{A}}{\forall x \mathcal{A}}$$

v-E

$$\frac{\forall x \mathcal{A}}{\exists x \mathcal{A}}$$

∃-I

$$\frac{\exists a \mathcal{A}}{\exists x \mathcal{A}}$$

∃-E

$$\frac{\exists x \mathcal{A} \quad \begin{array}{l} [\exists a] \\ \mathcal{C} \end{array}}{\mathcal{C}}$$

⊃-I

$$\frac{\begin{array}{l} [\mathcal{A}] \\ \mathcal{B} \end{array}}{\mathcal{A} \supset \mathcal{B}}$$

⊃-E

$$\frac{\mathcal{A} \quad \mathcal{A} \supset \mathcal{B}}{\mathcal{B}}$$

¬-I

$$\frac{\begin{array}{l} [\mathcal{A}] \\ \wedge \\ \neg \mathcal{A} \end{array}}{\neg \mathcal{A}}$$

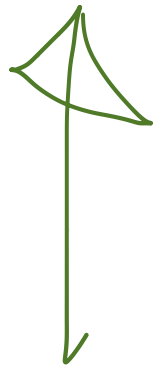
¬-E

$$\frac{\mathcal{A} \quad \neg \mathcal{A} \quad \wedge}{\wedge} \quad \frac{\wedge}{\mathcal{D}}$$

Example proof



$$\frac{\frac{\frac{}{\{B \wedge A\} \vdash B \wedge A} \text{Id}}{\{B \wedge A\} \vdash A} \wedge\text{-E}_2 \quad \frac{\frac{}{\{B \wedge A\} \vdash B \wedge A} \text{Id}}{\{B \wedge A\} \vdash B} \wedge\text{-E}_1}{\frac{}{\{B \wedge A\} \vdash A \wedge B} \wedge\text{-I}}{\frac{}{\{\} \vdash (B \wedge A) \rightarrow (A \wedge B)} \rightarrow\text{-I}} \quad \frac{\frac{}{\{B\} \vdash B} \text{Id} \quad \frac{}{\{A\} \vdash A} \text{Id}}{\{A, B\} \vdash B \wedge A} \wedge\text{-I}}{\frac{}{\{A, B\} \vdash A \wedge B} \rightarrow\text{-E}}$$



Gentzen's rules for Natural deduction

$$\begin{array}{c}
 \frac{\Gamma, B \vdash A}{\Gamma \vdash B \rightarrow A} \rightarrow\text{-I} \\
 \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge\text{-I} \\
 \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-E}_1 \\
 \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-E}_2 \\
 \frac{\Gamma \vdash B \rightarrow A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A} \rightarrow\text{-E} \\
 \frac{}{A \vdash A} \text{Id}
 \end{array}$$

Modern notation (Prawitz, Girard, ...):

- *Hypothesis:* A

- *Introductions:*

$$\frac{\begin{array}{c} \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ B \end{array}}{A \wedge B} \wedge\text{I}$$

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \Rightarrow B} \Rightarrow\text{I}$$

- *Eliminations:*

$$\frac{\begin{array}{c} \vdots \\ A \wedge B \end{array}}{A} \wedge\text{1E}$$

$$\frac{\begin{array}{c} \vdots \\ A \wedge B \end{array}}{B} \wedge\text{2E}$$

$$\frac{\begin{array}{c} \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ A \Rightarrow B \end{array}}{B} \Rightarrow\text{E}$$

Example proof

$$\begin{array}{c}
 \frac{\overline{\{B \wedge A\} \vdash B \wedge A} \text{ Id}}{\{B \wedge A\} \vdash A} \wedge\text{-E}_2 \quad \frac{\overline{\{B \wedge A\} \vdash B \wedge A} \text{ Id}}{\{B \wedge A\} \vdash B} \wedge\text{-E}_1 \\
 \hline
 \frac{\{B \wedge A\} \vdash A \quad \{B \wedge A\} \vdash B}{\{B \wedge A\} \vdash A \wedge B} \wedge\text{-I} \\
 \frac{\{B \wedge A\} \vdash A \wedge B}{\{\} \vdash (B \wedge A) \rightarrow (A \wedge B)} \rightarrow\text{-I} \\
 \hline
 \frac{\{\} \vdash (B \wedge A) \rightarrow (A \wedge B) \quad \frac{\overline{\{B\} \vdash B} \text{ Id} \quad \overline{\{A\} \vdash A} \text{ Id}}{\{A, B\} \vdash B \wedge A} \wedge\text{-I}}{\{A, B\} \vdash A \wedge B} \rightarrow\text{-E}
 \end{array}$$

$$\begin{array}{c}
 \boxed{\frac{\overline{\{B \wedge A\} \vdash B \wedge A} \text{ Id}}{\{B \wedge A\} \vdash A} \wedge\text{-E}_2} \quad \boxed{\frac{\overline{\{B \wedge A\} \vdash B \wedge A} \text{ Id}}{\{B \wedge A\} \vdash B} \wedge\text{-E}_1} \\
 \hline
 \frac{\{B \wedge A\} \vdash A \quad \{B \wedge A\} \vdash B}{\{B \wedge A\} \vdash A \wedge B} \wedge\text{-I} \\
 \frac{\{B \wedge A\} \vdash A \wedge B}{\{\} \vdash (B \wedge A) \rightarrow (A \wedge B)} \rightarrow\text{-I} \\
 \hline
 \frac{\{\} \vdash (B \wedge A) \rightarrow (A \wedge B) \quad \frac{\overline{\{B\} \vdash B} \text{ Id} \quad \overline{\{A\} \vdash A} \text{ Id}}{\{A, B\} \vdash B \wedge A} \wedge\text{-I}}{\{A, B\} \vdash A \wedge B} \rightarrow\text{-E}
 \end{array}$$

Proof simplification

$\lambda z. (\tau_1(z), \tau_2(z))$

$$\frac{\frac{[B \wedge A]_z}{A} \wedge\text{-E}_1 \quad \frac{[B \wedge A]_z}{B} \wedge\text{-E}_0}{A \wedge B} \wedge\text{-I} \quad \Rightarrow\text{-I}_z$$

$$\frac{[B]^y \quad [A]^x}{B \wedge A} \wedge\text{-I} \quad \Rightarrow\text{E}$$

$$\frac{\frac{[B]^y \quad [A]^x}{B \wedge A} \wedge\text{-I} \quad \frac{[B]^y \quad [A]^x}{B \wedge A} \wedge\text{-I}}{A \quad B} \wedge\text{-E}_1 \quad \wedge\text{-E}_0 \quad \wedge\text{-I}$$

$$\frac{[A]^x \quad [B]^y}{A \wedge B} \wedge\text{-I}$$

$\lambda x y. (x, y)$

