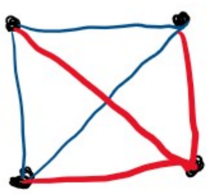
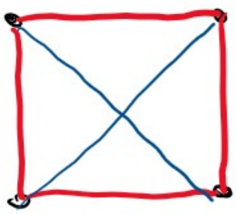
The probabilistic method

Ramsey number

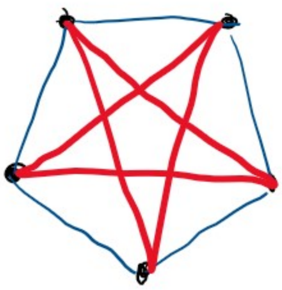
Ramsey number $R(k,t)$ is the smallest n, such that each 2-coloring of the edges of K_n (complete graph of n vertices) has a red subgraph K_k or a blue subgraph K_t.

$\boxed{R(3,3)} > 4$

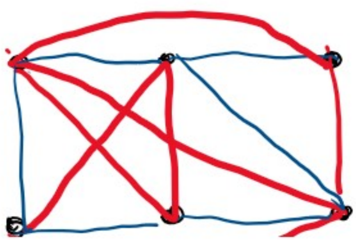




$R(3,3) > 5$

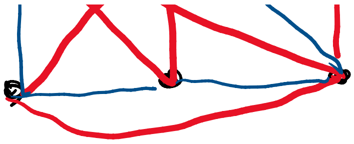


$R(3,3) = 6$



How many colorings of K_6 are there?

$\binom{6}{2} = 15$

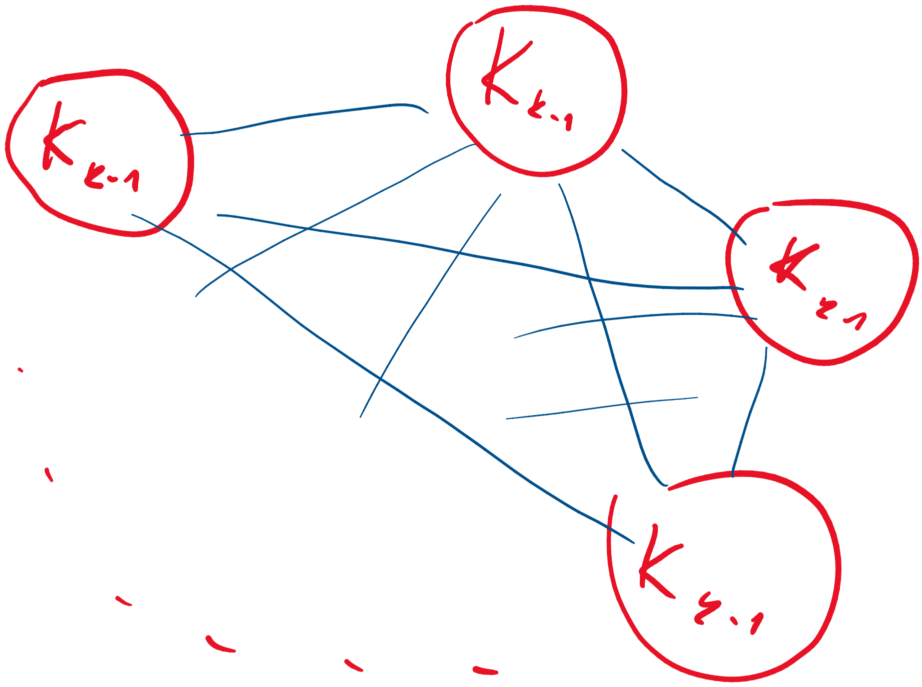


$\binom{6}{2} = 15$

$2^{15} \approx 32000$

How does $R(k,k)$ scale with k.

Can you find a constructive lower bound?



$R(k,k) > (k-1)^2$

$R(3,3) > 4$

$R(4,4) = 18 > 9$

$43 \leq R(5,5) \leq 49$

$\vee$

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PROBABILISTIC ARGUMENT

$\Rightarrow$ Color each edge at random and if probability of a 'counter example' is larger than 0, the counterexample must exist $\Rightarrow$ lower bound on $R(k,k)$.

Thm (from slides)

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1 \Rightarrow R(k,k) > n$$

Let us consider the following random experiment:

Color each edge of K_n red w.p. $1/2$
blue w.p. $1/2$

For each $S \subset V$, $|S| = k$

$r_S = 1$ if graph induced by S is all red

$b_S = 1$ ———— ———— all blue

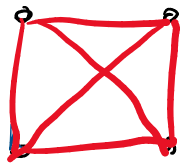
$$\forall S \quad \Pr(r_S = 1) = \frac{1}{2}^{\binom{k}{2}}$$

$$\forall S \quad \Pr(b_S = 1) = \frac{1}{2}^{\binom{k}{2}}$$

$$\forall S \quad \Pr(r_S = 1 \vee b_S = 1) = 2 \cdot \frac{1}{2}^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

What is the probability that some $S \subset V_1$, $|S| = k$ is monochromed?

$$\boxed{\Pr\left(\bigvee_{S\subset V_1,\, |S|=k} r_S = 1 \vee b_S = 1\right) < \binom{n}{k} \cdot 2^{1-\binom{k}{2}}}$$



$$1 - \Pr\left(\bigvee_{S\subset V_1,\, |S|=k} r_S = 1 \vee b_S = 1\right)$$ $\rightarrow$ probability of obtaining a 'counter example'

We need $$1 - \Pr\left(\bigvee_{S\subset V_1,\, |S|=k} r_S = 1 \vee b_S = 1\right) > 0$$

$$\Pr\left(\bigvee_{S\subset V_1,\, |S|=k} r_S = 1 \vee b_S = 1\right) < 1$$

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

if $n = \lfloor 2^{\frac{k}{2}} \rfloor$ then $R(k,k) \geq n$

Plug $n = \lfloor 2^{\frac{k}{2}} \rfloor$ into and check if it holds.

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} \cdot 2^{1-\frac{k(k-1)}{2}} \qquad \frac{n!}{k!\cancel{(n-k)}!} \qquad n^k$$

$\binom{n}{k} \cdot 2 \quad \leq \frac{}{k!} \cdot 2$ $\quad\quad k!\cancel{(n-k)!}$

$(n \cdot (n-1) \cdots (n-k+1))$ $\quad n^{}$

$$\leq \frac{\left(2^{\frac{k}{2}}\right)^k}{k!} \cdot \frac{2}{2^{\frac{k(k-1)}{2}}}$$

$$\leq \frac{\cancel{2^{\frac{k^2}{2}}}}{k!} \cdot \frac{2}{\cancel{2^{\frac{k^2}{2}}} \cdot 2^{-\frac{k}{2}}}$$

$$\leq \frac{2 \cdot 2^{\frac{k}{2}}}{k!} = \frac{2^{1+\frac{k}{2}}}{k!}$$

$k=3 \quad \frac{2^{3/2}}{3} = \frac{\sqrt{8}}{3} \approx 0.9$

$k=4 \quad \frac{2^{1+2}}{4!} = \frac{8}{24} = 1/3$

$R(3,3) \geq \lfloor 2^{\frac{3}{2}} \rfloor = \lfloor \sqrt{8} \rfloor = 2$

$R(4,4) \geq \lfloor 2^{\frac{4}{2}} \rfloor = 4$

$R(8,8) > \lfloor 2^4 \rfloor = 16$

$(k-1)^2 < 2^{\frac{k}{2}}$

$k=16$

$2^{\frac{k}{2}} > (k-1)^2$

$$2^{\frac{s}{2}} > (s-1)^2$$

$$256 > 225$$

Thm

if $\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$ for some $0 \le p \le 1$

then $R(s,t) > n$.

$\boxed{R(4,t)}$

$$\binom{n}{4} \cdot p^6 + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1 \qquad \boxed{p = n^{-2/3}}$$

$\wedge$

$$\frac{\cancel{n^4}}{4!} \cdot \cancel{n^{-4}} + \binom{n}{t}\left(1-n^{-2/3}\right)^{\binom{t}{2}} < 1$$

$$\binom{n}{t}\left(1-n^{-2/3}\right)^{\binom{t}{2}} < \frac{23}{24}$$

$$n = \lfloor 2^{\frac{t}{2}} \rfloor$$