Probability

PA154 Language Modeling (1.2)

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Source: Introduction to Natural Language Processing (600.465)
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Experiments & Sample Spaces

- Experiment, process, test, ...
- Set of possible basic outcomes: sample space Ω základní prostor obsahující možné výsledky)
  - coin toss (Ω = $\{$head, tail$\}$), die (Ω = $\{1..6\}$)
  - yes/no opinion poll, quality test (bad/good) (Ω = $\{0,1\}$)
  - lottery ($|\Omega| \cong 10^7..10^{12}$)
  - # of traffic accidents somewhere per year (Ω = N)
  - spelling errors ($\Omega = \mathrm{Z}^*$), where Z is an aplhabet, and Z^* is set of possible strings over such alphabet
  - missing word ($|\Omega| \cong$ vocabulary size)

Events

- Event jev) A is a set of basic outcomes
- Usually $A \subset \Omega$, and all $A \in 2^\Omega$ (the event space, jevové pole)
  - Ω is the certain event jistý jev), $\emptyset$ is the impossible event nemožný jev)
- Example:
  - experiment: three times coin toss
    - **Ω = $\{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$**
  - count cases with exactly two tails: then
    - **A = $\{$HTT, THT, TTH$\}$**
  - all heads:
    - **A = $\{$HHH$\}$**

Probability

- Repeat experiment many times, record how many times a given event A occured ("count" c_1).
- Do this whole series many times; remember all c_is.
- Observation: if repeated really many times, the ratios of $\frac{c_i}{T_i}$ (where T_i is the number of experiments run in the *i-th* series) are close to some (unknown but) **constant** value.
- Call this constant a **probability of A**. Notation: **p(A)**

Estimating Probability

- Remember: ...close to an *unknown* constant.
- We can only estimate it:
  - from a single series (typical case, as mostly the outcome of a series is given to us we cannot repeat the experiment):

$$p(A) = \frac{c_1}{T_1}$$

  - otherwise, take the weighted average of all $\frac{c_i}{T_i}$ (or, if the data allows, simply look at the set of series as if it is a single long series).
- This is the **best** estimate.

Example

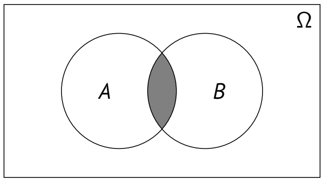
- Recall our example:
  - experiment: three times coin toss
    - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
  - count cases with exactly two tails: $A = \{HTT, THT, TTH\}$
- Run an experiment 1000 times (i.e. 3000 tosses)
- Counted: 386 cases with two tails (**HTT, THT or TTH**)
- estimate: p(A) = 386/1000 = .386
- Run again: 373, 399, 382, 355, 372, 406, 359
  - p(A) = .379 (weighted average) or simply 3032/8000
- *Uniform* distribution assumption: p(A) = 3/8 = .375

Basic Properties

- Basic properties:
  - p: $2^{\Omega} \rightarrow [0, 1]$
  - p(Ω) = 1
  - Disjoint events: p($\cup$ A_i) = $\sum_i$p(A_i)
- NB: *axiomatic definiton* of probability: take the above three conditions as axioms
- Immediate consequences:
  - P($\emptyset$) = 0
  - p($\bar{A}$) = 1 - p(A)
  - A $\subseteq$ B $\Rightarrow$ p(A) $\leq$ P(B)
  - $\sum_{a\in\Omega}$p(a) = 1

Joint and Conditional Probability

- $p(A, B) = p(A \cap B)$
- $p(A|B) = \frac{p(A, B)}{p(B)}$
  - Estimating form counts:
    - $p(A|B) = \frac{p(A, B)}{p(B)} = \frac{\frac{c(A \cap B)}{T}}{\frac{c(B)}{T}} = \frac{c(A \cap B)}{c(B)}$

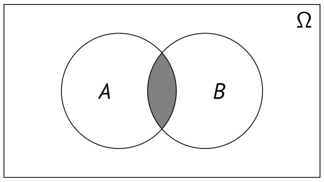


Bayes Rule

- p(A,B) = p(B,A) since p(A $\cap$ B) = p(B $\cap$ A)
  - therefore $p(A|B)p(B) = p(B|A)p(A)$, and therefore:

Bayes Rule

$$p(A|B) = \frac{p(B|A) \times p(A)}{p(B)}$$



Independence

- Can we compute p(A,B) from p(A) and p(B)?
- Recall from previous foil:

$$p(A|B) = \frac{p(B|A) \times p(A)}{p(B)}$$

$$p(A|B) \times p(B) = p(B|A) \times p(A)$$

$$p(A, B) = p(B|A) \times p(A)$$

...we're almost there: how $p(B|A)$ relates to p(B)?

  - p(B|A) = p(B) iff A and B are **independent**
- Example: two coin tosses, weather today and weather on March 4th 1789;
- Any two events for which p(B|A) = P(B)!

Chain Rule

$$p(A_1, A_2, A_3, A_4, \ldots, A_n) =$$
$$p(A_1|A_2, A_3, A_4, \ldots, A_n) \times p(A_2|A_3, A_4, \ldots, A_n) \times$$
$$\times p(A_3|A_4, \ldots, A_n) \times \cdots \times p(A_{n-1}|A_n) \times p(A_n)$$

- this is a direct consequence of the Bayes rule.

The Golden Rule of Classic Statistical NLP

- Interested in an event A given B (where it is not easy or practical or desirable) to estimate $p(A|B)$:
- take Bayes rule, max over all Bs:
- $argmax_A p(A|B) = argmax_A \frac{p(B|A) \times p(A)}{p(B)} =$

  $\boxed{argmax_A (p(B|A) \times p(A))}$
- ...as p(B) is constant when changing As

Random Variables

- is a function $X : \Omega \rightarrow Q$
  - in general $Q = R^n$, typically R
  - easier to handle real numbers than real-world events
- random variable is *discrete* if Q is countable (i.e. also if finite)
- Example: *die*: natural "numbering" [1,6], *coin*: $\{0,1\}$
- Probability distribution:
  - $p_X(x) = p(X = x) =_{df} p(A_x)$ where $A_x = \{a \in \Omega : X(a) = x\}$
  - often just $p(x)$ if it is clear from context what X is

Expectation
Joint and Conditional Distributions

- is a mean of a random variable (weighted average)
  - $E(X) = \sum_{x \in X(\Omega)} x.p_X(x)$
- Example: one six-sided die: 3.5, two dice (sum): 7
- Joint and Conditional distribution rules:
  - analogous to probability of events
- Bayes: $p_{X|Y}(x,y) =_{notation} p_{XY}(x|y) =_{\text{even simpler notation}}$

$$\boxed{p(x|y) = \frac{p(y|x).p(x)}{p(y)}}$$

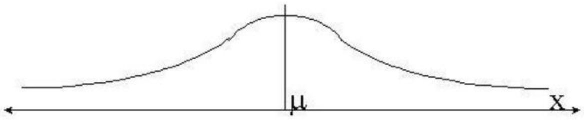
- Chain rule: $\boxed{p(w,x,y,z) = p(z).p(y|z).p(x|y,z).p(w|x,y,z)}$

Standard Distributions

- Binomial (discrete)
  - outcome: 0 or 1 (thus *bi*nomial)
  - make n trials
  - interested in the (probability of) numbers of successes r
- Must be careful: it's not uniform!
- $p_b(r|n) = \frac{\binom{n}{r}}{2^n}$ (for equally likely outcome)
- $\binom{n}{r}$ counts how many possibilities there are for choosing r objects out of n;
- $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

Continuous Distributions

- The normal distribution ("Gaussian")
- $p_{norm}(x|\mu,\sigma) = exp\left[\frac{\frac{-(x-\mu)^2}{2\sigma^2}}{\sigma\sqrt{2\pi}}\right]$
- where:
  - μ is the mean (x-coordinate of the peak) (0)
  - σ is the standard deviation (1)



- other: hyperbolic, t