IA081 Lambda-Calculus

1. History of Math & Motivations

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Nature of Mathematical Objects

- Plato's "Academy": *"Let no one who is not a geometer enter"* – Plato (427-348 BCE)

- Platonism:

1. There are mathematical objects
2. These are abstract objects existing outside of space and time
3. Math objects always existed & are entirely independent od people
4. Math objects do not interact with the physical world in any "causal" way – we cannot change them, and they cannot change us, yet
5. We are somehow able to gain knowledge of them

Prehistory of Formal Reasoning

- Aristotle (384-322 BCE) – Analytica Posteriora as a deductive science from basic truths or axioms
  * Deductive proofs as demonstration arguments in Euclid geometry
  * Logic in the form of syllogisms independent of mathematical/geometrical proofs

- Sufficient for more than two millenia

Positional number systems and calculations

- Numbers tied to areas

- Independent knowledge of Pythagoras (c. 580–500 BCE) Theorem (China - Zhou Bi Suan Jing, 1046–256 BCE, Babylonia – including sexagecimal approximation for the square root of 2 from 18th century BCE from Yale tablet, India – 2nd century BCE Bakhshali manuscript), hence construction of square roots

- Positional number system developped in India by Āryabhata (476–550), then adopted by Arabic Mathematicians via Brahmagupta by Al-Khwarizmi ("Algorithmi", c. 780–847+, founder of the Algebra)

- Fibonacci (1170–1250) brought it to Europe, first only as a tool for scientific calculations

- French national Audit Office used Roman numerals as late as in 18th century

Algebra

- al-Khwarizmi ("Algorithmi", c. 780–847+)
- the book "al-Kitab al-Mukhtasar fi Hisab al-Jebr wal-Muqabala" (The Compendium on Calculation by Restoration and Balancing)
- Full algebraic treatment of mathematical problem solving, including quadratic equations and systems of equations
- Translated to Latin independently by Robert of Chester (England) and Gerard of Cremona (Italy) in the 12th century
- Known to Fibonacci who quotes al-Khwarizmi in his "Liber Abaci" of 1202 (under latinicized version of his first name, Maumeth)
- al-Jebr gave rise to "algebra"
- latinicized surname Algorithmi gave rise to "algorithm" (solutions in al-Jebr are actually presented as algorithms to solve algebraic problems)

Logicism

- Gottfried Wilhelm Leibniz (1646-1716)

Mathematical facts as truths of reason

Hoped in "calculus ratiocinator" as a systematic calculational logic for representation of human reasoning (similar to the differential calculus for mathematical physics)

Introduced logical operations (but did not publish this)

- Immanuel Kant (1724-1804) – Critique of Pure Reason

Arithmetic and Geometry as synthetic a priori issues akin to Metaphysics

- Richard J. W. Dedekind (1831-1916) - culmination of arithmetization of Arithmetic and Geometry

Logicism

- Gottlob Frege (1848-1925)
- Begriffsschrift, 1879 (concept notation for pure thought / logic)
- Grundlagen der Arithmetic, 1884
- ➢ Analyticity of arithmetic truths derived from their
- ➢ 7+5=12 as analytical truth (contrary to Kant)
- Grundsetze der Arithmetic, Vol. 1, 1893
- ➢ distinction between sense and reference
- ➢ introduction of notation for concepts and semantics
- ➢ symbolic language for expressing everything explicitly & finite set of rules
- Vol. 2 was at the publisher in 1903, when Russel wrote to Frege about the Russel Paradox ($A=\{B: B$ is a set & $B \notin B\}$, and both $A \in A$, and $A \notin A$)
- ❖ Problems with infinite sets in logicism, also Gödel's incompleteness

Formalization of logical inference

- Giuseppe Peano (1858-1932)

Around 1890 formalization of logical inference

Formal rules based on axioms and Modus Ponens (A=>B, A |- B)

- Bertrand Russel (1872-1970)

Principia Mathematica, 1910-13, with Whitehead: expressing axioms as basic truths, and deriving logical truths by Modus Ponens and universal generalization

Combinatory Logic

- Moses Iljich Schönfinkel (1888–1942)

- „Elemente der Logic" (1920) –a talk in Hilbert's group in Gottingen

- replacement of functions of more arguments by higher-order unary ones (f(x,y)=g(x)(y) with g(x): y |-> f(x,y)

- Combinatory Logic for defining funcions without any use of variables (**I**,**K**,**S**: **I**M=M, **K**MN=M, and **S**MNL=ML(NL))

- Haskell Curry (1900–1982) – development of Combinatory Logic as a whole system (see also programing languages Haskell, Brook, and Curry)

David Hilbert (1862-1943) Grundlagen der Geometrie,1899

Four foundational problems:

1. Formalization of mathematical theory
2. Proof of consistency of the the axioms
3. Independence and completeness of the axioms
4. The decision problem: is there a method answering any question in the theory?

David Hilbert

- By 1920's, "Hilbert style" axiomatic approach dominates

- Propositional logic proved complete and decidable

- Predicate logic presented in Hilbert style by Ackermann by 1920

Hilbert Program ~1920

- **Hilbert Program**: expressing higher mathematics in terms of elementary Arithmetics;
  formalizing all Mathematics in axiomatic form together with a proof of completeness
  (finitistic methods, purely intuitive basis of concrete signs)
- P. Bernays, W. Ackermann, J. von Neumann, J. Herbrand

- Ackermann and von Neumann – proof of consistency of number theory, Ackermann thought near completion for analysis
- Hilbert claimed in 1928 in Bologna that the work is essentially completed

Kurt Gödel: completeness of First-Order Predicate Logic

- Completeness of First-Order Predicate Logic stated by Hilbert and Ackermann in 1928

- Kurt Gödel tackled this in his doctoral thesis in 1929

- Thm: Every logical expression is either satisfiable or refutable, aka Every valid logical expression is provable

- Presented as his forthcoming PhD thesis in September 1930 in Königsberg

Kurt Gödel: incompleteness of formal systems

- Also in September 1930 in Königsberg, presented as an "aside" (not a talk on the conference programme)

- The First Incompleteness Thm showing arithmetic is neither provable nor refutable in Peano arithmetic

- The Second Incompleteness Thm showing that consistency of arithmetic cannot be proved in Arithmetics itself, $Con(P) \equiv \neg \, Prov(\lceil 0 \rceil = \lceil 1 \rceil)$

- von Neumann interrupted his lectures on Hilbert proof theory in the Fall of 1930; seeing Hilbert program could not be achieved at all
  → Destruction of Hilbert Program (impossibility of proving consistence of a formal system inside of it)

Intuitionism (constructivism)

- L.E.J. Brower (1881-1966): mathematical knowledge comes from constructing mathematical objects within human intuition; belief that the law of excluded middle, or indirect existential proofs are dangerous to the coherence of Mathematics

- Around 1908 – clash with Hilbert, also in "The untrustworthiness of the principles of logic" challenged the belief that the rules of classical logic which came from Aristotle have absolute validity

- Arend Heyting (1898-1980) was his PhD student developing intuitionism further

Gödel and Gentzen

- Translation from Peano arithmetic to intuitionistic Heyting arithmetic by Gödel, in parallel with Gentzen, 1932-33

Gentzen

- Gentzen thesis (1934-35) on analysis of mathematical proofs
- Natural Deduction (intro & elim rules, esp. suitable for intuitionistic logic)

$$\frac{A \quad B}{A \,\&\, B} \;\&\text{I}$$

$$\frac{A \,\&\, B}{A} \;\&\text{E}_1 \qquad \frac{A \,\&\, B}{B} \;\&\text{E}_2$$

$$\frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \supset B} \;\supset\text{I}$$

$$\frac{A \supset B \quad A}{B} \;\supset\text{E}$$

Gentzen

- Gentzen thesis (1934-35) on analysis of mathematical proofs
- "Sequenzenkalkul", Sequent Calculus
- Normalization and cut-elimination

Conjunction

$$\frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \& B} \text{ R\&} \quad \frac{A, \Gamma \to \Delta}{A \& B, \Gamma \to \Delta} \text{ L\&}_1 \quad \frac{B, \Gamma \to \Delta}{A \& B, \Gamma \to \Delta} \text{ L\&}_2$$

Implication

$$\frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B} \text{ R}\supset \quad \frac{\Gamma \to \Theta, A \quad B, \Delta \to \Lambda}{A \supset B, \Gamma, \Delta \to \Theta, \Lambda} \text{ L}\supset$$

$$\frac{\Gamma \to A \quad A, \Delta \to C}{\Gamma, \Delta \to C} \text{ Cut}$$

Gödel and Gentzen

- Gentzen gave an alternative proof of the Incompleteness Thm (written 1939, published 1943) by showing a formula unprovable in Peano arithmetic (thus also showing consistency of Peano arithmetic)
- Gödel's proof of consistency using Dialectica interpretation

(Unclear mutual interaction in 1939.)

Computability and Undecidable problems

- Hilbert (1928): "Entscheidungsproblem": Is there a general effective procedure deciding whether or not a given formula A of a calculus is provable?

- 1936: Alonzo Church proved on the basis of λ-calculus and Alan Turing on the basis of Turing machines several months later that

  the answer to the Entscheidungsproblem is negative

- Existence of *undecidable problems* in Informatics

Alonzo Church (1903–1995) and his λ-calculus

- λ-abstraction: making bound variables in function definitions explicit

- λx(M) means definition of a function mapping an argument a into M[x/a] ("formal parameters" in programming languages)

- Operational semantics via rewriting: λx(M)N –> M[x/N]

- Fixed-point Theorem: For each F there is X s.t. FX= X

- There is a fixed-point combitator **Y** s.t. F(**Y**F) = **Y**F

- Proof: **Y** = λf(λx.f(xx))(λx.f(xx))

Computability and Undecidable problems

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- Existence of *undecidable problems* in Informatics
- Church-Turing thesis (Kleene, 1952) – computable functions / computability

Church, Turing, and Gödel

- Church using λ-calculus as a formal tool tried to formalize mathematics

- Learning about Gödel's result, claimed that it does not apply to this system

- Kleene - recursive functions

- Rosser - reductions

- Church-Turing thesis (Kleene, 1952) for computable functions / computability:
  - Turing machines
  - λ-definability
  - Gödel's general recursive functions (Princeton, 1934)

Inconsistency in the presence of Logic –
the Curry paradox (1942, after Kleene&Rosser, 1935)

- Logical principles:

    P⊃P (**1**) and

    (P⊃(P⊃Q)) ⊃(P⊃Q) (**2**) together with

    Modus Ponens (from P and P⊃Q infer Q)

- Now construct F such that

    F = F⊃A (**3**)

By taking F = λx(x⊃A) in the **Fixed-Point theorem**.

- By (**1**) we get

    F ⊃ F

- And applying (**2**) to the second F yields

    F⊃ (F⊃A)

- Now using (**2**) and **Modus Ponens** one gets

    F⊃A

- And by (**3**) in the reverse we receive

    F

- And by **Modus Ponens** we get

    A, for any A,   hence a paradox.

Completing Gödel's rupture in the structure of mathematics

- Gödel opened a crack in the foundations of Mathematics

- The complement of this rupture lays outside of combinatorial formulation; still within the real of Mathematics

- The "inside" of this crack opened up a new discipline, Informatics; may be thought of as a camouflage of formal logic (proof theory) into fairly applicable computational tool

Proof theory as a basis for constructivistic formulation

- Frege vs. Hilbert concerning Platonism vs. Formalism (Hilbert – Mathematics is invented and best viewed as formal symbolic games without intrinsic meaning)

- Hilbert vs. Brower concerning Formalism vs. Intuitionism

- Constructivism (Kronecker (1823-1891) with "God made the natural numbers, all else is the work of man", and esp. Andrej Markov who claimed Mathematics should deal exclusively with constructive objects)

- Proof generation as object construction

- Proof simplification/normalization as a computational mechanism (even without underlying formula semantics)

Completion Requirements

➢ Essay on a topic using this concept, developing necessary mathematical details.

➢ Structure of the essay corresponding to an article introducing the topic, and developing details.

➢ This may either be something relevant to your work/interest, or e.g. taking some of the results concerning λ-Calculus or Combinatory Logic within a suitable formal system, and completing proofs in sufficient mathematical detail.

➢ 3-5 thousand words (approx. 6-12 pages).