IA038 Types and Proofs

4. The Curry-Howard Isomorphism

Jiří Zlatuška

March 14, 2023

3.1.2 Terms

$[x_i : T_i]^p$ Hypothesis [and its parcel]

$$\dfrac{\begin{array}{c}[x:U]^x \\ \vdots \\ v:V\end{array}}{\lambda x.v : U \to V}\ \to\text{-I}^x \qquad \dfrac{t : U \to V \quad u : U}{tu : V}\ \to\text{-E}$$

MP

$$\dfrac{u : U \quad v : V}{\langle u, v\rangle : U \times V}\ \times\text{-I} \qquad \dfrac{t : U \times V}{\pi^1 t : U}\ \times\text{-1E} \qquad \dfrac{t : U \times V}{\pi^2 t : V}\ \times\text{-2E}$$

3.1.2. Terms using another formulation

$$\frac{}{x_i : T_i \vdash x_i : T_i}\,\mathrm{Id}_p \qquad \text{Axiom [parcel } p\text{]}$$

$$\frac{\Gamma, x : U \vdash v : V}{\Gamma \vdash \lambda x.v : U \to V}\,\to\text{-I} \qquad \frac{\Gamma \vdash t : U \to V \quad \Delta \vdash u : U}{\Gamma, \Delta \vdash tu : V}\,\to\text{-E}$$

$$\frac{\Gamma \vdash u : U \quad \Delta \vdash v : V}{\Gamma, \Delta \vdash \langle u, v\rangle : U \times V}\,\times\text{-I} \qquad \frac{\Gamma \vdash t : U \times V}{\Gamma \vdash \pi^1 t : U}\,\times\text{-1E} \qquad \frac{\Gamma \vdash t : U \times V}{\Gamma \vdash \pi^2 t : V}\,\times\text{-2E}$$

or expressed in logic notation

$$\frac{}{x_i : T_i \vdash x_i : T_i}\,\mathrm{Id}_p \qquad \text{Axiom [parcel } p\text{]}$$

$$\frac{\Gamma, x : U \vdash v : V}{\Gamma \vdash \lambda x.v : U \Rightarrow V}\,\Rightarrow\text{-I} \qquad \frac{\Gamma \vdash t : U \Rightarrow V \quad \Delta \vdash u : U}{\Gamma, \Delta \vdash tu : V}\,\Rightarrow\text{-E}$$

$$\frac{\Gamma \vdash u : U \quad \Delta \vdash v : V}{\Gamma, \Delta \vdash \langle u, v\rangle : U \wedge V}\,\wedge\text{-I} \qquad \frac{\Gamma \vdash t : U \wedge V}{\Gamma \vdash \pi^1 t : U}\,\wedge\text{-1E} \qquad \frac{\Gamma \vdash t : U \wedge V}{\Gamma \vdash \pi^2 t : V}\,\wedge\text{-2E}$$

3.1.4 Conversion

Expressed using Natural Deduction derivation trees:

$$\dfrac{\dfrac{\begin{array}{c}[x:U]^x\\ \vdots\\ v:V\end{array}}{\lambda x.v : U\to V}\,\to\text{-I}^x \qquad u:U}{(\lambda x.v)u : V}\,\to\text{-E} \quad\Longrightarrow\quad v[u/x]:V \qquad \dfrac{\begin{array}{c}\vdots\\ U\end{array}\qquad \dfrac{\begin{array}{c}U\\ \vdots\\ V\end{array}}{U\Rightarrow V}\,\Rightarrow\text{-I}}{V}\,\Rightarrow\text{E} \quad\Longrightarrow\quad \begin{array}{c}\vdots\\ U\\ \vdots\\ V\end{array}$$

$$\dfrac{\dfrac{u:U \qquad v:V}{\langle u,v\rangle : U\times V}\,\times\text{-I}}{\pi^1\langle u,v\rangle : U}\,\times\text{-1E} \quad\Longrightarrow\quad u:U \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ U\end{array}\quad \begin{array}{c}\vdots\\ V\end{array}}{U\wedge V}\,\wedge\text{-I}}{U}\,\wedge\text{-1E} \quad\Longrightarrow\quad \begin{array}{c}\vdots\\ U\end{array}$$

$$\dfrac{\dfrac{u:U \qquad v:V}{\langle u,v\rangle : U\times V}\,\times\text{-I}}{\pi^2\langle u,v\rangle : U}\,\times\text{-2E} \quad\Longrightarrow\quad v:V \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ U\end{array}\quad \begin{array}{c}\vdots\\ V\end{array}}{U\wedge V}\,\wedge\text{-I}}{V}\,\wedge\text{-2E} \quad\Longrightarrow\quad \begin{array}{c}\vdots\\ V\end{array}$$

Conversion expressed using alternative logical system for ND

$$\dfrac{\dfrac{\Gamma, x : U \vdash v : V}{\Gamma \vdash \lambda x.v : U \to V}\,\to\text{-I} \qquad \cancel{\Delta} \vdash u : U}{\Gamma, \Delta \vdash (\lambda x.t)u : V}\,\to\text{-E} \quad \Longrightarrow \quad \Gamma, \Delta \vdash t[u/x] : V$$

$$\dfrac{\dfrac{\Gamma \vdash u : U \quad \Delta \vdash v : V}{\Gamma, \Delta \vdash \langle u, v\rangle : U \times V}\,\times\text{-I}}{\Gamma, \Delta \vdash \pi^1\langle u, v\rangle : U}\,\times\text{-1E} \quad \Longrightarrow \quad \Gamma \vdash u : U$$

$$\dfrac{\dfrac{\Gamma \vdash u : U \quad \Delta \vdash v : V}{\Gamma, \Delta \vdash \langle u, v\rangle : U \times V}\,\times\text{-I}}{\Gamma, \Delta \vdash \pi^2\langle u, v\rangle : U}\,\times\text{-2E} \quad \Longrightarrow \quad \Delta \vdash v : V$$

A few bits of history dates

1934 Gentzen's Natural Deduction

1940 Church's Lambda-Calculus

1956 Prawitz published normalization of Natural deduction proofs directly (Gentzen used Sequent Calculus, even though a direct proof in ND by him was recently discovered in his writings 1)

1956 Curry and Feys published Combinatory Logic, a system based purely on combinators

without variables; the types of basic combibators (I, K, S) corresponded to Hilbert axioms for Propositional Logic

1969 Howard combined results of Prawitz, Curry and Feys into the correspondence/isomorphism

1980 Howard's worh published in "To H. B. Curry", a festschrift for Curry's 80 birthday

[1] Plato and Gentzen: Gentzen's Proof of Normalization for Natural Deduction, The Bulletin of Symbolic Logic, Vol. 14, No. 2, June 2008, 240-257

Some essential pieces of the correspondence

Lambda-calculus	Proof Theory
variable (in terms)	assumption
term	proof (construction)
type variable	propositional variable
type	formula
type constructor	logical operation
typable term	construction of a proposition
redex	redundancy within a proof structure, lemma usage
reduction/computation	normalization
value	proof/construction in normal form
computation	normalization

Aside: Combinatory Logic and Hilbert-style systems

Combinatorial terms over alphabet consisting of constants $\mathbf{K}_{A\Rightarrow(B\Rightarrow A)}$, $\mathbf{S}_{(A\Rightarrow(B\Rightarrow C)\Rightarrow((A\Rightarrow B)\Rightarrow(A\Rightarrow C))}$, for every A, B, $C \in Typ$, and typed variables $x_T \in \mathcal{C}^T$:

$$\frac{}{\mathbf{K}_{A\Rightarrow(B\Rightarrow A)} \in \mathcal{C}^{A\Rightarrow(B\Rightarrow A)}} \qquad \frac{}{\mathbf{S}_{(A\Rightarrow(B\Rightarrow C))\Rightarrow((A\Rightarrow B)\Rightarrow(A\Rightarrow C))} \in \mathcal{C}^{(A\Rightarrow(B\Rightarrow C))\Rightarrow((A\Rightarrow B)\Rightarrow(A\Rightarrow C))}} \qquad \frac{}{x \in \mathcal{C}^T}\ x \text{ var of type } T$$

$$\frac{P \in \mathcal{C}_{A\Rightarrow B} \qquad Q \in \mathcal{C}_A}{(PQ) \in \mathcal{C}_B}$$

Hilbert system
Two axiom schemes

$$A \Rightarrow (B \Rightarrow A)$$
$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

and Modus Ponens:

$$\frac{A \qquad A \Rightarrow B}{B}$$

Example
Proof of $A \Rightarrow A$ in Hilbert system:

$$(\mathbf{S}_{(A\Rightarrow(B\Rightarrow A)\Rightarrow A)\Rightarrow(A\Rightarrow(B\Rightarrow A))\Rightarrow(A\Rightarrow A)}\mathbf{K}_{A\Rightarrow((B\Rightarrow A)\Rightarrow A)})\mathbf{K}_{A\Rightarrow(B\Rightarrow A)} \in \mathcal{C}^{A\Rightarrow A}$$

Hilbert system proof:

$$\dfrac{\dfrac{(A \Rightarrow ((B \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A)) \qquad A \Rightarrow ((B \Rightarrow A) \Rightarrow A)}{(A \Rightarrow (B \Rightarrow A) \Rightarrow (A \Rightarrow A)} \qquad A \Rightarrow (B \Rightarrow A)}{A \Rightarrow A}$$

This corresponds to a term $(\mathbf{SK})\mathbf{K} : A \Rightarrow A$:

$$\dfrac{\dfrac{\mathbf{S}_{(A\Rightarrow((B\Rightarrow A)\Rightarrow A))\Rightarrow((A\Rightarrow(B\Rightarrow A))\Rightarrow(A\Rightarrow A))} \qquad \mathbf{K}_{A\Rightarrow((B\Rightarrow A)\Rightarrow A)}}{\mathbf{SK} \in \mathcal{C}^{(A\Rightarrow(B\Rightarrow A))\Rightarrow(A\Rightarrow A)}} \qquad \mathbf{K}_{A\Rightarrow(B\Rightarrow A)}}{(\mathbf{SK})\mathbf{K} \in \mathcal{C}^{A\Rightarrow A}}$$

Conversion between proofs in Hilbert system and Natural Deduction follows from

1. $\lambda x.x = (\boldsymbol{SK})\boldsymbol{K}$,
2. $\lambda x.M = \boldsymbol{K}M$, for $x \notin \mathrm{FV}(M)$,
3. $\lambda x.MN = \boldsymbol{S}(\lambda x.M)(\lambda x.N)$.

Example of term normalization

$$(\lambda z.\langle\pi^2 z,\pi^1 z\rangle)(\langle y,x\rangle)\longrightarrow_\beta\langle\pi^2\langle y,x\rangle,\pi^1\langle y,x\rangle\rangle\longrightarrow_{second}\langle x,\pi^1\langle y,x\rangle\rangle\longrightarrow_{first}\langle x,y\rangle$$

Natural Deduction definition of conversion/proof simplification

$$\dfrac{\dfrac{\begin{matrix}[x:U]^x \\ \vdots \\ v:V\end{matrix}}{\lambda x.v : U \Rightarrow V}\Rightarrow\text{-I}^x \qquad u:U}{(\lambda x.v)u : V}\Rightarrow\text{-E} \quad \Longrightarrow \quad v[u/x]:V$$

$$\dfrac{\dfrac{u:U \quad v:V}{\langle u,v\rangle : U \wedge V}\wedge\text{-I}}{\pi^1\langle u,v\rangle : U}\wedge\text{-1E} \quad \Longrightarrow \quad u:U$$

$$\dfrac{\dfrac{u:U \quad v:V}{\langle u,v\rangle : U \wedge V}\wedge\text{-I}}{\pi^2\langle u,v\rangle : U}\wedge\text{-2E} \quad \Longrightarrow \quad v:V$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[z : B \times A]^z}{\pi^2 z : A}\,\times\text{-2E} \qquad \dfrac{[z : B \times A]^z}{\pi^1 z : A}\,\times\text{-1E}}{\langle \pi^2 z, \pi^1 z \rangle : A \times B}\,\times\text{-I}}{\lambda z.\langle \pi^2 z, \pi^1 z \rangle : (B \times A) \to (A \times B)}\,\to\text{-I}_z \qquad \dfrac{[y : B]^y \qquad [x : A]^x}{\langle y, x \rangle : B \times A}\,\times\text{-I}}{(\lambda z.\langle \pi^2 z, \pi^1 z \rangle)(\langle y, x \rangle) : A \times B}\,\to\text{-E}$$

$$\Big\Downarrow \quad \beta\text{-conversion}$$

$$\dfrac{\dfrac{\dfrac{[y : B]^y \qquad [x : A]^x}{\langle y, x \rangle : B \times A}\,\times\text{-I}}{\pi^2 \langle y, x \rangle : A}\,\times\text{-2E} \qquad \dfrac{\dfrac{[y : B]^y \qquad [x : A]^x}{\langle y, x \rangle : B \times A}\,\times\text{-I}}{\pi^1 \langle y, x \rangle : A}\,\times\text{-1E}}{\langle \pi^2 \langle y, x \rangle, \pi^1 \langle y, x \rangle \rangle : A \times B}\,\times\text{-I}$$

$$\Big\Downarrow\Big\Downarrow \quad \text{pairing}$$

$$\dfrac{[x : A]^x \qquad [y : B]^y}{\langle x, y \rangle : A \times B}\,\times\text{-I}$$

Using logic notation

$$\dfrac{\dfrac{\Gamma, x : U \vdash v : V}{\Gamma \vdash \lambda x.v : U \Rightarrow V}\Rightarrow\text{-I} \qquad \Delta \vdash u : U}{\Gamma, \Delta \vdash (\lambda x.t)u : V}\Rightarrow\text{-E} \quad \Longrightarrow \quad \Gamma, \Delta \vdash t[u/x] : V$$

$$\dfrac{\dfrac{\Gamma \vdash u : U \quad \Delta \vdash v : V}{\Gamma, \Delta \vdash \langle u, v\rangle : U \wedge V}\wedge\text{-I}}{\Gamma, \Delta \vdash \pi^1\langle u, v\rangle : U}\wedge\text{-1E} \quad \Longrightarrow \quad \Gamma \vdash u : U$$

$$\dfrac{\dfrac{\Gamma \vdash u : U \quad \Delta \vdash v : V}{\Gamma, \Delta \vdash \langle u, v\rangle : U \wedge V}\wedge\text{-I}}{\Gamma, \Delta \vdash \pi^2\langle u, v\rangle : U}\wedge\text{-2E} \quad \Longrightarrow \quad \Delta \vdash v : V$$

Proof simplification using the logic-based system

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id$_z$}
\UnaryInfC{$z : B \wedge A \vdash z : B \wedge A$}
\RightLabel{$\wedge$-2E}
\UnaryInfC{$z : B \wedge A \vdash \pi^2 z : A$}
\AxiomC{}
\RightLabel{Id$_z$}
\UnaryInfC{$z : B \wedge A \vdash z : B \wedge A$}
\RightLabel{$\wedge$-1E}
\UnaryInfC{$z : B \wedge A \vdash \pi^1 z : A$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$z : B \wedge A \vdash \langle \pi^2 z, \pi^1 z \rangle : A \wedge B$}
\RightLabel{$\Rightarrow$-I$_z$}
\UnaryInfC{$\vdash \lambda z.\langle \pi^2 z, \pi^1 z \rangle : (B \wedge A) \Rightarrow (A \wedge B)$}
\AxiomC{}
\RightLabel{Id$_y$}
\UnaryInfC{$y : B \vdash y : B$}
\AxiomC{}
\RightLabel{Id$_x$}
\UnaryInfC{$x : A \vdash x : A$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$x : A, y : B \vdash \langle y, x \rangle : B \wedge A$}
\RightLabel{$\Rightarrow$-E}
\BinaryInfC{$x : A, y : B \vdash (\lambda z.\langle \pi^2 z, \pi^1 z \rangle)(\langle y, x \rangle) : A \wedge B$}
\end{prooftree}
$$

$$\Downarrow \quad \beta\text{-conversion}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id$_y$}
\UnaryInfC{$y : B \vdash y : B$}
\AxiomC{}
\RightLabel{Id$_x$}
\UnaryInfC{$x : A \vdash x : A$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$x : A, y : B \vdash \langle y, x \rangle : B \wedge A$}
\RightLabel{$\wedge$-2E}
\UnaryInfC{$x : A, y : B \vdash \pi^2 \langle y, x \rangle : A$}
\AxiomC{}
\RightLabel{Id$_y$}
\UnaryInfC{$y : B \vdash y : B$}
\AxiomC{}
\RightLabel{Id$_x$}
\UnaryInfC{$x : A \vdash x : A$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$x : A, y : B \vdash \langle y, x \rangle : B \wedge A$}
\RightLabel{$\wedge$-1E}
\UnaryInfC{$x : A, y : B \vdash \pi^1 \langle y, x \rangle : A$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$x : A, y : B \vdash \langle \pi^2 \langle y, x \rangle, \pi^1 \langle y, x \rangle \rangle : A \wedge B$}
\end{prooftree}
$$

$$\Downarrow\Downarrow \quad \text{pairing}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Id$_x$}
\UnaryInfC{$x : A \vdash x : A$}
\AxiomC{}
\RightLabel{Id$_y$}
\UnaryInfC{$y : B \vdash y : B$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$x : A, y : B \vdash \langle x, y \rangle : A \wedge B$}
\end{prooftree}
$$