IA169 Model Checking

Automata-based LTL model checking

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Automata-based LTL model checking

system K

automaton A_K
representing runs of K

specification formula φ

automaton $A_{\neg\varphi}$
representing runs violating φ

Do A_K and $A_{\neg\varphi}$ represent
disjoint sets of runs?

YES,
all runs of K satisfy φ

NO + counterexample,
i.e. a run of K violating φ

Agenda and sources

agenda

- formalization of the state-based LTL model checking problem: (fair) Kripke structure and LTL
- Büchi automata (BA) and generalized Büchi automata (GBA)
- transformation of finite (fair) Kripke structures to (G)BA
- translation of LTL to BA via self-loop alternating automata
- algorithms checking disjointness of A_K and $A_{\neg\varphi}$
  - algorithm based on SCC decomposition
  - nested DFS algorithm
  - optimizations
- action-based version of LTL model checking

sources

- Chapter 7 of E. M. Clarke, O. Grumberg, D. Kroening, D. Peled, and H. Veith: *Model Checking*, Second Edition, MIT, 2018.
- M. Y. Vardi: *An Automata-Theoretic Approach to Linear Temporal Logic*, LNCS 1043, Springer, 1995.

Formalization of the state-based LTL model checking problem

Atomic propositions

atomic propositions

- basic observable properties of each state of the system
- for example: $x \geq y + 10$, *z is even*, *gate is open*, *program is at line 10*
- the validity of each atomic proposition in each state of the system has to be fully determined by the state
- specification talks only about validity of atomic proposition during system runs
- *AP* denotes a countable set of atomic propositions

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- basic formalism for state-based systems is a Kripke structure

Kripke structure

Definition (Kripke structure)

A Kripke structure is a tuple $K = (S, T, S_0, L)$, where

- S is a set of states,
- $T \subseteq S \times S$ is a transition relation,
- $S_0 \subseteq S$ is a set of initial states,
- $L : S \rightarrow 2^{AP}$ is a labeling function associating to each state $s \in S$ the set of atomic propositions that are true in s.

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- Kripke structures are typically described in an implicit way
- formats for implicit description typically offer
  - programs, processes, finite-state machines
  - synchronous or asynchronous composition
  - communication and synchronization mechanisms
  - nondeterminism or inputs

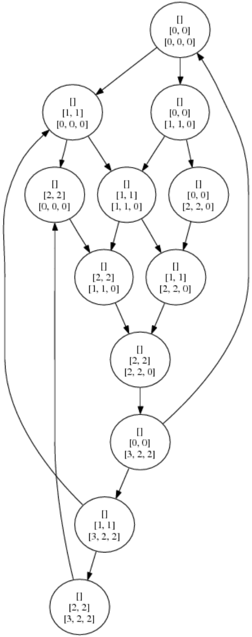
Example in the modelling language DVE

```
channel {byte} c[0];

process A {
    byte a;
    state q1,q2,q3;
    init q1;
    trans
    q1→q2 { effect a=a+1; },
    q2→q3 { effect a=a+1; },
    q3→q1 { sync c!a; effect a=0; };
}

process B {
    byte b,x;
    state p1,p2,p3,p4;
    init p1;
    trans
    p1→p2 { effect b=b+1; },
    p2→p3 { effect b=b+1; },
    p3→p4 { sync c?x; },
    p4→p1 { guard x==b; effect b=0, x=0; };
}

system async;
```



Example of a simple mutual exclusion system

cobegin $P_0 \parallel P_1$ **coend**

P_0:: l_0: **while** *true* **do**
NC_0: **wait** $(turn = 0)$;
CR_0: $turn := 1$
end while

P_1:: l_1: **while** *true* **do**
NC_1: **wait** $(turn = 1)$;
CR_1: $turn := 0$
end while

assume that *turn* is initially 0 or 1

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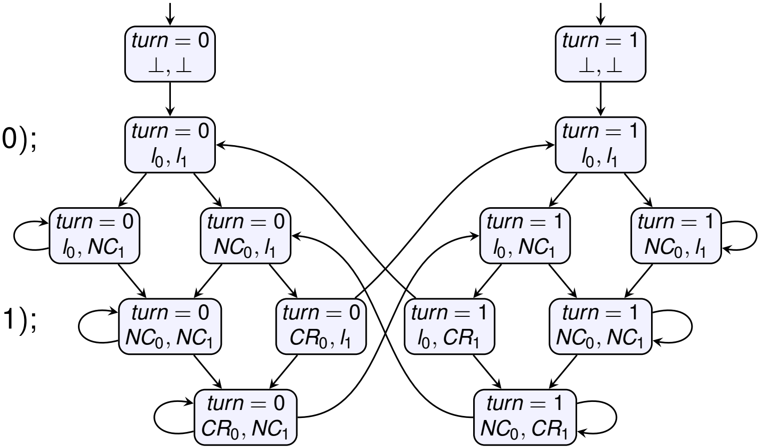

Run of a Kripke structure

Definition (run)

Let $K = (S, T, S_0, L)$ be a Kripke structure. A run of K is an infinite sequence $\pi = s_0 s_1 s_2 \ldots$ of states such that $s_0 \in S_0$ and $(s_i, s_{i+1}) \in T$ holds for each $i \geq 0$.

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- linear time model checking decides whether all runs satisfy the specification
- the set of infinite sequences of states is denoted by S^ω
- to consider also finite runs, we can define a run as a maximal sequence $\pi = s_0 s_1 s_2 \cdots \in S^+ \cup S^\omega$ of successive states starting in an initial state, where maximal means infinite or ending in a state without any successor
- it is usually assumed that there are no states without any successors: any system can be transformed to this form by adding self-loops to such states

Linear temporal logic (LTL)

Definition (linear temporal logic, LTL)

Formulae of Linear Temporal Logic (LTL) are defined by

$$\varphi \ ::= \ \top \ \mid \ a \ \mid \ \neg\varphi \ \mid \ \varphi_1 \wedge \varphi_2 \ \mid \ \mathrm{X}\varphi \ \mid \ \varphi_1 \,\mathrm{U}\, \varphi_2$$

where $\top$ stands for true and a ranges over a countable set AP.

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abbreviations and alternative notation

- $\bot \ \equiv \ \neg\top$
- $\varphi \lor \psi \ \equiv \ \neg(\neg\varphi \land \neg\psi)$
- $\varphi \Rightarrow \psi \ \equiv \ \neg\varphi \lor \psi$
- $\varphi \Leftrightarrow \psi \ \equiv \ \varphi \Rightarrow \psi \land \varphi \Leftarrow \psi$
- $\bigcirc\varphi \ \equiv \ \mathsf{X}\varphi$
- $\mathsf{F}\varphi \ \equiv \ \Diamond\varphi \ \equiv \ \top \,\mathsf{U}\, \varphi$
- $\mathsf{G}\varphi \ \equiv \ \Box\varphi \ \equiv \ \neg\mathsf{F}\neg\varphi$

Intuitive semantic of LTL

	operator name	intuitive meaning
$\mathsf{X}a$	next	$\bullet\ a\ \bullet\ \bullet\ \bullet\ \ldots$
$a\,\mathsf{U}\,b$	until	$a\ a\ \ldots\ a\ b\ \bullet\ \bullet\ \bullet\ \ldots$
$\mathsf{F}a$	eventually	$\bullet\ \bullet\ \ldots\ \bullet\ a\ \bullet\ \bullet\ \bullet\ \ldots$
$\mathsf{G}a$	always or globally	$a\ a\ a\ a\ \ldots$

Semantics of LTL

- we interpret LTL on infinite words $w = w(0)w(1)\ldots \in (2^{AP})^\omega$
- by w_i we denote the suffix of w of the form $w(i)w(i+1)w(i+2)\ldots$

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Definition

The relation $w \models \varphi$, meaning that w satisfies φ, is defined inductively as follows.

$$
\begin{array}{lll}
w \models \top & & \\
w \models a & \text{iff} & a \in w(0) \\
w \models \neg\varphi & \text{iff} & w \not\models \varphi \\
w \models \varphi_1 \wedge \varphi_2 & \text{iff} & w \models \varphi_1 \wedge w \models \varphi_2 \\
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w \models \varphi_1 \,\mathrm{U}\, \varphi_2 & \text{iff} & \exists i \geq 0 \,.\, w_i \models \varphi_2 \;\wedge\; \forall 0 \leq j < i \,.\, w_j \models \varphi_1
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$$

By $AP(\varphi)$ we denote the set of atomic propositions appearing in φ.
The language of φ is defined as $L(\varphi) = \{w \in \Sigma^\omega \mid w \models \varphi\}$, where $\Sigma = 2^{AP(\varphi)}$.

The goal of LTL model checking

Definition

Let $K = (S, T, S_0, L)$ be a Kripke structure and φ be an LTL formula.
A run $\pi = s_0 s_1 s_2 \ldots$ of K satisfies φ, written $\pi \models \varphi$, if $L(s_0)L(s_1)L(s_2)\ldots \models \varphi$.
K satisfies φ, written $K \models \varphi$, if $\pi \models \varphi$ holds for every run π of K.

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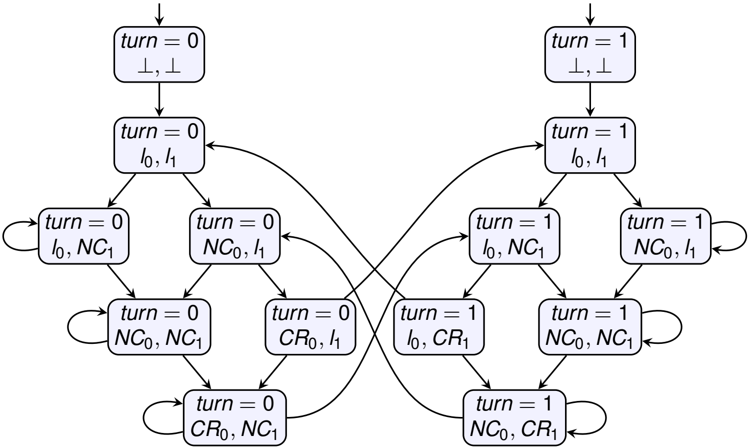
Given a Kripke structure K and an LTL formula φ, the goal of LTL model checking is to decide whether $K \models \varphi$ or not. In the negative case, model checking should provide a counterexample, i.e., a run π of K such that $\pi \not\models \varphi$.

Example



which formulae are satisfied?

- $\mathrm{G}\neg(CR_0 \wedge CR_1)$
- $\mathrm{GF}\,turn = 0 \wedge \mathrm{GF}\,turn = 1$

Extension with fairness

- fairness allows to add additional restrictions on the system runs
- can reflect properties of process schedulers

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Definition (fair Kripke structure)

A fair Kripke structure is a tuple $K = (S, T, S_0, L, \mathcal{F})$, where (S, T, S_0, L) is a Kripke structure and $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ is a finite set of fairness constraints such that $F_i \subseteq S$ for each $1 \leq i \leq n$.

A sequence $\pi = s_0 s_1 s_2 \in S^\omega$ is called a fair run of K if it is a run of (S, T, S_0, L) and it visits each $F_i \in \mathcal{F}$ infinitely often, i.e., $s_j \in F_i$ for infinitely many j.

K fairly satisfies an LTL formula φ, written $K \models_F \varphi$, if each fair run of K satisfies φ.

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- add reasonable fairness constraint to the mutual exclusion system

Büchi automata (BA) and generalized Büchi automata (GBA)

Büchi automaton (BA)

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A Büchi automaton (BA) is a tuple $A = (Q, \Sigma, \delta, Q_0, F)$, where

- Q is a finite set of states,
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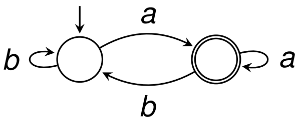
- we write $p \xrightarrow{a} q$ instead of $(p, a, q) \in \delta$



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A run of A over an infinite word $w = a_1 a_2 \ldots \in \Sigma^\omega$ is a sequence of states $\pi = s_0 s_1 \ldots \in Q^\omega$ satisfying $s_0 \in Q_0$ and $s_{i-1} \stackrel{a_i}{\to} s_i$ for each $i \geq 1$.

A run π is accepting if $\mathtt{inf}(\pi) \cap F \neq \emptyset$.

A word $w \in \Sigma^\omega$ is accepted by A if there exists an accepting run of A over w.
A language represented by A is the set $L(A) \subseteq \Sigma^\omega$ of words accepted by A.

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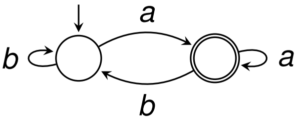
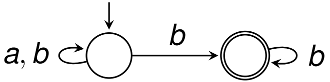


$L(A) = \{w \in \{a,b\}^\omega \mid a \in \mathsf{inf}(w)\}$

Properties of Büchi automata

- languages represented by Büchi automata are called ω-regular
- the class of ω-regular languages is closed under $\cup$, $\cap$, and complement (though complementation of Büchi automata is highly non-trivial)
- deterministic Büchi automata are less expressive than nondeterministic ones: for example $\{a, b\}^*.\{b\}^\omega$ cannot be described by any deterministic BA



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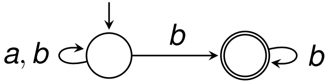

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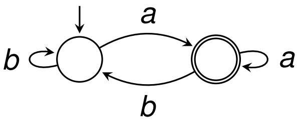
- the class of languages represented by deterministic Büchi automata is not closed under complement



$$L(B) = \{w \in \{a, b\}^\omega \mid a \in \inf(w)\}$$
$$L(B) = \{a, b\}^\omega \smallsetminus L(A)$$

Generalized Büchi automaton (GBA)

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A generalized Büchi automaton (GBA) is a tuple $A = (Q, \Sigma, \delta, Q_0, \mathcal{F})$, where Q, Σ, δ, Q_0 have the same meaning as in BA and $\mathcal{F} = \{F_1, \ldots, F_n\}$ is a finite set of accepting sets satisfying $F_i \subseteq Q$ for each $F_i \in \mathcal{F}$.

The definition of run is the same as for BA.

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The definition of an accepted word and language is the same as for BA.

- each BA $(Q, \Sigma, \delta, Q_0, F)$ can be seen as a GBA $(Q, \Sigma, \delta, Q_0, \{F\})$
- each GBA can be transformed into a BA representing the same language
- GBAs can be more succinct

Transformation of finite (fair) Kripke structures to (G)BA

Kripke structure $\rightarrow$ BA

- since now on, we consider only Kripke structures K with finitely many states
- assume that we know the set $AP(\varphi)$, which is always finite
- when deciding $K \overset{?}{\models} \varphi$, we can ignore atomic propositions outside $AP(\varphi)$
- we transform K into a Büchi automaton A_K with alphabet $\Sigma = 2^{AP(\varphi)}$ representing the language

$$L_K^\Sigma = \{a_0 a_1 a_2 \ldots \in \Sigma^\omega \mid \begin{array}{l}\text{there exists a run } s_0 s_1 s_2 \ldots \text{ of } K \text{ such that} \\ a_i = L(s_i) \cap AP(\varphi) \text{ for each } i \geq 0\}\end{array}$$

corresponding to runs of K projected to $AP(\varphi)$

Kripke structure $\rightarrow$ BA

input: a set $AP(\varphi)$ and a Kripke structure $K = (S, T, S_0, L)$
output: a BA $A_K = (S, 2^{AP(\varphi)}, \delta, S_0, S)$ representing L_K^{Σ}, where $\Sigma = 2^{AP(\varphi)}$

- $\delta = \{(p, a, q) \mid (p, q) \in T \text{ and } a = L(p) \cap AP(\varphi)\}$

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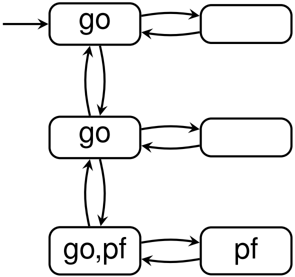

$AP(\varphi) = \{go\}$

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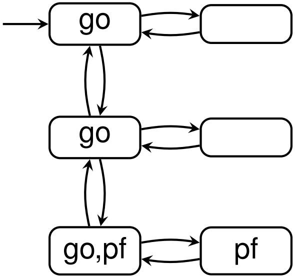

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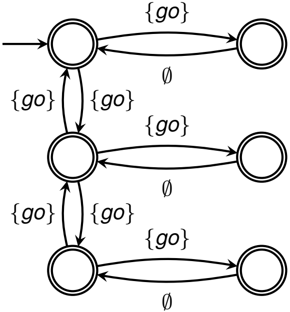

Fair Kripke structure $\to$ GBA

- similarly, we transform a fair Kripke structure K into a generalized Büchi automaton A_K with alphabet $\Sigma = 2^{AP(\varphi)}$ representing the language

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input: a set $AP(\varphi)$ and a fair Kripke structure $K = (S, T, S_0, L, \mathcal{F})$
output: a GBA $A_K = (S, 2^{AP(\varphi)}, \delta, S_0, \mathcal{F})$ representing L_K^Σ, where $\Sigma = 2^{AP(\varphi)}$

- $\delta = \{(p, a, q) \mid (p, q) \in T \text{ and } a = L(p) \cap AP(\varphi)\}$

Translation of LTL to BA via self-loop alternating automata

LTL $\to$ BA translations in general

- translates an LTL formula φ into a BA A_φ accepting $L(\varphi)$
- many LTL $\to$ BA translations
  - LTL $\to$ GBA $\to$ BA (Spin)
  - LTL $\to$ transition-based GBA (TGBA) $\to$ BA (Spot)
  - LTL $\to$ self-loop alternating BA $\to$ TGBA $\to$ BA (LTL2BA, LTL3BA)
  - LTL $\to$ self-loop alternating BA $\to$ BA
  - ...
- translations via self-loop alternating automata offer
  - size-reducing optimizations of self-loop alternating automata
  - smaller resulting BA (in some cases)

Translation of LTL to BA via self-loop alternating automata

Alternating automata

Positive Boolean formulae

Definition (positive boolean formulae)

Positive Boolean formulae over set Q, denoted with $\mathcal{B}^+(Q)$, are defined by

$$\varphi ::= \top \mid \bot \mid q \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2$$

where $\top$ stands for true, $\bot$ stands for false, and q ranges over Q.

Positive Boolean formulae

Definition (positive boolean formulae)

Positive Boolean formulae over set Q, denoted with $\mathcal{B}^+(Q)$, are defined by

$$\varphi \ ::= \ \top \mid \bot \mid q \mid \ \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2$$

where $\top$ stands for true, $\bot$ stands for false, and q ranges over Q.

$S \subseteq Q$ is a model of φ $\iff$ the valuation assigning true just to elements of S satisfies φ

S is a minimal model of φ (written $S \models \varphi$) $\iff$ S is a model of φ and no proper subset of S is a model of φ

Examples of positive Boolean formulae

formulae of $\mathcal{B}^+(\{p, q, r\})$	(minimal) models
$\bot$	no model
$\top$	$\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \ldots$
$p \wedge q$	$\{p, q\}, \{p, q, r\}$
$p \vee (q \wedge r)$	$\{p\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}$
$p \wedge (q \vee r)$	$\{p, q\}, \{p, r\}, \{p, q, r\}$

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- minimal models correspond to clauses in disjunctive normal form (without superfluous clauses)

$$\varphi \equiv \bigvee_{S \models \varphi} \left(\bigwedge_{p \in S} p \right)$$

Alternating Büchi automaton

Definition (alternating Büchi automaton)

An alternating Büchi automaton is a tuple $A = (Q, \Sigma, \delta, Q_0, F)$, where

- Q is a finite set of states,
- Σ is a finite alphabet,
- $\delta : Q \times \Sigma \rightarrow \mathcal{B}^+(Q)$ is a transition function,
- $Q_0 \subseteq Q$ is a set of initial states,
- $F \subseteq Q$ is a set of accepting states.

Trees

Definition (tree, Q-labeled tree)

A tree is a set $T \subseteq \mathbb{N}_0^*$ such that if $xc \in T$, where $x \in \mathbb{N}_0^*$ and $c \in \mathbb{N}_0$, then also

- $x \in T$ and
- $xc' \in T$ for all $0 \leq c' < c$.

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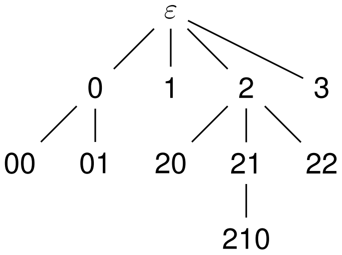

$$T = \{\varepsilon, 0, 1, 2, 3, 00, 01, 20, 21, 22, 210\}$$

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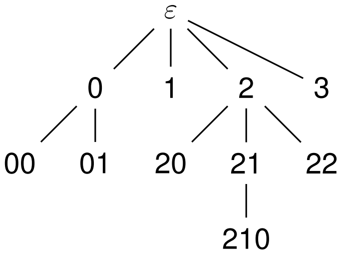
A Q-labeled tree is a pair (T, r) of a tree T and a labeling function $r : T \to Q$.



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Alternating Büchi automaton

Definition (run, language)

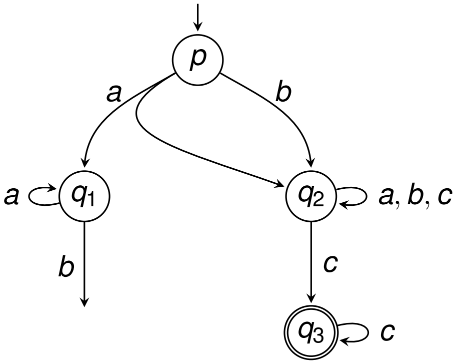
A run of an alternating BA $A = (Q, \Sigma, \delta, Q_0, F)$ on word $w = a_0 a_1 \ldots \in \Sigma^\omega$ is a Q-labeled tree (T, r) such that

- $r(\varepsilon) \in Q_0$ and
- for each $x \in T$: $\{r(xc) \mid c \in \mathbb{N}_0, xc \in T\} \models \delta(r(x), a_{|x|})$.

A run (T, r) is accepting iff for each infinite branch σ in T it holds that infinitely many nodes of the branch are labeled with a state in F.

A word $w \in \Sigma^\omega$ is accepted by A iff there exists an accepting run of $\mathcal{A}$ over w.
A language represented by A is the set $L(A) \subseteq \Sigma^\omega$ of words accepted by A.

Example of an alternating Büchi automaton



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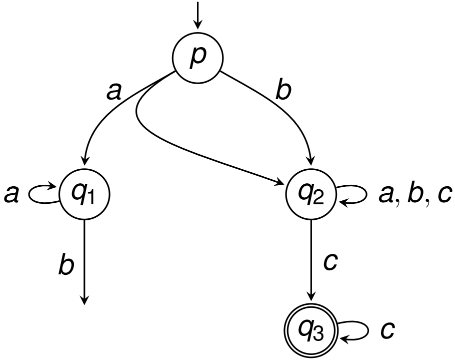

$$L(A) = \{a\}^*.\{b\}.\{a,b,c\}^*.\{c\}^\omega$$

Self-loop alternating Büchi automaton

Intuitively, an alternating BA is self-loop (or 1-weak or linear or very weak, written SLAA or A1W or VWAA) if it contains no cycles except self-loops.

Self-loop alternating Büchi automaton

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Definition (self-loop alternating BA)

Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an alternating BA. For each $p \in Q$ we define the set of all successors of p as

$$Succ(p) = \{q \mid \exists a \in \Sigma, S \subseteq Q : S \cup \{q\} \models \delta(p, a)\}.$$

Automaton A is self-loop (or 1-weak or linear or very weak) if there exists a partial order $\leq$ on Q such that for all $p, q \in Q$ it holds:

$$q \in Succ(p) \implies q \leq p$$

Notes

- standard Büchi automata are alternating Büchi automata where each $\delta(p, a)$ is $\perp$ or a disjunction of states
- self-loop alternating BA have the same expressive power as LTL

Translation of LTL to BA via self-loop alternating automata

LTL $\rightarrow$ self-loop alternating BA

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input: an LTL formula φ
output: self-loop alternating BA $A = (Q, \Sigma, \delta, \{q_\varphi\}, F)$ accepting $L(\varphi)$

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- $\Sigma = 2^{AP(\varphi)}$

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- δ is defined as follows (where $\overline{\alpha} \in \mathcal{B}^+(Q)$ satisfies $\overline{\alpha} \equiv \neg\alpha$)

$$
\begin{aligned}
\delta(q_\top, I) &= \top \\
\delta(q_a, I) &= \top \text{ if } a \in I, \ \bot \text{ otherwise} \\
\delta(q_{\neg\psi}, I) &= \overline{\delta(q_\psi, I)} \\
\delta(q_{\psi\wedge\rho}, I) &= \delta(q_\psi, I) \wedge \delta(q_\rho, I) \\
\delta(q_{\mathsf{X}\psi}, I) &= q_\psi \\
\delta(q_{\psi\mathsf{U}\rho}, I) &= \delta(q_\rho, I) \vee (\delta(q_\psi, I) \wedge q_{\psi\mathsf{U}\rho})
\end{aligned}
\qquad
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\overline{\top} &= \bot \\
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\overline{q_\psi} &= q_{\neg\psi} \\
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- $F = \{q_{\neg(\psi\mathsf{U}\rho)} \mid \psi\,\mathsf{U}\,\rho \text{ is a subformula of } \varphi\}$

LTL $\rightarrow$ self-loop alternating BA

Note that every infinite path of a run of A has a suffix labeled with a state of the form $q_{\psi \mathsf{U} \rho}$ or $q_{\neg(\psi \mathsf{U} \rho)}$ (other states have no loops and can appear at most once on a path). F is defined to prevent the first case: $\psi \mathsf{U} \rho$ is satisfied only if ρ eventually holds.

Theorem

Given an LTL formula φ, one can construct an self-loop alternating BA A accepting $L(\varphi)$ and such that the number of states of A is linear in the length of φ.

Translation of LTL to BA via self-loop alternating automata

Self-loop alternating BA $\rightarrow$ BA

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input: a self-loop alternating BA $A = (Q, \Sigma, \delta, Q_0, F)$
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Intuitively, A' tracks states on each level of the computation tree of A. Moreover, A' has to divide the set of states into two sets: states labeling paths with recent occurrence of an accepting state, and states labeling the other paths.

Self-loop alternating BA $\rightarrow$ BA

input: a self-loop alternating BA $A = (Q, \Sigma, \delta, Q_0, F)$
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- $Q' = 2^Q \times 2^Q$

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- $\delta'((U, V), l)$ is defined as:
  - if $U \neq \emptyset$ then

$$\begin{aligned}\delta'((U,V),l) = \{(U',V') \mid\ & \exists X, Y \subseteq Q \text{ such that} \\ & X \models \textstyle\bigwedge_{q \in U} \delta(q,l) \text{ and} \\ & Y \models \textstyle\bigwedge_{q \in V} \delta(q,l) \text{ and} \\ & U' = X \setminus F \text{ and } V' = Y \cup (X \cap F)\}\end{aligned}$$

  - if $U = \emptyset$ then

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- $F' = \{\emptyset\} \times 2^Q$

Self-loop alternating BA $\rightarrow$ BA

Theorem

Given a self-loop alternating BA $A = (Q, \Sigma, \delta, Q_0, F)$, one can construct a BA A' accepting $L(A)$ and such that the number of states of A' is $2^{\mathcal{O}(|Q|)}$.

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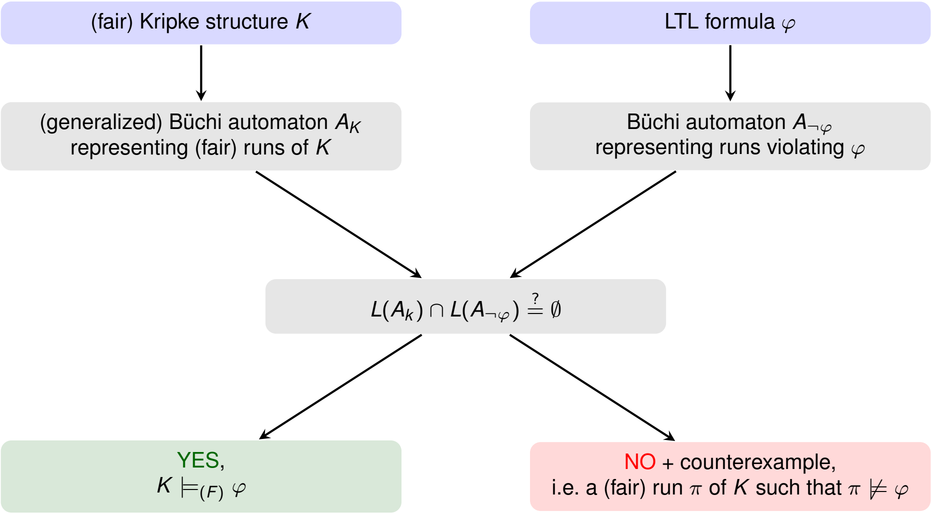
Corollary

Given an LTL formula φ and an alphabet Σ, one can construct a BA A' accepting $L(\varphi)$ and such that the number of states of A' is $2^{\mathcal{O}(|\varphi|)}$.

Algorithms checking disjointness of A_K and $A_{\neg\varphi}$

Automata-based LTL model checking



(fair) Kripke structure K

LTL formula φ

(generalized) Büchi automaton A_K
representing (fair) runs of K

Büchi automaton $A_{\neg\varphi}$
representing runs violating φ

product generalized Büchi automaton B
$L(B) = L(A_K) \cap L(A_{\neg\varphi})$

$L(B) \stackrel{?}{=} \emptyset$

YES,
$K \models_{(F)} \varphi$

NO + counterexample,
i.e. a (fair) run π of K such that $\pi \not\models \varphi$

Construction of product automaton

input: GBAs $A_1 = (Q_1, \Sigma, \delta_1, Q_{01}, \mathcal{F}_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, Q_{02}, \mathcal{F}_2)$
output: a GBA $B = (Q_1 \times Q_2, \Sigma, \delta, Q_{01} \times Q_{02}, \mathcal{F})$ representing $L(A_1) \cap L(A_2)$

- $\delta = \{((p_1, p_2), a, (q_1, q_2)) \mid (p_1, a, q_1) \in \delta_1 \text{ and } (p_2, a, q_2) \in \delta_2\}$
- $\mathcal{F} = \{F_{1i} \times Q_2 \mid F_{1i} \in \mathcal{F}_1\} \cup \{Q_1 \times F_{2i} \mid F_{2i} \in \mathcal{F}_2\}$

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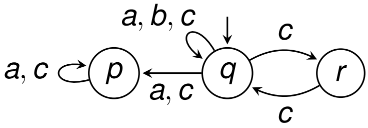
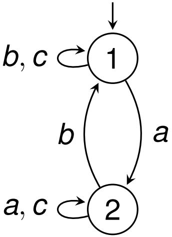


$\mathcal{F}_2 = \{\{p, r\}\}$



$\mathcal{F}_1 = \{\{1\}, \{2\}\}$

Construction of product automaton

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Lemma

$L(B) = L(A_1) \cap L(A_2)$.

Emptiness of a GBA

Theorem

Let $B = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ be a GBA. The following conditions are equivalent.

1. $L(B) \neq \emptyset$
2. *There exists a nontrivial SCC of B reachable from Q_0 and such that the SCC contains at least one state of each $F_i \in \mathcal{F}$.*
3. *There exists an accepting run of B of the form $\tau.\rho^\omega$ (so-called lasso-shaped).*

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Proof.

1 $\Longrightarrow$ 2 Assume that $L(B) \neq \emptyset$. Hence, there exists an accepting run π. The run has to contain an infinite suffix contained in a single nontrivial SCC of B reachable form Q_0. As the run visits each $F_i \in \mathcal{F}$ infinitely often, this SCC has to contain at least one state of each $F_i \in \mathcal{F}$.

□

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Proof.

$2 \Longrightarrow 3$ Assume that B has a nontrivial SCC reachable from Q_0 and containing at least one state of each $F_i \in \mathcal{F}$. Let τ be a sequence of successive states starting in Q_0 and leading to a state q of the SCC. Due to the properties of the SCC, there exists a sequence ρ of states of the SCC which starts in some successor of q, ends in q, and contains some state of each $F_i \in \mathcal{F}$. Then $\tau.\rho^\omega$ is an accepting run. $\square$

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Proof.

$3 \implies 1$ Obvious.

□

Algorithms checking disjointness of A_K and $A_{\neg\varphi}$

Algorithm based on SCC decomposition

Emptiness check by SCC decomposition

input : a GBA $B = (Q, \Sigma, \delta, Q_0, \mathcal{F})$
output: *true* if $L(B) = \emptyset$; *false* otherwise

procedure isGBAempty
    remove unreachable states from the automaton
    decompose the automaton into SCCs
    if some nontrivial SCC contains at least one state of each $F_i \in \mathcal{F}$ **then**
        return *false*
    else
        return *true*

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- if $L(B) \neq \emptyset$, a counterexample accepted by a lasso-shaped run $\tau.\rho^\omega$ can be constructed such that τ reaches the found SCC from Q_0 and ρ is a loop containing all states of the SCC
- the corresponding accepted word $u.v^\omega \in L(B)$ is also lasso-shaped

Emptiness check by SCC decomposition

pros

- simple
- SCC decomposition can be done in time $\mathcal{O}(|Q| + |\delta|)$

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- in model checking, GBA is a product of A_K and $A_{\neg\varphi}$, where A_K is typically very large and described implicitly

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on-the-fly model checking algorithms

- the emptiness check explores the product automaton gradually and can detect nonemptiness without knowing the whole product
- the states and transitions of the product are constructed from $A_{\neg\varphi}$ and the implicit description of A_K only on demand

Algorithms checking disjointness of A_K and $A_{\neg\varphi}$

Nested DFS algorithm

Nested DFS check

- also called double DFS
- allows on-the-fly model checking
- checks emptiness of a BA (not generalized)
- can be easily used for model checking of a (not fair) Kripke structure K
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construction of product BA for a BA with all states accepting and another BA

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output: a BA $B = (Q_1 \times Q_2, \Sigma, \delta, Q_{01} \times Q_{02}, F)$ representing $L(A_1) \cap L(A_2)$

- $\delta = \{((p_1, p_2), a, (q_1, q_2)) \mid (p_1, a, q_1) \in \delta_1$ and $(p_2, a, q_2) \in \delta_2\}$
- $F = Q_1 \times F_2$

Emptiness of a BA

Theorem

Let $B = (Q, \Sigma, \delta, Q_0, F)$ *be a BA. The* $L(B) \neq \emptyset \iff$ *there exist a run of the form* $\tau.\rho^\omega$ *where* ρ *starts with a state of* F*.*

Emptiness of a BA

Theorem

Let $B = (Q, \Sigma, \delta, Q_0, F)$ be a BA. The $L(B) \neq \emptyset \iff$ there exist a run of the form $\tau.\rho^\omega$ where ρ starts with a state of F.

Proof.

$\Longleftarrow$ Follows directly from the fact that $\tau.\rho^\omega$ is an accepting run of B.

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$\Longrightarrow$ Assume that $L(B) \neq \emptyset$. There exists an accepting run $\pi = s_0 s_1 \ldots \in Q^\omega$. As π is accepting, there exists a state $q \in \inf(\pi) \cap F$. Let $i < j$ be such that s_i, s_j are the first two occurrences of q in π. Further, let $\tau = s_0 s_1 \ldots s_{i-1}$ and $\rho = s_i s_{i+1} \ldots s_{j-1}$. Then $\tau.\rho^\omega = s_0 s_1 \ldots s_{i-1}.(s_i s_{i+1} \ldots s_{j-1})^\omega$ is a run of B and ρ starts with $s_i \in F$.

$\square$

Nested DFS algorithm

- the algorithm uses two nested instances of depth-first search
- the first DFS searches for reachable accepting states
- the nested DFS looks for a cycle from accepting states
- the algorithm terminates when a cycle from an accepting state is found
- all executions of the nested DFS share the information about visited states: without this feature, the overall complexity of nested DFS executions would be $\mathcal{O}(|F| \cdot (|Q| + |\delta|))$

Nested DFS algorithm

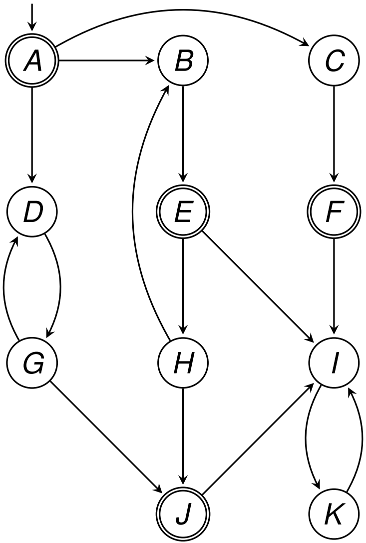
input : a BA $B = (Q, \Sigma, \delta, Q_0, F)$
output: *true* if $L(B) = \emptyset$;
false otherwise

procedure isBAempty
  visited1 $\leftarrow \emptyset$
  visited2 $\leftarrow \emptyset$
  onStack $\leftarrow \emptyset$
  forall $q_0 \in Q_0$ **do**
    dfs1(q_0)
  terminate *true*

procedure dfs1(q)
  visited1 $\leftarrow$ visited1 $\cup \{q\}$
  onStack $\leftarrow$ onStack $\cup \{q\}$
  forall successors q' of q **do**
    if $q' \notin$ visited1 **then** dfs1(q')
  if $q \in F$ **then** dfs2(q)
  onStack $\leftarrow$ onStack $\setminus \{q\}$

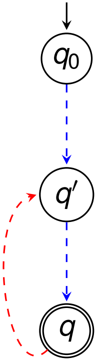
procedure dfs2(q)
  visited2 $\leftarrow$ visited2 $\cup \{q\}$
  forall successors q' of q **do**
    if $q' \in$ onStack **then terminate** *false*
    if $q' \notin$ visited2 **then** dfs2(q')

Example



Nested DFS algorithm

- if the algorithm returns *false*, it can produce a counterexample corresponding to the lasso-shaped accepting run given by the current content of DFS stacks
- let q be the accepting state from which the last nested DFS was executed
- let q' be the state on stack discovered by the nested DFS



stack of the first DFS

stack of the nested DFS

accepting lasso-shaped run:

$q_0 \dashrightarrow q' \dashrightarrow (q \dashrightarrow q' \dashrightarrow)^{\omega}$

Correctness of the nested DFS algorithm

Theorem

The nested DFS algorithm returns false $\iff L(B) \neq \emptyset$.

Correctness of the nested DFS algorithm

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The nested DFS algorithm returns false $\iff$ $L(B) \neq \emptyset$.

Proof.

$\Longrightarrow$ is obvious. We prove $\Longleftarrow$ by contradiction. Assume that $L(B) \neq \emptyset$ and the algorithm returns *true*. As $L(B) \neq \emptyset$, there is a run $\tau.\rho^\omega$ where ρ starts with a state $q \in F$. When the nested DFS is started from q, there has to be a state q' on the stack of the first DFS reachable from q. Nested DFS has not found the cycle because q' is reachable only via $r \in$ visited2. Assume that q is the first such a state and that r is added to visited2 during the nested DFS started from $q'' \in F$.

1. If q'' is reachable from q, then there is a cycle $q'' \dashrightarrow r \dashrightarrow q \dashrightarrow q''$ which is the contradiction with the assumption that q is the first such state.
2. If q'' is not reachable from q, then q is reachable from q'' via $q'' \dashrightarrow r \dashrightarrow q$. We have the contradiction with the fact that the first DFS backtracks from a state only after it backtracks from all states reachable from them and thus nested DFS from q'' cannot be executed before the nested DFS from q. □

Complexity of the nested DFS algorithm

complexity of the first DFS

- time: $\mathcal{O}(|Q| + |\delta|)$
- space: $\mathcal{O}(|Q|)$

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overall complexity

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Algorithms checking disjointness of A_K and $A_{\neg\varphi}$

Optimizations

Terminal and weak BA

Definition (terminal BA, weak BA)

Let B be a Büchi automaton with alphabet Σ. A Büchi automaton is terminal if each accepting state has a loop transition under each $a \in \Sigma$.

A Büchi automaton is weak if each strongly connected component consists either of accepting states or of nonaccepting states.

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- many LTL properties translate to terminal or weak BA
- if this is the case, simpler emptiness checks can be used

Optimization for terminal automata

- assume that $A_{\neg\varphi}$ is a terminal BA and each state of BA A_K is accepting and has a successor
- let B be the product BA of $A_{\neg\varphi}$ and A_K
- $L(B) \neq \emptyset$ iff B has a reachable accepting state
- instead of nested DFS, emptiness of $L(B)$ can be decided by a single DFS checking the reachability of an accepting state
- properties φ with terminal $A_{\neg\varphi}$ are called safety properties
- typical safety property: $\mathrm{G}\neg \mathit{err}$

Optimization for weak automata

- assume that $A_{\neg\varphi}$ is a weak BA and each state of BA A_K is accepting
- let B be the product BA of $A_{\neg\varphi}$ and a BA A_K
- each cycle of B contains either only accepting states or no accepting state
- instead of nested DFS, emptiness of $L(B)$ can be decided by a single DFS that looks for a cycle and if a cycle is found, it checks whether the current state is accepting
- typical property φ with weak $A_{\neg\varphi}$: $\mathsf{G}(a \implies \mathsf{F}b)$ (responsivity)

Extending LTL with release

- another derived LTL operator release: $\varphi \,\mathsf{R}\, \psi \;\equiv\; \neg(\neg\varphi \,\mathsf{U}\, \neg\psi)$
- equivalently: $\varphi \,\mathsf{R}\, \psi \;\equiv\; \mathsf{G}\psi \;\vee\; \psi \,\mathsf{U}\, (\psi \wedge \varphi)$

$$a \,\mathsf{R}\, b \qquad b\; b\; b\; b\; \ldots \qquad \text{or} \qquad b\; b\; \ldots\; b\; (ab)\; \ldots$$

- by adding $\bot$, $\vee$, and R to the basic syntax of LTL, we can push all negations towards atomic propositions using equivalences

$$\begin{aligned}
\neg(\varphi \,\mathsf{U}\, \psi) &\equiv \neg\varphi \,\mathsf{R}\, \neg\psi \\
\neg(\varphi \,\mathsf{R}\, \psi) &\equiv \neg\varphi \,\mathsf{U}\, \neg\psi \\
\neg\mathsf{X}\varphi &\equiv \mathsf{X}\neg\varphi \\
\neg(\varphi \vee \psi) &\equiv \neg\varphi \;\wedge\; \neg\psi \\
\neg(\varphi \wedge \psi) &\equiv \neg\varphi \;\vee\; \neg\psi \\
\neg\neg a &\equiv a
\end{aligned}$$

Hierarchy of LTL classes

Definition (hierarchy of LTL classes)

- $\Sigma_0 = \Pi_0$ is the smallest set of LTL formulas containing all atomic propositions and closed under application of $\wedge$, $\vee$, $\neg$, and X.
- Σ_{i+1} is the smallest set of LTL formulas containing Π_i and closed under application of $\wedge$, $\vee$, X, and U.
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- for each $\varphi \in \Pi_i$, $\neg\varphi$ can be transformed (by pushing negations towards atomic propositions) to an equivalent formula $\psi \in \Sigma_i$
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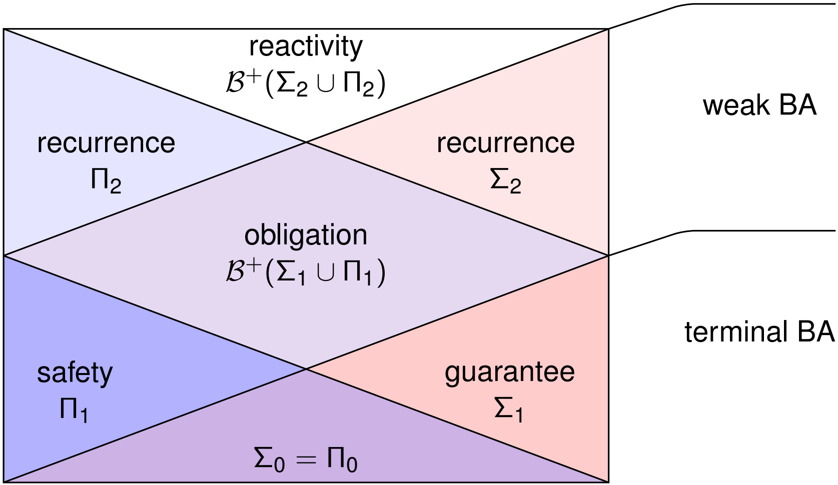
Properties corresponding to LTL classes

- Σ_1 describes guarantee properties
- Π_1 describes safety properties
- $\mathcal{B}^+(\Sigma_1 \cup \Pi_1)$ describes obligation properties
- Σ_2 describes persistence properties
- Π_2 describes recurrence (or response) properties
- $\mathcal{B}^+(\Sigma_2 \cup \Pi_2)$ describes reactivity properties
- the LTL classes are sometimes called guarantee, safety, ... formulae

Hierarchy of properties



- each language definable in LTL is definable in $\mathcal{B}^+(\Sigma_2 \cup \Pi_2)$
- formulae of Σ_1 can be translated to terminal BA
- formulae of Σ_2 can be translated to weak BA

Fighting state-space explosion

state-space explosion problem

- Kripke structure (and thus also the product automaton) can have enormous number of states, often exponential in the size of its implicit description
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  - abstraction
- symbolic representation of sets of states (by formulae or BDDs)
- parallel and distributed algorithms

Action-based version of LTL model checking

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- basic observable information attached to each transition of the system
- for example: *gate openning*, *process P entered critical section*
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- basic formalism for action-based systems is a labeled transition system

Labeled transition system

Definition (labeled transition system, LTS)

A labeled transition systems (LTS) is a tuple $M = (S, Act', \delta, S_0)$, where

- S is a set of states,
- $Act' \subseteq Act$ is a finite set of actions,
- $\delta \subseteq S \times Act' \times S$ is a transition relation,
- $S_0 \subseteq S$ is a set of initial states.

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Definition (run, trace)

Let $M = (S, Act', \delta, S_0)$ be an LTS. A run of M is an infinite sequence $\pi = (s_0, a_0, s_1)(s_1, a_1, s_2)(s_2, a_2, s_3)\ldots \in \delta^\omega$ of adjacent transitions such that $s_0 \in S_0$.

The trace of π is then the infinite word $\sigma(\pi) = a_0 a_1 a_2 \ldots$.

Modified syntax and semantics of LTL

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modified semantics of LTL

- we interpret LTL on infinite words $w = w(0)w(1)\ldots \in Act^\omega$
- the only change in the inductive definition of $w \models \varphi$ is the line

$$w \models a \quad \text{iff} \quad a = w(0) \qquad \text{(instead of} \quad w \models a \quad \text{iff} \quad a \in w(0))$$

The goal of action-based LTL model checking

Definition

Let $M = (S, Act', \delta, S_0)$ be an LTS and φ be an LTL formula.
A run π of M satisfies φ, written $\pi \models \varphi$, if $\sigma(\pi) \models \varphi$.
M satisfies φ, written $M \models \varphi$, if $\pi \models \varphi$ holds for every run π of M.

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The goal of action-based LTL model checking is to decide whether a given LTS M satisfies a given LTL formula φ. In the negative case, model checking should provide a counterexample, i.e., a run π of M such that $\pi \not\models \varphi$.

Automata-based approach to action-based LTL model checking

The automata-based approach to LTL model checking of finite LTS is basically identical as for finite Kripke structures.

changes

- Büchi automata use the alphabet $\Sigma = Act'$ given by the LTS
- we assume that φ contains only actions from Act' (otherwise, we extend Act')
- LTS $M = (S, Act', \delta, S_0)$ is transformed into a BA $A_M = (S, Act', \delta, S_0, S)$
- modification of LTL $\rightarrow$ self-loop alternating BA translation

$\delta(q_a, l) = \top$ if $a = l$, $\bot$ otherwise (instead of $\delta(q_a, l) = \top$ if $a \in l$, $\bot$ otherwise)