Definition 1. Binary relation *on set* A *is a set* $R \subseteq A \times A$.

- R *is reflexive* $\Leftrightarrow$ *for all* $a \in A$ *holds* $(a, a) \in R$
- R *is symmetric* $\Leftrightarrow$ *for all* $a, b \in A$ *holds: if* $(a, b) \in R$*, then* $(b, a) \in R$
- R *is transitive* $\Leftrightarrow$ *for all* $a, b, c \in A$ *holds: if* $(a, b), (b, c) \in R$*, then* $(a, c) \in R$
- R *is an equivalence relation* $\Leftrightarrow$ R *is reflexive, symmetric and transitive*

Exercise 1

Give an example of a set A and a binary relation R on A such that

1. R is reflexive and symmetric, but not transitive.
2. R is reflexive and transitive, but not symmetric.
3. R is symmetric and transitive, but not reflexive.

Exercise 2

Let $R, S \subseteq A \times A$ be equivalences on A. Prove or disprove the following statements:

1. $R \cap S$ is equivalence relation.
2. $R \cup S$ is equivalence relation.
3. $R \setminus S$ is equivalence relation.

Exercise 3

Prove or disprove the following statement: If $R, S \subseteq A \times A$ are equivalence relations on A, then $R \circ S$ is equivalence.

Exercise 4

How many equivalence relations on $\emptyset$ exist?

Definition 2. *Function from the set* A *to the set* B *is a relation* $f \subseteq A \times B$ *satisfying the following. For all* $a \in A$ *there is exactly one* $b \in B$ *cush that* $(a, b) \in f$.

We write $f : A \to B$ and we also use $f(a) = b$ instead of $(a, b) \in f$.

1. $f : A \to B$ is injective $\Leftrightarrow$ for all $a, b \in A$ such that $a \neq b$ we have $f(a) \neq f(b)$.
2. $f : A \to B$ is surjective $\Leftrightarrow$ for all $b \in B$ there is $a \in A$ such that $f(a) = b$.
3. $f : A \to B$ is bijective $\Leftrightarrow$ f is both injective and surjective.

Exercise 5

Let $f : A \to B$ be a function. Prove that f is injective iff there is $g : B \to A$ such that $g \circ f = id_A$ ($id_A(a) = a$ for all $a \in A$).

Definition 3. *Let* $R \subseteq A \times A$ *be a relation on* A.

- R *is antisymmetric* $\Leftrightarrow$ *for all* $a, b \in A$ *the following holds: if* $(a, b), (b, a) \in R$*, then* $a = b$
- R *is an ordering* $\Leftrightarrow$ R *is reflexive, transitive and antisymmetric.*

We write aRb instead of $(a, b) \in R$.

Definition 4. *Ordered set is a tuple* $(A, \sqsubseteq)$ *where* A *is a set and* $\sqsubseteq \subset A \times A$ *is an ordering on* A.

Two ordered sets $(A, \sqsubseteq)$ and $(B, \preceq)$ are isomorphic iff there is a bijection $f : A \to B$ such that for all $a, b \in A$ we have $a \sqsubseteq b \Leftrightarrow f(a) \preceq f(b)$.

Exercise 6

Find two ordering $\sqsubseteq$ and $\preceq$ on $\mathbb{N}_0 = \{0, 1, \ldots\}$ such that $(\mathbb{N}_0, \sqsubseteq)$ and $(\mathbb{N}_0, \preceq)$ are not isomorphic.

Exercise 7

Find infinitely many orderings $\preceq_0, \preceq_1, \ldots$ on $\mathbb{N}_0$ such that for all $i \neq j$ the ordered sets $(\mathbb{N}_0, \preceq_i)$ and $(\mathbb{N}_0, \preceq_j)$ are not isomorphic.

Exercise 8

Let $\mathbb{U}$ be an uncountable set. Find an ordering $\preceq_s$ on $\mathbb{N}_0$ for each $s \in \mathbb{U}$ such that for any $s_1, s_2 \in \mathbb{U}$ satisfying $s_1 \neq s_2$ the ordered sets $(\mathbb{N}_0, \preceq_{s_1})$ and $(\mathbb{N}_0, \preceq_{s_2})$ are not isomorphic.

Exercise 9

Find two linear orderings $\sqsubseteq$ and $\preceq$ on $\mathbb{N}_0$ such that $(\mathbb{N}_0, \sqsubseteq)$ and $(\mathbb{N}_0, \preceq)$ are not isomorphic.

Exercise 10

Find infinitely many linear orderings $\preceq_0, \preceq_1, \ldots$ on $\mathbb{N}_0$ such that for all $i \neq j$ the ordered sets $(\mathbb{N}_0, \preceq_i)$ and $(\mathbb{N}_0, \preceq_j)$ are not isomorphic.

Exercise 11

Let $\mathbb{U}$ be an uncountable set. Find a linear ordering $\preceq_s$ on $\mathbb{N}_0$ for each $s \in \mathbb{U}$ such that for any $s_1, s_2 \in \mathbb{U}$ satisfying $s_1 \neq s_2$ the ordered sets $(\mathbb{N}_0, \preceq_{s_1})$ and $(\mathbb{N}_0, \preceq_{s_2})$ are not isomorphic.

Exercise 12

Let $(A, \sqsubseteq)$ be a finite ordered set. Prove that there is a linear ordering $\preceq$ on A such that $\sqsubseteq \subseteq \preceq$.