př.: $\displaystyle\int \frac{7}{\sqrt{x^2-4x-5}}\,dx =$

$$= \boxed{\frac{1}{\sqrt{a}} \cdot \ln\left|2ax+b+2\sqrt{a}\cdot\sqrt{ax^2+bx+c}\right| + k}$$

$$= \underline{\underline{7 \cdot \ln\left|2x-4+2\cdot\sqrt{x^2-4x-5}\right| + K}}$$

$$\int \frac{7}{\sqrt{x^2-4x-5}}\,dx = 7\int \frac{1}{\sqrt{(x+1)\cdot(x-5)}}\,dx =$$

$$= 7\cdot\int \frac{1}{\sqrt{(x+1)^2\cdot\frac{x-5}{x+1}}}\,dx = 7\cdot\int \frac{1}{(x+1)\cdot\sqrt{\frac{x-5}{x+1}}}\,dx =$$

$$= \left|\begin{array}{ll|l} t^2 = \frac{x-5}{x+1} & \longrightarrow & x t^2 + t^2 = x-5 \\ 2t\,dt = \frac{6}{(x+1)^2}\,dx & & x\cdot(t^2-1) = -5-t^2 \\ \frac{1}{x+1}\,dx = \frac{1}{3}\cdot(x+1)\cdot t\,dt & & x = \frac{t^2-5}{1-t^2} \end{array}\right| =$$

$$= 7 \cdot \int \frac{1}{\cancel{t}} \cdot \frac{1}{3} \left(\frac{t^2+5}{1-t^2} + 1 \right) \cdot \cancel{t} \, dt =$$

$$= \frac{7}{3} \cdot \int \frac{t^2+5+1-t^2}{1-t^2} \, dt = \frac{7}{3} \cdot \int \frac{6}{1-t^2} \, dt =$$

$$= \frac{7}{3} \cdot 6 \cdot \int \frac{1}{(1-t)\cdot(1+t)} \, dt =$$

$$= 14 \cdot \int \frac{\frac{1}{2}}{1-t} + \frac{\frac{1}{2}}{1+t} \, dt =$$

$$= -7 \cdot \ln|1-t| + 7 \cdot \ln|1+t| + C =$$

$$= 7 \cdot \ln\left|\frac{1+t}{1-t}\right| + C =$$

$$= 7 \cdot \ln\left|\frac{1+\sqrt{\frac{x-5}{x+1}}}{1-\sqrt{\frac{x-5}{x+1}}}\right| + c = 7 \cdot \ln\left|\frac{\frac{\sqrt{x+1}+\sqrt{x-5}}{\cancel{\sqrt{x+1}}}}{\frac{\sqrt{x+1}-\sqrt{x-5}}{\cancel{\sqrt{x+1}}}}\right| + C$$

$$= 7 \cdot \ln\left|\frac{\sqrt{x+1}+\sqrt{x-5}}{\sqrt{x+1}-\sqrt{x-5}} \cdot \frac{\sqrt{x+1}+\sqrt{x-5}}{\sqrt{x+1}+\sqrt{x-5}}\right| + C$$

$$= 7 \cdot \ln\left|\frac{x+1+2\cdot\sqrt{(x+1)\cdot(x-5)}+x-5}{\cancel{x}+1-\cancel{x}+5}\right| + c$$

$$= 7 \cdot \ln \left| \frac{1}{6} \cdot \left(2x - 4 + 2 \cdot \sqrt{x^2 - 4x - 5} \right) \right| + c =$$

$$= 7 \cdot \ln \left| 2x - 4 + 2 \cdot \sqrt{x^2 - 4x - 5} \right| + \underbrace{7 \cdot \ln \frac{1}{6} + c}_{=: K}$$

ř.:

$$\int \frac{7}{\sqrt{5+4x-x^2}}\,dx =$$

$$\left|\; 5+4x-x^2 = -(x^2-4x-5) = -\left((x-2)^2-4-5\right) = -(x-2)^2+9 = 9-(x-2)^2 \;\right|$$

$$= 7\cdot\int \frac{1}{\sqrt{9-(x-2)^2}}\,dx = \frac{7}{3}\cdot\int \frac{1}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}}\,dx$$

$$= \left|\begin{array}{l} t = \frac{x-2}{3} \\ dt = \frac{1}{3}\,dx \\ dx = 3\,dt \end{array}\right| = 7\cdot\int \frac{1}{\sqrt{1-t^2}}\,dt =$$

$$= 7 \cdot \arcsin t + c =$$

$$= 7 \cdot \arcsin \frac{x-2}{3} + c$$

ř.:

$$y \cdot y' = \frac{\operatorname{arctg} x}{y^2 - 2y + 1}$$

$$y \cdot \frac{dy}{dx} = \frac{\operatorname{arctg} x}{y^2 - 2y + 1}$$

$$\int y \cdot (y^2 - 2y + 1)\, dy = \int \operatorname{arctg} x \, dx$$

$$\int \operatorname{arctg} x \, dx = \int 1 \cdot \operatorname{arctg} x \, dx =$$

$$= \left| \begin{array}{ll} u = \operatorname{arctg} x & u' = \frac{1}{1+x^2} \\ v' = 1 & v = x \end{array} \right| = x \cdot \operatorname{arctg} x -$$

$$- \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \cdot \ln(1+x^2) + C_1$$

$$\int (y^3 - 2y^2 + y)\, dy = \ldots$$

$$\frac{y^4}{4} - 2 \cdot \frac{y^3}{3} + \frac{y^2}{2} = x \cdot \operatorname{arctg} x - \frac{1}{2} \cdot \ln(1+x^2) + C_2$$

$$3y^4 - 8y^3 + 6y^2 = 12x \cdot \operatorname{arctg} x - 6 \cdot \ln(1+x^2) + C$$

Ř.:

$$x \cdot y' = y \cdot \ln \frac{y}{x}$$

$$y' = \frac{y}{x} \cdot \ln \frac{y}{x}$$

$$u(x) =: u = \frac{y}{x} \Rightarrow y = u \cdot x$$

$$y' = u' \cdot x + u \cdot 1$$

$$y' = u' \cdot x + u$$

$$u' x + u = u \cdot \ln u$$

$$u' x = u \cdot \ln u - u$$

$$\frac{du}{dx} \cdot x = u \cdot \ln u - u$$

$$\int \frac{1}{x\cdot(\ln x - 1)}\,dx = \int \frac{1}{x}\,dx \quad \Big/\ x\cdot(\ln x - 1) \neq 0$$

$$= \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\,dx \end{array} \right| = \int \frac{1}{t-1}\,dt = \ln|t-1| + c_1 =$$

$$= \ln|\ln x - 1| + c_1$$

$$\Rightarrow \ln|\ln x - 1| = \ln|x| + c_2 \quad \Big/ \exp.$$

$$|\ln x - 1| = |x| \cdot e^{c_2}$$

$$\ln x - 1 = x \cdot K$$

$\ln u = x \cdot K + 1 \quad / \exp$

$u = e^{x \cdot K + 1}$

$\| $

$\frac{y}{x}$

$$\boxed{y = x \cdot e^{x \cdot K + 1}}$$

$u \cdot (\ln u - 1) \neq 0$

$\sim = 0 \Leftrightarrow u = 0 \quad \vee \quad \ln u = 1$

$\cancel{y = 0}$

$u = e$

$\frac{y}{x} = e \Rightarrow y = x \cdot e$ ✓

$K = 0$

př.: $y' + 2x\,y = x \cdot e^{-x^2}$

$a(x)$

$$\mu(x) = e^{\int a(x)\,dx} = e^{\int 2x\,dx} = e^{2\cdot\frac{x^2}{2} + c} =$$

$$= e^{x^2 + c} = e^{x^2} \cdot e^{c} = e^{x^2} \cdot \underbrace{K}_{1}$$

$$y' + 2x\,y = x \cdot e^{-x^2} \quad / \cdot \mu(x) \quad / \cdot e^{x^2}$$

$$\underbrace{y' \cdot e^{x^2} + 2xy\,e^{x^2}} = x \cdot e^{-x^2} \cdot e^{x^2}$$

$$\left(y\,e^{x^2}\right)' = x$$

$$y\, e^{x^2} = \int x\, dx + c_1$$

$$y \cdot e^{x^2} = \frac{x^2}{2} + c_2$$

$$\underline{\underline{y = e^{-x^2} \cdot \left(\frac{x^2}{2} + c_2 \right)}}$$

ř.:

$$y' - \frac{2y}{x} = \frac{1}{x^3}$$

$$\mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\cdot\ln|x|} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

$$y'\cdot\frac{1}{x^2} - \frac{2y}{x}\cdot\frac{1}{x^2} = \frac{1}{x^3}\cdot\frac{1}{x^2}$$

$$\underbrace{\frac{y'}{x^2} - \frac{2y}{x^3}} = \frac{1}{x^5}$$

$$\left(\frac{y}{x^2}\right)' = \frac{1}{x^5}$$

$$\frac{y}{x^2} = \int x^{-5}\,dx + c_1 = \frac{x^{-4}}{-4} + c =$$

$$= \frac{-1}{4x^4} + c$$

$$\underline{\underline{y = \frac{-1}{4x^2} + cx^2}}$$

Př.: $2y' + y = x \quad / \cdot \frac{1}{2}$

$$y' + a(x) \cdot y = b(x)$$

$$y' + \frac{1}{2} y = \frac{x}{2} \quad / \cdot \mu(x)$$

$$\mu(x) = e^{\int \frac{1}{2} \, dx} = e^{\frac{x}{2}}$$

$$\underbrace{y' \cdot e^{\frac{x}{2}} + \frac{y}{2} \cdot e^{\frac{x}{2}}} = \frac{x}{2} \cdot e^{\frac{x}{2}}$$

$$\left(y \cdot e^{\frac{x}{2}}\right)' = \frac{x}{2} e^{\frac{x}{2}}$$

$$y \cdot e^{\frac{x}{2}} = \int \frac{x}{2} \cdot e^{\frac{x}{2}} \, dx + c_1 = \left| \begin{array}{l} t = \frac{x}{2} \\ dt = \frac{1}{2} dx \\ dx = 2\,dt \end{array} \right| =$$

$$= 2 \cdot \int t \cdot e^{t} \, dt + c_1 =$$

$$= \left| \begin{array}{ll} u = t & u' = 1 \\ v' = e^{t} & v = e^{t} \end{array} \right| = 2 \cdot \left(\underline{\underline{t \cdot e^{t}}} - \int e^{t} \, dt \right) + c_2$$

$$= 2 \cdot t \cdot e^{t} - 2 \cdot e^{t} + c_3 =$$

$$= 2 \cdot \frac{x}{2} \cdot e^{\frac{x}{2}} - 2 \cdot e^{\frac{x}{2}} + c_3 = \underline{\underline{e^{\frac{x}{2}} \cdot (x-2) + c_3}}$$

$$y \cdot e^{\frac{x}{2}} = e^{\frac{x}{2}} \cdot (x-2) + c_3$$

$$y = x - 2 + c_3 \cdot e^{-\frac{x}{2}}$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_1 y' + a_0 = 0$$

1. CHAR. POLY: $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$

2. KOŘENY CHAR. POLY: $\lambda_1, \dots, \lambda_n$

3. FUND. SYST. ŘEŠ.:

(i) λ_1 JE 1-NÁS. KOŘEN $\Rightarrow \underbrace{y_1 = e^{\lambda_1 x}}$

(ii) $\lambda_2 = \lambda_3 = \lambda_4 \Rightarrow y_2 = e^{\lambda_2 \cdot x}, \; y_3 = x \cdot e^{\lambda_2 x},$

$y_4 = x^2 \cdot e^{\lambda_2 x}$

(iii) λ_5, λ_6 ... KOMPL. SDRUŽ.

$$\lambda_5 = \alpha + i\beta$$

$$\lambda_6 = \alpha - i\beta$$

$$\Rightarrow \quad y_5 = e^{\alpha x} \cdot \cos \beta x$$

$$y_6 = e^{\alpha x} \cdot \sin \beta x$$

$$e^{(\alpha + \beta i) \cdot x} = e^{\alpha x + \beta x i} = e^{\alpha x} \cdot \underbrace{e^{\beta x i}} =$$

$$= e^{\alpha x} \cdot (\cos \beta x + i \cdot \sin \beta x)$$

④ OBECNÉ ŘEŠ.:

$$y = C_1 y_1 + C_2 \cdot y_2 + \dots + C_n \cdot y_n$$

Ř.: $y'' + y' - 2y = 0$

$$\lambda^2 + \lambda - 2 = 0$$

$$D = 1 + 8 = 9$$

$$\lambda_{1,2} = \frac{-1 \pm 3}{2} = \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -2 \end{cases}$$

$$y_1 = e^{x}, \quad y_2 = e^{-2x}$$

$$\underline{\underline{y = c_1 \cdot e^{x} + c_2 \cdot e^{-2x}}}$$

Ř.:

$$y^{(4)} + 6y^{(3)} + 9y^{(2)} = 0$$

$$\lambda^4 + 6\lambda^3 + 9\lambda^2 = 0$$

$$\lambda^2 \cdot (\lambda^2 + 6\lambda + 9) = 0$$

$$\lambda^2 \cdot (\lambda + 3)^2 = 0$$

$$\lambda_{1,2} = 0 \quad , \quad \lambda_{3,4} = -3$$

$$y_1 = e^{\lambda_1 x} = e^{0x} = 1 \quad , \quad y_2 = x \cdot e^{0x} = x$$

$$y_3 = e^{-3x} \quad , \quad y_4 = x \cdot e^{-3x}$$

$$y = c_1 + c_2 x + c_3 \cdot e^{-3x} + c_4 \cdot x \cdot e^{-3x}$$

Př.: $y''' + 8y'' + 25y' + 26 = 0$

$$\lambda^3 + 8\lambda^2 + 25\lambda + 26 = 0$$

↳ $\cancel{\pm 1}, \pm 2, \pm 13, \pm 26$

	1	8	25	26	
~~2~~	~~1~~	10	~~45~~	$2 \cdot 45 + 26 \neq 0$	×
−2	1	6	13	0	✓

$\lambda_1 = -2$

$$\Rightarrow (\lambda + 2) \cdot (\lambda^2 + 6\lambda + 13) = 0$$

$$D = 36 - 4 \cdot 13 = 36 - 52 = -16$$

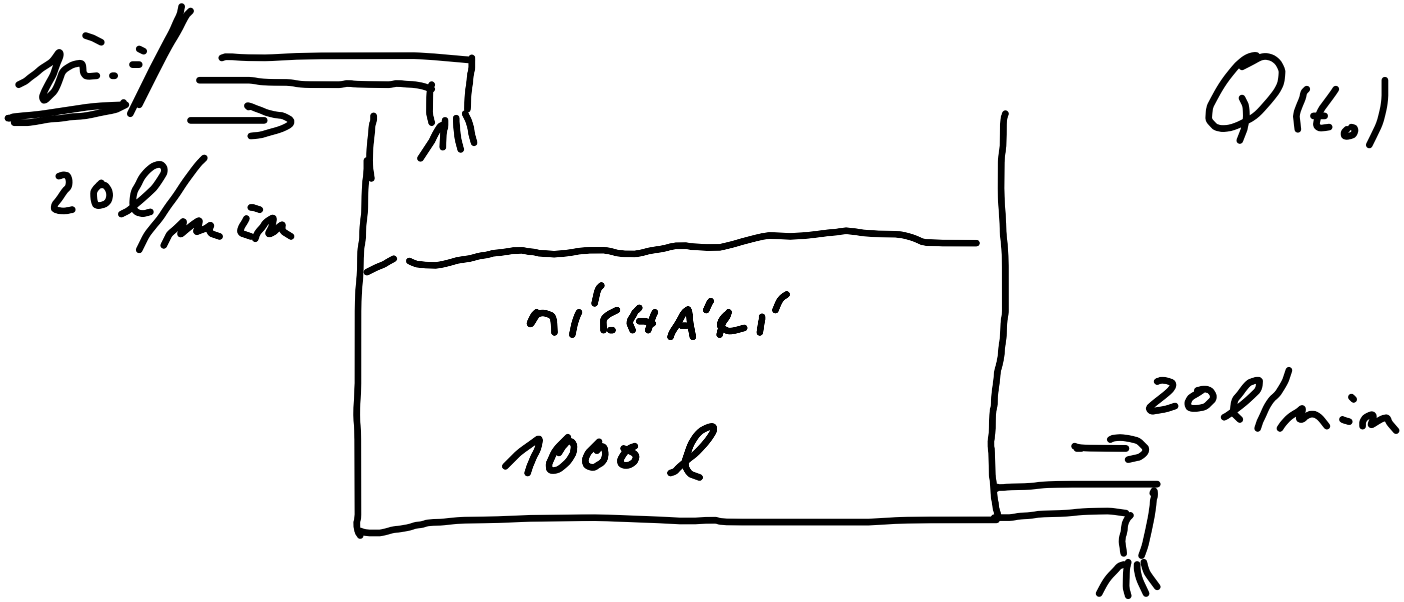
$$\lambda_{2,3} = \frac{-6 \pm \sqrt{16 i^2}}{2} = -3 \pm 2i$$

$$y_1 = e^{-2x}$$

$$y_2 = e^{-3x} \cdot \cos 2x$$

$$y_3 = e^{-3x} \cdot \sin 2x$$

$$y = c_1 \cdot e^{-2x} + c_2 \cdot e^{-3x} \cdot \cos 2x + c_3 \cdot e^{-3x} \cdot \sin 2x$$



$Q(t_0) = 0$

$Q(t)$ = MNOŽSTVÍ SOLI [g] V NÁDRŽI V ČASE t [min]

$\Rightarrow$ $Q'(t)$ = JAK RYCHLE SE MĚNÍ MNOŽSTVÍ $Q(t)$

$$Q'(t) = \begin{pmatrix} \text{RYCHLOST, S JAKOU} \\ \text{SŮL PŘITÉKÁ} \end{pmatrix} - \begin{pmatrix} \text{R. S JAKOU} \\ \text{ODTÉKÁ} \end{pmatrix}$$

PŘITÉKÁ RYCHL.:

$$c \cdot v = 50\,[g/l] \cdot 20\,[l/min] = 1000\,[g/min]$$

VYTÉKÁ RYCHL.: $\quad$ (KONC. V NÁDRŽI) $\cdot\, v$ =

$$= \frac{Q(t)}{V} \cdot v = \frac{Q(t)}{1000} \cdot 20 =$$

$$= \frac{Q(t)}{50} \quad [g/min]$$

CELKEM:

$$\boxed{\begin{aligned} Q'(t) &= 1000 - \frac{Q(t)}{50} \\ Q(0) &= 0 \end{aligned}}$$

$$Q' + \frac{1}{50} Q = 1000$$

$$\mu(x) = e^{\int \frac{1}{50} dt} = e^{\frac{t}{50}}$$

$$\underbrace{Q' \cdot e^{t/50} + \frac{1}{50} \cdot Q \cdot e^{t/50}} = 1000 \cdot e^{t/50}$$

$$\left(Q \cdot e^{t/50}\right)' = 1000\, e^{t/50}$$

$$Q \cdot e^{t/50} = \int 1000 \cdot e^{t/50}\, dt = 1000 \cdot e^{t/50} \cdot 50 + c$$

$$Q = 50\,000 + c \cdot e^{-\frac{t}{50}}, \quad c \in \mathbb{R}$$

$$Q(t) = 50000 + c \cdot e^{-\frac{t}{50}}$$

$$Q(0) = 0$$

t

$$0 = Q(0) = 50\,000 + c \cdot e^{-\frac{0}{50}}$$

$$0 = 50\,000 + c$$

$$c = -50\,000$$

$$\underline{\underline{Q(t) = 50\,000 \cdot \left(1 - e^{-t/50}\right)}}$$

$V = 1000\,l$

$\rho_{vody} \doteq 1\ [kg./l] \Rightarrow$ VODY MÁME 1000 kg

$3\% \Rightarrow 30\ kg$ SOLI

TEDY: $30\,000 = 50\,000 \cdot (1 - e^{-\frac{t}{50}})$

$$\frac{3}{5} = 1 - e^{-\frac{t}{50}}$$

$$e^{-t/50} = \frac{2}{5} = 0{,}4 \quad / \ln$$

$$-\frac{t}{50} = \ln 0{,}4$$

$$t = -\frac{\ln 0{,}4}{0{,}02} \doteq 45{,}815 \ [\mathrm{min}]$$