Untyped lambda calculus

terms and values

$$M ::= x \mid MM' \mid \lambda x.M$$

$$V ::= \lambda x.M$$

full β-reduction evaluation rules

$$\frac{}{(\lambda x.M)N \to_\beta M[x := N]} \qquad \frac{M_1 \to_\beta M_2}{\lambda x.M_1 \to_\beta \lambda x.M_2}$$

$$\frac{M_1 \to_\beta M_2}{M_1 N \to_\beta M_2 N} \qquad \frac{N_1 \to_\beta N_2}{M N_1 \to_\beta M N_2}$$

call-by-value evaluation rules

$$\frac{}{(\lambda x.M)V \to_\beta M[x := V]} \qquad \frac{M_1 \to_\beta M_2}{M_1 N \to_\beta M_2 N} \qquad \frac{N_1 \to_\beta N_2}{V N_1 \to_\beta V N_2}$$

Church numerals and booleans

$\underline{0} := \lambda f.\lambda x.x$

$\underline{1} := \lambda f.\lambda x.f\ x$

$\underline{n} := \underbrace{f(f \ldots (f}_{\text{n-times}}(x) \ldots) = \lambda f.\lambda x.f^n(x)$

true $:= \lambda x.\lambda y.x$

false $:= \lambda x.\lambda y.y$

Simply-typed lambda calculus $\lambda^\to$

with Booleans

terms and values

$$\mathrm{t} ::= x \mid \mathrm{t}\ \mathrm{t}' \mid \lambda x : \mathrm{T}.\mathrm{t} \mid \mathrm{true} \mid \mathrm{false} \mid \mathrm{if\ t\ then\ t'\ else\ t''}$$

$$\mathrm{v} ::= \lambda x : \mathrm{T}.\mathrm{t} \mid \mathrm{true} \mid \mathrm{false}$$

types

$$\mathrm{T} ::= \mathrm{T} \to \mathrm{T} \mid \mathrm{Bool}$$

typing rules

$$\Gamma \vdash \mathrm{true} : \mathrm{Bool}\ \text{(T-True)}$$

$$\Gamma \vdash \mathrm{false} : \mathrm{Bool}\ \text{(T-False)}$$

$$\frac{x : \mathrm{T} \in \Gamma}{\Gamma \vdash x : \mathrm{T}}\ \text{(T-Var)}$$

$$\frac{\Gamma, x : \mathrm{T}_1 \vdash \mathrm{t} : \mathrm{T}_2}{\Gamma \vdash \lambda x.\mathrm{t} : \mathrm{T}_1 \to \mathrm{T}_2}\ \text{(T-Abs)}$$

$$\frac{\Gamma \vdash \mathrm{t}_1 : \mathrm{T}_1 \to \mathrm{T}_2 \qquad \Gamma \vdash \mathrm{t}_2 : \mathrm{T}_1}{\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : \mathrm{T}_2}\ \text{(T-App)}$$

$$\frac{\Gamma \vdash \mathrm{t}_1 : \mathrm{Bool} \qquad \Gamma \vdash \mathrm{t}_2 : \mathrm{T} \qquad \Gamma \vdash \mathrm{t}_3 : \mathrm{T}}{\Gamma \vdash \mathrm{if\ t_1\ then\ t_2\ else\ t_3} : \mathrm{T}}\ \text{(T-If)}$$

Hindley-Milner type system

term and values

$$\mathrm{t} ::= x \mid \mathrm{t}\ \mathrm{t}' \mid \lambda x.\mathrm{t} \mid \mathrm{let}\ \ x = \mathrm{t}\ \ \mathrm{in}\ \ \mathrm{t}$$

types

$$\mathrm{T} ::= \alpha \mid \mathrm{T} \to \mathrm{T}$$

$$\mathrm{S} ::= \mathrm{T} \mid \forall \vec{\alpha}.\mathrm{T}$$

typing rules

$$\frac{x : \mathrm{S} \in \Gamma}{\Gamma \vdash x : \mathrm{S}}\ \text{(T-Var)}$$

$$\frac{\Gamma, x : \mathrm{T}_1 \vdash \mathrm{t} : \mathrm{T}_2}{\Gamma \vdash \lambda x.\mathrm{t} : \mathrm{T}_1 \to \mathrm{T}_2}\ \text{(T-Abs)}$$

$$\frac{\Gamma \vdash \mathrm{t}_1 : \mathrm{T}_1 \to \mathrm{T}_2 \qquad \Gamma \vdash \mathrm{t}_2 : \mathrm{T}_1}{\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : \mathrm{T}_2}\ \text{(T-App)}$$

$$\frac{\Gamma \vdash \mathrm{t}_1 : \mathrm{S} \qquad \Gamma, x : \mathrm{S} \vdash \mathrm{t}_2 : \mathrm{T}}{\Gamma \vdash \mathrm{let}\ \ x = \mathrm{t}_1\ \ \mathrm{in}\ \ \mathrm{t}_2 : \mathrm{T}}\ \text{(T-Let)}$$

$$\frac{\Gamma \vdash \mathrm{t} : \mathrm{S}' \qquad \mathrm{S}' \sqsubseteq \mathrm{S}}{\Gamma \vdash \mathrm{t} : \mathrm{S}}\ \text{(T-Inst)}$$

$$\frac{\Gamma \vdash \mathrm{t} : \mathrm{S} \qquad \alpha \notin FV(\Gamma)}{\Gamma \vdash \mathrm{t} : \forall \alpha.\mathrm{S}}\ \text{(T-Gen)}$$

System F

term and values

$$\mathrm{t} ::= x \mid \mathrm{t}\ \mathrm{t}' \mid \lambda x : \mathrm{T}.\mathrm{t} \mid \Lambda\alpha.\mathrm{t} \mid \mathrm{t}\ [\mathrm{T}]$$

$$v ::= \lambda x : \mathrm{T}.\mathrm{t} \mid \alpha.\mathrm{t}$$

types

$$\mathrm{T} ::= \alpha \mid \mathrm{T} \to \mathrm{T} \mid \forall \vec{\alpha}.\mathrm{T}$$

typing rules *(in addition to $\lambda^{\to}$)*

$$\frac{\Gamma \vdash \mathrm{t} : \mathrm{T} \qquad \alpha \notin FV(\Gamma)}{\Gamma \vdash \Lambda\alpha.\mathrm{t} : \forall \alpha.\mathrm{T}}\ \text{(T-TAbs)} \qquad \textit{(T-Gen)}$$

$$\frac{\Gamma \vdash \mathrm{t} : \forall \alpha.\mathrm{T}_1}{\Gamma \vdash \mathrm{t}\ [\mathrm{T}_2] : \{\mathrm{T}_2/\alpha\}\mathrm{T}_1}\ \text{(T-TApp)} \qquad \textit{(T-Inst)}$$

evaluation rules *(in addition to $\lambda^{\to}$)*

$$\frac{\mathrm{t} \to \mathrm{t}'}{\mathrm{t}\ [\mathrm{T}] \to \mathrm{t}'\ [\mathrm{T}]}\ \text{(E-Tapp)}$$

$$(\Lambda\alpha.\mathrm{t})\ [\mathrm{T}] \to \{\mathrm{T}/\alpha\}\mathrm{t}\ \text{(E-TappTabs)}$$