Bayesian Games – Nature & Common Values

A *Bayesian Game (with nature and common values)* consists of

- a set of players $N = \{1, \ldots, n\}$,
- a set of *states of nature* Ω,
- a set of *actions* A_i available to player i,
- a set of *possible types* T_i of player i,
- a *type function* $\tau_i : \Omega \to T_i$ assigning a type of player i to every state of nature,
- a payoff function u_i for every player i

$$u_i : A_1 \times \cdots \times A_n \times \Omega \to \mathbb{R}$$

- a probability distribution P over Ω called *common prior*.

As before, a *pure strategy* for player i is a function $s_i : T_i \to A_i$.

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Given a pure strategy s_i of player i and a state of nature $\omega \in \Omega$, we denote by $s_i(\omega)$ the action $s_i(\tau_i(\omega))$ chosen by player i when the state is ω.

We denote by $s(\omega)$ the action profile $(s_1(\tau_1(\omega)), \ldots, s_n(\tau_n(\omega)))$.

Given a set $A \subseteq \Omega$ of states of nature and a type $t_i \in T_i$ of player i, we denote by $P(A \mid t_i)$ the conditional probability of A conditioned on the event that player i has type t_i.

We define the expected payoff for player i by

$$u_i(s_1, \ldots, s_n; t_i) = \mathbb{E}_{\omega \sim P}\left[u_i(s(\omega); \omega) \mid \tau_i(\omega) = t_i\right]$$

Here the right hand side is the expected payoff of player i with respect to the probability distribution P conditioned on his type t_i.

Definition 93

A pure strategy profile $s = (s_1, \ldots, s_n) \in S$ in the Bayesian game is a *pure strategy Bayesian Nash equilibrium* if for each player i and each type $t_i \in T_i$ and every pure strategy s_i' of player i we have that

$$u_i(s_i, s_{-i}; t_i) \geq u_i(s_i', s_{-i}; t_i)$$

Adverse Selection

- A firm C is taking over a firm D.
- The true value d of D is not known to C, assume that it is uniformly distributed on $[0, 1]$.

  This is of course a bit artificial, more precise analysis can be done with a different distribution.
- It is known that D's value will flourish under C's ownership: it will rise to λd where $\lambda > 1$.
- All of the above is a common knowledge.

Let us model the situation as a Bayesian game (with common values).

Adverse Selection (Cont.)

- $N = \{C, D\}$,
- $\Omega = [0, 1]$ where $d \in \Omega$ expresses the true value of D,
- $A_C = [0, 1]$ where $c \in A_C$ expresses how much is the firm C willing to pay for the firm D,

  $A_D = \{yes, no\}$ (sell or not to sell),
- $T_C = \{t_1\}$ (a trivial type) and $T_D = \Omega = [0, 1]$,
- $\tau_C(d) = t_1$ and $\tau_D(d) = d$ for all $d \in \Omega$,
- $u_C(c, yes; d) = \lambda d - c$ and $u_C(c, no; d) = 0$

  $u_D(c, yes; d) = c$ and $u_D(c, no; d) = d$,
- P is the uniform distribution on $[0, 1]$.

Is there a BNE?

Adverse Selection (Cont.)

What is the best response of firm D to an action $c \in [0, 1]$ of firm C?

Such a best response must satisfy:

- say *yes* if $d < c$
- say *no* if $d > c$

So the expected *value* of the firm D (in the eyes of C) *assuming that D says yes* is $c/2$.
Indeed, assuming that the firm D says yes, the value d is uniformly distributed between 0 and c, so the average is $c/2$.

Therefore, the expected payoff of C is

$$\lambda(c/2) - c = c\left(\frac{\lambda}{2} - 1\right)$$

which is negative for $\lambda \leq 2$. So it is not profitable (on average) for the firm C to buy unless the target D more than doubles in value after the takeover!

Committe Voting

Consider a very simple model of a jury made up of two players (jurors) who must collectively decide whether to acquit (A), or to convict (C) a defendant who can be either guilty (G) or innocent (I).

Each player casts a sealed vote (A or C), and the defendant is convicted if and only if both vote C.

A prior probability that the defendant is guilty is $q > \frac{1}{2}$ (i.e., $P(G) = q$) and is common knowledge.

Assume that each player gets payoff 1 for a right decision and 0 for incorrect decision. We consider risk neutral players who maximize their expected payoff.

We may model this situation using a strategic-form game:

	A	C
A	$1-q, 1-q$	$1-q, 1-q$
C	$1-q, 1-q$	q, q

Is there a dominant strategy?

Committee Voting (Cont.)

Let's make things a bit more complicated.

Assume that each juror has a different expertise and, when observing the evidence, gets a private signal $t_i \in \{\theta_G, \theta_I\}$ that contains a valuable piece of information. That is if the defendant is guilty, θ_G is more probable, if innocent, θ_I is more probable. For $i \in \{1, 2\}$:

$$P(t_i = \theta_G \mid G) = P(t_i = \theta_I \mid I) = p > \frac{1}{2}$$

$$P(t_i = \theta_G \mid I) = P(t_i = \theta_I \mid G) = 1 - p < \frac{1}{2}$$

We also assume that the players get their signals independently conditional on the defendants condition:

$$P(t_1 = \theta_X \wedge t_2 = \theta_Y \mid Z) = P(t_1 = \theta_X \mid Z) \cdot P(t_2 = \theta_Y \mid Z)$$

for all $X, Y, Z \in \{G, I\}$.

Committe Voting (Cont.)

We obtain a Bayesian game:

- $N = \{1, 2\}$
- $A_1 = A_2 = \{A, C\}$
- $\Omega = \{(Z, \theta_X, \theta_Y) \mid Z, X, Y \in \{G, I\}\}$
- $T_1 = T_2 = \{\theta_G, \theta_I\}$
- $\tau_1(Z, \theta_X, \theta_Y) = \theta_X$ and $\tau_2(Z, \theta_X, \theta_Y) = \theta_Y$
- For arbitrary $U, V \in \{A, C\}$ and $X, Y \in \{G, I\}$ we have that

$$u_i(U, V; (G, \theta_X, \theta_Y)) = \begin{cases} 1 & \text{if } U = V = C, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_i(U, V; (I, \theta_X, \theta_Y)) = \begin{cases} 0 & \text{if } U = V = C, \\ 1 & \text{otherwise.} \end{cases}$$

- $P(Z, \theta_X, \theta_Y) = P(Z)P(t_1 = \theta_X \mid Z)P(t_2 = \theta_Y \mid Z)$

  I.e., $P(Z, \theta_X, \theta_Y)$ is the probability of choosing (Z, θ_X, θ_Y) as follows: First, $Z \in \{G, I\}$ is randomly chosen ($Z = G$ has probability q). Then, conditioned on Z, θ_X and θ_Y are independently chosen.

Committee Voting (Cont.)

Now consider just one player i. If the player i would be able to decide by himself, how does his decision depend on his type $t_i \in \{\theta_G, \theta_I\}$?

If $t_i = \theta_G$, then how probable is that the defendant is guilty?

$$P(G \mid t_i = \theta_G) = \frac{P(t_i = \theta_G \mid G)P(G)}{P(t_i = \theta_G)} = \frac{pq}{qp + (1-q)(1-p)} > q$$

so that the posterior probability of G is even higher.
If θ_I is received, then how probable is that the defendant is guilty?

$$P(G \mid t_i = \theta_I) = \frac{P(t_i = \theta_I \mid G)P(G)}{P(t_i = \theta_I)} = \frac{(1-p)q}{q(1-p) + (1-q)p} < q$$

which means, clearly, that the player is less sure about G.
In particular, player i chooses I instead of G if

$$P(G \mid t_i = \theta_I) = \frac{q(1-p)}{q(1-p) + (1-q)p} < \frac{1}{2}$$

which holds iff $p > q$.

Committee Voting (Cont.)

So if $p > q$ each player would choose to vote according to his signal.

Denote by XY the strategy of player i in which he chooses X if $t_i = \theta_G$ and Y if $t_i = \theta_I$.

Question: Is (CA, CA) BNE assuming that $p > q$?

$$
\begin{aligned}
u_1(CA, CA; \theta_I) &= P(I \mid t_1 = \theta_I) \\
&= P(I \mid t_1 = \theta_I \wedge t_2 = \theta_G)P(t_2 = \theta_G \mid t_1 = \theta_I) \\
&\quad + P(I \mid t_1 = \theta_I \wedge t_2 = \theta_I)P(t_2 = \theta_I \mid t_1 = \theta_I)
\end{aligned}
$$

$$
\begin{aligned}
u_1(CC, CA; \theta_I) &= P(G \wedge t_2 = \theta_G \mid t_1 = \theta_I) + P(I \wedge t_2 = \theta_I \mid t_1 = \theta_I) \\
&= P(G \mid t_1 = \theta_I \wedge t_2 = \theta_G)P(t_2 = \theta_G \mid t_1 = \theta_I) \\
&\quad + P(I \mid t_1 = \theta_I \wedge t_2 = \theta_I)P(t_2 = \theta_I \mid t_1 = \theta_I)
\end{aligned}
$$

Note that the blue expressions are equal, so the payoff depends only on the red ones, where player 2 is assumed to consider the defendant guilty.
Intuitively, if player 2 chooses A, then the decision of player 1 does not have any impact. On the other hand, if player 2 chooses C, then the decision is, in fact, up to player 1 (we say that he is *pivotal*).

Committee Voting (Cont.)

So what is the probability that the defendant is guilty assuming that the vote of player 1 counts? That is, assuming $t_2 = \theta_G$ and $t_1 = \theta_I$?

$$
\begin{aligned}
P(G \mid t_1 = \theta_I \wedge t_2 = \theta_G) &= \frac{P(t_1 = \theta_I \wedge t_2 = \theta_G \mid G)P(G)}{P(t_1 = \theta_I \wedge t_2 = \theta_G)} \\
&= \frac{(1-p)pq}{p(1-p)} \\
&= q > \frac{1}{2} > (1-q) \\
&= P(I \mid t_1 = \theta_I \wedge t_2 = \theta_G)
\end{aligned}
$$

which means that player 1 is more convinced that the defendant is guilty contrary to the signal! This means that even though individual decision would be "innocent", taking into account that the vote should have some value gives "guilty".

Hence $u_1(CA, CA; \theta_I) < u_1(CC, CA; \theta_I)$ and thus playing CC is a better response to CA.

By the way, is (CC, CA) a BNE?

Winner's Curse

An auction for a new oil field (of unknown size), assume only two firms competing (two players).

The field is either small (worth \$10 million), medium (worth \$20 million), large (worth \$30 million).

That is, the real value v of the field satisfies $v \in \{10, 20, 30\}$.

Assume some prior information about the size of the filed:

$$P(v = 10) = P(v = 30) = \frac{1}{4} \qquad\qquad P(v = 20) = \frac{1}{2}$$

The government is selling the field in the second-price sealed-bid auction, so that in the case of a tie, the winner is chosen randomly (and pays his bid). That is, in effect, in case of a tie, the payoff of each player is $(v - b)/2$ where v is the value, b the (common) bid.

Using the same argument as for the "ordinary" second-price auction with private values one may show that playing the true private value weakly dominates all other bids.

Winner's Curse (Cont.)

Each of the firms performs a (free) exploration that will provide the type $t_i \in \{L, H\}$ (low or high), correlated with the size as follows:

- If $v = 10$, then $t_1 = t_2 = L$
- If $v = 30$, then $t_1 = t_2 = H$
- If $v = 20$, then for $i \in \{1, 2\}$, conditioned on $v = 20$, the exploration results are uniformly distributed:
  There are four possible results, (L, L), (L, H), (H, L), (H, H), each with probability $\frac{1}{4}$.

Given the signal t_i, player i may estimate the true value of the field:

$$P(v = 10 \mid t_i = L) = \frac{1}{2} \qquad P(v = 10 \mid t_i = H) = 0$$

$$P(v = 20 \mid t_i = L) = \frac{1}{2} \qquad P(v = 20 \mid t_i = H) = \frac{1}{2}$$

$$P(v = 30 \mid t_i = L) = 0 \qquad P(v = 30 \mid t_i = H) = \frac{1}{2}$$

Thus $\mathbb{E}(v \mid t_i = L) = \frac{1}{2}10 + \frac{1}{2}20 = 15$.
and $\mathbb{E}(v \mid t_i = H) = \frac{1}{2}20 + \frac{1}{2}30 = 25$

Winner's Curse (Cont.)

Is it a good idea to bid the expected value?

Define a strategy s_i for player i by

- $s_i(L) = \mathbb{E}(v \mid t_i = L)$
- $s_i(H) = \mathbb{E}(v \mid t_i = H)$

Is (s_1, s_2) a Nash equilibrium?

Consider $t_1 = L$. Then player 1 bids 15. What is his expected payoff?

Note that if $t_2 = H$, then player 2 bids 25 and wins, which means that player 1 gets payoff 0. So player 1 can get a non-zero value only if $t_2 = L$. This implies that

$$\begin{aligned} u_1(s_1, s_2; L) &= P(v = 20 \wedge t_2 = L \mid t_1 = L) \cdot (20 - 15)/2 \\ &\quad + P(v = 10 \wedge t_2 = L \mid t_1 = L) \cdot (10 - 15)/2 \\ &= P(v = 20 \wedge t_2 = L \mid t_1 = L) \cdot 5/2 \\ &\quad + P(v = 10 \wedge t_2 = L \mid t_1 = L) \cdot (-5)/2 \end{aligned}$$

Winner's Curse (Cont.)

In what follows we show that

$$P(v = 20 \wedge t_2 = L \mid t_1 = L) = \frac{1}{4} \tag{31}$$

$$P(v = 10 \wedge t_2 = L \mid t_1 = L) = \frac{1}{2} \tag{32}$$

which means that

$$\begin{aligned} u_1(s_1, s_2; L) &= P(v = 20 \wedge t_2 = L \mid t_1 = L) \cdot 5/2 \\ &\quad + P(v = 10 \wedge t_2 = L \mid t_1 = L) \cdot (-5)/2 \\ &= \frac{1}{4}\frac{5}{2} + \frac{1}{2}\frac{(-5)}{2} = \frac{-5}{8} < 0 \end{aligned}$$

and player 1 would be better off by bidding 0 and always losing!!

Intuition: Player 1 wins only if the signal of player 2 is L, which in effect means, that assuming win, the *effective* expected value of the field is *lower* than the predicted expected value.

In the rest of the proof we heavily use the Bayes' theorem and the law of total probability.

Winner's Curse (Cont.) : Proof of Equation (31)

$$
\begin{aligned}
& P(v = 20 \wedge t_2 = L \mid t_1 = L) = \\
&\quad = P(v = 20 \wedge t_2 = L \mid t_1 = L \wedge t_2 = L) \cdot P(t_2 = L \mid t_1 = L) \\
&\quad + P(v = 20 \wedge t_2 = L \mid t_1 = L \wedge t_2 = H) \cdot P(t_2 = H \mid t_1 = L) \\
&\quad = P(v = 20 \mid t_1 = L \wedge t_2 = L) \cdot P(t_2 = L \mid t_1 = L)
\end{aligned}
$$

Here

$$
\begin{aligned}
& P(t_2 = L \mid t_1 = L) = \\
&\quad = P(t_2 = L \mid t_1 = L \wedge v = 10) \cdot P(v = 10 \mid t_1 = L) \\
&\quad + P(t_2 = L \mid t_1 = L \wedge v = 20) \cdot P(v = 20 \mid t_1 = L) \\
&\quad = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}
\end{aligned}
$$

(Here we used the fact that t_1 and t_2 are independent assuming a fixed v)

We show (see next slide) that

$$P(v = 20 \mid t_1 = L \wedge t_2 = L) = \frac{1}{3}$$

and thus

$$P(v = 20 \wedge t_2 = L \mid t_1 = L) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

Winner's Curse (Cont.) : Proof of Equation (31)

First, note that

$$P(t_1 = L \wedge t_2 = L \mid v = 10) = 1$$

$$P(t_1 = L \wedge t_2 = L \mid v = 20) = \frac{1}{4}$$

Now by Bayes' theorem

$$\begin{aligned}
&P(v = 20 \mid t_1 = L \wedge t_2 = L) = \\
&\quad = [\, P(t_1 = L \wedge t_2 = L \mid v = 20) \cdot P(v = 20)\,] \;/\; P(t_1 = L \wedge t_2 = L) = \\
&\quad = \frac{\frac{1}{4} \cdot \frac{1}{2}}{P(t_1 = L \wedge t_2 = L)} = \frac{1}{8 \cdot P(t_1 = L \wedge t_2 = L)}
\end{aligned}$$

But by the law of total probability

$$\begin{aligned}
&P(t_1 = L \wedge t_2 = L) = \\
&\quad = P(t_1 = L \wedge t_2 = L \mid v = 10) P(v = 10) + \\
&\quad + P(t_1 = L \wedge t_2 = L \mid v = 20) P(v = 20) \\
&\quad = 1 \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}
\end{aligned}$$

which gives $P(v = 20 \mid t_1 = L \wedge t_2 = L) = \frac{1}{3}$.

Winner's Curse (Cont.) : Proof of Equation (32)

Finally, similarly as for (31),

$$
\begin{aligned}
&P(v = 10 \wedge t_2 = L \mid t_1 = L) = \\
&\quad = P(v = 10 \wedge t_2 = L \mid t_1 = L \wedge t_2 = L) \cdot P(t_2 = L \mid t_1 = L) \\
&\quad\ + P(v = 10 \wedge t_1 = L \mid t_1 = L \wedge t_2 = H) \cdot P(t_2 = H \mid t_1 = L) \\
&\quad = P(v = 10 \mid t_1 = L \wedge t_2 = L) \cdot P(t_2 = L \mid t_1 = L) \\
&\quad = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}
\end{aligned}
$$

Here $P(v = 10 \mid t_1 = L \wedge t_2 = L) = \frac{2}{3}$ follows from $P(v = 20 \mid t_1 = L \wedge t_2 = L) = \frac{1}{3}$ and $P(v = 30 \mid t_1 = L \wedge t_2 = L) = 0$.