Iterated Strict Dominance in Pure Strategies

We know that no rational player ever plays strictly dominated strategies.

As each player knows that each player is rational, each player knows that his opponents will not play strictly dominated strategies and thus all opponents know that *effectively* they are facing a "smaller" game.

As rationality is a common knowledge, everyone knows that everyone knows that the game is effectively smaller.

Thus everyone knows, that nobody will play strictly dominated strategies in the smaller game (and such strategies may indeed exist).

Because it is a common knowledge that all players will perform this kind of reasoning again, the process can continue until no more strictly dominated strategies can be eliminated.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 10
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is an *IESDS equilibrium* if each s_i survives IESDS.

A game is *IESDS solvable* if it has a unique IESDS equilibrium.

Remark: If all S_i are *finite*, then in 2. we may remove only some of the strictly dominated strategies (not necessarily all). The result is *not* affected by the order of elimination since strictly dominated strategies remain strictly dominated even after removing some other strictly dominated strategies.

IESDS Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only one surviving the first round of IESDS.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

all strategies survive all rounds (i.e. IESDS $\equiv$ anything may happen, sorry)

A Bit More Interesting Example

	L	C	R
L	$4,3$	$5,1$	$6,2$
C	$2,1$	$8,4$	$3,6$
R	$3,0$	$9,6$	$2,8$

IESDS on greenboard!

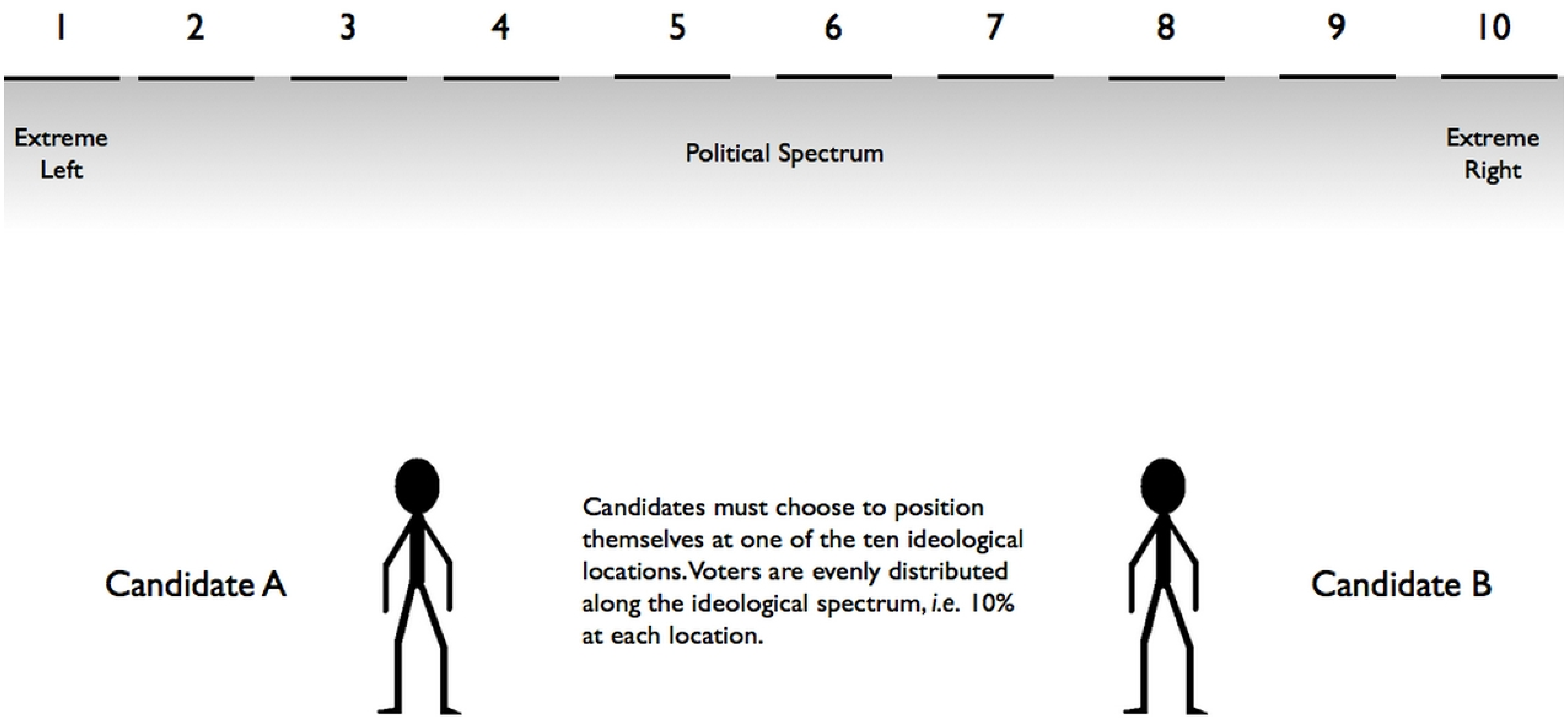
Political Science Example: Median Voter Theorem

Hotelling (1929) and Downs (1957)

- $N = \{1, 2\}$
- $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (political and ideological spectrum)
- 10 voters belong to each position

  (Here 10 means ten percent in the real-world)
- Voters vote for the closest candidate. If there is a tie, then $\frac{1}{2}$ got to each candidate
- Payoff: The number of voters for the candidate, each candidate (selfishly) strives to maximize this number

Political Science Example: Median Voter Theorem



- 1 and 10 are the (only) strictly dominated strategies $\Rightarrow$ $D_1^1 = D_2^1 = \{2, \ldots, 9\}$
- in G_{DS}^1, 2 and 9 are the (only) strictly dominated strategies $\Rightarrow$ $D_1^2 = D_2^2 = \{3, \ldots, 8\}$
- $\ldots$
- only $5, 6$ survive IESDS

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

What if we rather want to actively preserve reasonable behavior? What is reasonable? .... what we believe is reasonable :-).

Intuition:

- Imagine that your colleague did something stupid
- What would you ask him? Usually something like "What were you thinking?"
- The colleague may respond with a reasonable description of his *belief* in which his action was (one of) the best he could do

  (You may of course question reasonableness of the belief)

Let us formalize this type of reasoning ....

Belief & Best Response

Definition 11
A *belief* of player i is a pure strategy profile $s_{-i} \in S_{-i}$ of his opponents.

Definition 12
A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \text{for all } s_i' \in S_i$$

Claim 3
A rational player who believes that his opponents will play $s_{-i} \in S_{-i}$ always chooses a best response to $s_{-i} \in S_{-i}$.

Definition 13
A strategy $s_i \in S_i$ is *never best response* if it is not a best response to any belief $s_{-i} \in S_{-i}$.

A rational player never plays any strategy that is never best response.

Best Response vs Strict Dominance

Proposition 1

If s_i is strictly dominated for player i, then it is never best response.

The opposite does not have to be true in pure strategies:

	X	Y
A	$1, 1$	$1, 1$
B	$2, 1$	$0, 1$
C	$0, 1$	$2, 1$

Here A is never best response but is strictly dominated neither by B, nor by C.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G^k_{Rat} the game obtained from G by restricting to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are best responses to some beliefs in G^k_{Rat}.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 14
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is a *rationalizable equilibrium* if each s_i is rationalizable.

We say that a game is *solvable by rationalizability* if it has a unique rationalizable equilibrium.

(Warning: For some reasons, rationalizable strategies are almost always defined using mixed strategies!)

Rationalizability Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only rationalizable equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

all strategies are rationalizable.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

What is a best response of player 1 to a given q_2 ?

Solve $\frac{\delta u_1}{\delta q_1} = \theta - 2q_1 - q_2 = 0$, which gives that $q_1 = (\theta - q_2)/2$ is the only best response of player 1 to q_2.
Similarly, $q_2 = (\theta - q_1)/2$ is the only best response of player 2 to q_1.

Since $q_2 \geq 0$, we obtain that q_1 is never best response iff $q_1 > \theta/2$.
Similarly q_2 is never best response iff $q_2 > \theta/2$.

Thus $R_1^1 = R_2^1 = [0, \theta/2]$.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
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- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Now, in G^1_{Rat}, we still have that $q_1 = (\theta - q_2)/2$ is the best response to q_2, and $q_2 = (\theta - q_1)/2$ the best resp. to q_1

Since $q_2 \in R^1_2 = [0, \theta/2]$, we obtain that q_1 is never best response iff $q_1 \in [0, \theta/4)$
Similarly q_2 is never best response iff $q_2 \in [0, \theta/4)$

Thus $R^2_1 = R^2_2 = [\theta/4, \theta/2]$.

....

Cournot Duopoly (cont.)

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

In general, after $2k$ iterations we have $R_i^{2k} = R_i^{2k} = [\ell_k, r_k]$ where

- $r_k = (\theta - \ell_{k-1})/2$ for $k \geq 1$
- $\ell_k = (\theta - r_k)/2$ for $k \geq 1$ and $\ell_0 = 0$

Solving the recurrence we obtain

- $\ell_k = \theta/3 - \left(\frac{1}{4}\right)^k \theta/3$
- $r_k = \theta/3 + \left(\frac{1}{4}\right)^{k-1} \theta/6$

Hence, $\lim_{k \to \infty} \ell_k = \lim_{k \to \infty} r_k = \theta/3$ and thus $(\theta/3, \theta/3)$ is the only rationalizable equilibrium.

Cournot Duopoly (cont.)

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Are $q_i = \theta/3$ Pareto optimal? NO!

$$u_1(\theta/3, \theta/3) = u_2(\theta/3, \theta/3) = \theta^2/9$$

but

$$u_1(\theta/4, \theta/4) = u_2(\theta/4, \theta/4) = \theta^2/8$$

IESDS vs Rationalizability in Pure Strategies

Theorem 15
Assume that S is finite. Then for all k we have that $R_i^k \subseteq D_i^k$. That is, in particular, all rationalizable strategies survive IESDS.

The opposite inclusion does not have to be true in pure strategies:

	X	Y
A	$1,1$	$1,1$
B	$2,1$	$0,1$
C	$0,1$	$2,1$

Recall that A is never best response but is strictly dominated by neither B, nor C. That is, A survives IESDS but is not rationalizable.

Proof of Theorem 15

By induction on k. For $k = 0$ we have that $R_i^0 = S_i = D_i^0$ by definition. Now assume that $R_i^k \subseteq D_i^k$ for some $k \geq 0$.

We prove that $R_i^{k+1} \subseteq D_i^{k+1}$ by showing the following:

$$\text{For all } s_i^* \in R_i^k \subseteq D_i^k: \text{ If } s_i^* \notin D_i^{k+1}, \text{ then } s_i^* \notin R_i^{k+1}$$

Let us fix $s_i^* \in R_i^k$ such that $s_i^* \notin D_i^{k+1}$. By definition, it *suffices to prove* that for **every** $s_{-i}^k \in R_{-i}^k$ there **exists** $s_i^k \in R_i$ such that

$$u_i(s_i^k, s_{-i}^k) > u_i(s_i^*, s_{-i}^k) \tag{1}$$

(In words, for every possible behavior of opponents of player i in G_{Rat}^k, player i has a strictly better strategy than s_i^* in G_{Rat}^k)

As $s_i^* \notin D_i^{k+1}$, the strategy s_i^* must be strictly dominated in G_{DS}^k by a strategy $\bar{s}_i$. That is for all $s_{-i}^k \in D_{-i}^k \supseteq R_{-i}^k$ we have

$$u_i(\bar{s}_i, s_{-i}^k) > u_i(s_i^*, s_{-i}^k) \tag{2}$$

(Now note that if $\bar{s}_i \in R_i^k \subseteq D_i^k$, then we are done. Indeed, it suffices to put $s_i^k := \bar{s}_i$ and the equation (1) will be satisfied for all $s_{-i}^k \in D_{-i}^k \supseteq R_{-i}^k$. However, it does not have to be the case that $\bar{s}_i \in R_i^k$)

Proof of Theorem 15 (cont.)

Clearly, there is $\ell \leq k$ such that $\bar{s}_i \in R_i^\ell$.
(Note that $\bar{s}_i$ *does not* have to strictly dominate s_i^* in G_{Rat}^ℓ since R_{-i}^ℓ may be larger than D_{-i}^k)

Recall that we need to find $s_i^k \in R_i^k$ for every *given* $s_{-i}^k \in R_{-i}^k$ so that the inequality (1) holds.
(That is, s_i^k may be different for different s_{-i}^k's)

Let us fix $s_{-i}^k \in R_{-i}^k \subseteq D_{-i}^k$.

Let $s_i^k \in R_i^\ell$ be a strategy maximizing $u_i(s_i, s_{-i}^k)$ over all $s_i \in R_i^\ell$.
In particular, we obtain the inequality (1):

$$u_i(s_i^k, s_{-i}^k) \geq u_i(\bar{s}_i, s_{-i}^k) > u_i(s_i^*, s_{-i}^k)$$

Finally, note that $s_i^k \in R_i^k$ follows immediately from the fact that s_i^k is a best response to s_{-i}^k in all games $G_{Rat}^\ell, \ldots, G_{Rat}^k$
(Indeed, even after removing some strategies (other than s_i^k and s_{-i}^k), s_i^k remains a best resp. to s_{-i}^k)

Pinning Down Beliefs – Nash Equilibria

Criticism of previous approaches:

- Strictly dominant strategy equilibria often do not exist
- IESDS and rationalizability may not remove any strategies

Typical example is Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Here all strategies are equally reasonable according to the above concepts.

But are all strategy profiles really equally reasonable?

Pinning Down Beliefs – Nash Equilibria

	O	F
O	$2,1$	$0,0$
F	$0,0$	$1,2$

Assume that each player has a belief about strategies of other players.

By Claim 3, each player plays a best response to his beliefs.

Is (O, F) as reasonable as (O, O) in this respect?

Note that if player 1 believes that player 2 plays O, then playing O is reasonable, and if player 2 believes that player 1 plays F, then playing F is reasonable. But such **beliefs cannot be correct together**!

(O, O) can be obtained as a profile where each player plays the best response to his belief and the **beliefs are correct**.

Nash Equilibrium

Nash equilibrium can be defined as a set of beliefs (one for each player) and a strategy profile in which every player plays a best response to his belief and each strategy of each player is consistent with beliefs of his opponents.

A usual definition is following:

Definition 16
A pure-strategy profile $s^* = (s_1^*, \dots, s_n^*) \in S$ is a (pure) Nash equilibrium if s_i^* is a best response to s_{-i}^* for each $i \in N$, that is

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in S_i \text{ and all } i \in N$$

Note that this definition is equivalent to the previous one in the sense that s_{-i}^* may be considered as the (consistent) belief of player i to which he plays a best response s_i^*

Nash Equilibria Examples

In the Prisoner's dilemma:

	C	S
C	$-5,-5$	$0,-20$
S	$-20,0$	$-1,-1$

(C, C) is the only Nash equilibrium.

In the Battle of Sexes:

	O	F
O	$2,1$	$0,0$
F	$0,0$	$1,2$

only (O, O) and (F, F) are Nash equilibria.

In Cournot Duopoly, $(\theta/3, \theta/3)$ is the only Nash equilibrium.

(Best response relations: $q_1 = (\theta - q_2)/2$ and $q_2 = (\theta - q_1)/2$ are both satisfied only by $q_1 = q_2 = \theta/3$)

Example: Stag Hunt

Story:

- Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt

  - stag (S) = a large tasty meal
  - hare (H) = also tasty but small

- Hunting stag is much more demanding and forces of both players need to be joined (hare can be hunted individually)

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former Pareto dominates the latter! Which one is more reasonable?

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former Pareto dominates the latter! Which one is more reasonable?

If each player believes that the other one will go for hare, then (H, H) is a reasonable outcome $\Rightarrow$ a society of individualists who do not cooperate at all.

If each player believes that the other will cooperate, then this anticipation is self-fulfilling and results in what can be called a cooperative society.

This is supposed to explain that in real world there are societies that have similar endowments, access to technology and physical environment but have very different achievements, all because of self-fulfilling beliefs (or *norms* of behavior).

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former Pareto dominates the latter! Which one is more reasonable?

Another point of view: (H, H) is less risky

Minimum secured by playing S is 0 as opposed to 3 by playing H
(We will get to this *minimax* principle later)

So it seems to be rational to expect (H, H) (?)

Nash Equilibria vs Previous Concepts

Theorem 17

1. *If s^* is a strictly dominant strategy equilibrium, then it is the unique Nash equilibrium.*
2. *Each Nash equilibrium is rationalizable and survives IESDS.*
3. *If S is finite, neither rationalizability, nor IESDS creates new Nash equilibria.*

Proof: Homework!

Corollary 18

Assume that S is finite. If rationalizability or IESDS result in a unique strategy profile, then this profile is a Nash equilibrium.

Interpretations of Nash Equilibria

Except the two definitions, usual interpretations are following:

- When the goal is to give advice to all of the players in a game (i.e., to advise each player what strategy to choose), any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad, in the sense that, if all other players followed the parts of the advice directed to them, it would be better for some player to do differently than he was advised. If the advice is an equilibrium, however, this will not be the case, because the advice to each player is the best response to the advice given to the other players.
- When the goal is prediction rather than prescription, a Nash equilibrium can also be interpreted as a potential stable point of a dynamic adjustment process in which individuals adjust their behavior to that of the other players in the game, searching for strategy choices that will give them better results.