# Lemke-Howson Algorithm – Notation

Fix a strategic-form two-player game $G = (\{1,2\}, (S_1, S_2), (u_1, u_2))$.

Assume that

- $S_1 = \{1, \ldots, m\}$
- $S_2 = \{m+1, \ldots, m+n\}$

(I.e., player 1 has $m$ pure strategies $1, \ldots, m$ and player 2 has $n$ pure strategies $m+1, \ldots, m+n$. In particular, each pure strategy determines the player who can play it.)

Assume that $u_1, u_2$ are positive, i.e., $u_1(k,\ell) > 0$ and $u_2(k,\ell) > 0$ for all $(k,\ell) \in S_1 \times S_2$.
This assumption is w.l.o.g. since any positive constant can be added to payoffs without altering the set of (mixed) Nash equilibria.

Mixed strategies of player 1 : $\sigma_1 = (\sigma(1), \ldots, \sigma(m)) \in [0,1]^m$
Mixed strategies of player 2 : $\sigma_2 = (\sigma(m+1), \ldots, \sigma(m+n)) \in [0,1]^n$
I.e. we omit the lower index of $\sigma$ whenever it is determined by the argument.
A strategy profile $\sigma = (\sigma_1, \sigma_2)$ can be seen as a vector
$\sigma = (\sigma_1, \sigma_2) = (\sigma(1), \ldots, \sigma(m+n)) \in [0,1]^{m+n}$.

# Running Example

| | 3 | 4 |
|---|---|---|
| 1 | 3, 1 | 2, 2 |
| 2 | 2, 3 | 3, 1 |

- Player 1 (row) plays $\sigma_1 = (\sigma(1), \sigma(2)) \in [0, 1]^2$
- Player 2 (column) plays $\sigma_2 = (\sigma(3), \sigma(4)) \in [0, 1]^2$
- A typical mixed strategy profile is $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$

For example: $\sigma_1 = (0.2, 0.8)$ and $\sigma_2 = (0.4, 0.6)$ give the profile $(0.2, 0.8, 0.4, 0.6)$.

# Characterizing Nash Equilibria

Recall that by Lemma 42 the following holds:

$(\sigma_1, \sigma_2) = (\sigma(1), \ldots, \sigma(m+n)) \in \Sigma$ is a Nash equilibrium **iff**

- For all $\ell = m+1, \ldots, m+n$ we have that

  $$u_2(\sigma_1, \ell) \leq u_2(\sigma_1, \sigma_2)$$

  and either $\sigma(\ell) = 0$, or $u_2(\sigma_1, \ell) = u_2(\sigma_1, \sigma_2)$
- For all $k = 1, \ldots, m$ we have that

  $$u_1(k, \sigma_2) \leq u_1(\sigma_1, \sigma_2)$$

  and either $\sigma(k) = 0$, or $u_1(k, \sigma_2) = u_1(\sigma_1, \sigma_2)$

This is equivalent to the following: $(\sigma_1, \sigma_2) = (\sigma(1), \ldots, \sigma(m+n)) \in \Sigma$ is a Nash equilibrium **iff**

- For all $\ell = m+1, \ldots, m+n$ we have that either $\sigma(\ell) = 0$, or $\ell$ is a best response to $\sigma_1$.
- For all $k = 1, \ldots, m$ we have that either $\sigma(k) = 0$, or $k$ is a best response to $\sigma_2$.

# Characterizing Nash Equilibria

Given a mixed strategy $\sigma_1 = (\sigma(1), \ldots, \sigma(m))$ of player 1 we define $L(\sigma_1) \subseteq \{1, 2, \ldots, m+n\}$ to consist of

- all $k \in \{1, \ldots, m\}$ satisfying $\sigma(k) = 0$
- all $\ell \in \{m+1, \ldots, m+n\}$ that are best responses to $\sigma_1$

Given a mixed strategy $\sigma_2 = (\sigma(m+1), \ldots, \sigma(m+n))$ of player 2 we define $L(\sigma_2) \subseteq \{1, 2, \ldots, m+n\}$ to consist of

- all $k \in \{1, \ldots, m\}$ that are best responses to $\sigma_2$
- all $\ell \in \{m+1, \ldots, m+n\}$ satisfying $\sigma(\ell) = 0$

**Proposition 3**

$\sigma = (\sigma_1, \sigma_2)$ *is a Nash equilibrium* **iff** $L(\sigma_1) \cup L(\sigma_2) = \{1, \ldots, m+n\}$.

We also label the vector $0^m := (0, \ldots, 0) \in \mathbb{R}^m$ with $\{1, \ldots, m\}$ and $0^n := (0, \ldots, 0) \in \mathbb{R}^n$ with $\{m+1, \ldots, m+n\}$.
We consider $(0^m, 0^n)$ as a special mixed strategy profile.

How many labels could possibly be assigned to one strategy?

# Running Example

| | 3 | 4 |
|---|---|---|
| 1 | 3, 1 | 2, 2 |
| 2 | 2, 3 | 3, 1 |

A strategy $\sigma_1 = (2/3, 1/3)$ of player 1 is labeled by $3, 4$ since both pure strategies $3, 4$ of player 2 are best responses to $\sigma_1$ (they result in the same payoff to player 2)

A strategy $\sigma_2 = (1/2, 1/2)$ of player 2 is labeled by $1, 2$ since both pure strategies $1, 2$ of player 1 are best responses to $\sigma_2$ (they result in the same payoff to player 1)

A strategy $\sigma_1 = (0, 1)$ of player 1 is labeled by $1, 3$ since the strategy 1 is played with zero probability in $\sigma_1$ and 3 is the best response to $\sigma_1$

A strategy $\sigma_1 = (1/10, 9/10)$ of player 1 is labeled by 3 since no pure strategy of player 1 is played with zero probability (and hence neither 1, nor 2 labels $\sigma_1$) and 3 is the best response to $\sigma_1$.

# Non-degenerate Games

Definition: $G$ is *non-degenerate* if for every $\sigma_1 \in \Sigma_1$ we have that $|supp(\sigma_1)|$ is at least the number of pure best responses to $\sigma_1$, and for every $\sigma_2 \in \Sigma_2$ we have that $|supp(\sigma_2)|$ is at least the number of pure best responses to $\sigma_2$.
"Most" games are non-degenerate, or can be made non-degenerate by a slight perturbation of payoffs

We assume that **the game $G$ is non-degenerate**.

Non-degeneracy implies that $L(\sigma_1) \leq m$ for every $\sigma_1 \in \Sigma_1$ and $L(\sigma_2) \leq n$ for every $\sigma_2 \in \Sigma_2$.
We say that a strategy $\sigma_1$ of player 1 (or $\sigma_2$ of player 2) is *fully labeled* if $|L(\sigma_1)| = m$ (or $|L(\sigma_2)| = n$, respectively).

**Lemma 50**
*Non-degeneracy of G implies the following:*

- *If $\sigma_i, \sigma_i' \in \Sigma_i$ are fully labeled, then $L(\sigma_i) \neq L(\sigma_i')$. There are at most $\binom{m+n}{m}$ fully labeled strategies of player 1, $\binom{m+n}{n}$ of player 2.*
- *For every fully labeled $\sigma_i \in \Sigma_i$ and a label $k \in L(\sigma_i)$ there is exactly one fully labeled $\sigma_i' \in \Sigma_i$ such that $L(\sigma_i) \cap L(\sigma_i') = L(\sigma_i) \setminus \{k\}$.*

# Examples

An example of a degenerate game:

| | 3 | 4 |
|---|---|---|
| 1 | 1, 1 | 1, 1 |
| 2 | 3, 3 | 4, 4 |

Note that there are two pure best responses to the strategy 1.

Are there fully labeled strategies in the following game?

| | 3 | 4 |
|---|---|---|
| 1 | 3, 1 | 2, 2 |
| 2 | 2, 3 | 3, 1 |

Yes, the strategy $(2/3, 1/3)$ of player 1 is labeled by $3, 4$ and the strategy $(1/2, 1/2)$ of player 2 is labeled by $1, 2$.

**Exercise:** Find all fully labeled strategies in the above example.

# Lemke-Howson (Idea)

Define a graph $H_1 = (V_1, E_1)$ where

$$V_1 = \{\sigma_1 \in \Sigma_1 \mid |L(\sigma_1)| = m\} \cup \{0^m\}$$

and $\{\sigma_1, \sigma_1'\} \in E_1$ iff $L(\sigma_1) \cap L(\sigma_1') = L(\sigma_1) \setminus \{k\}$ for some label $k$.

Note that $\sigma_1'$ is determined by $\sigma_1$ and $k$, we say that $\sigma_1'$ is obtained from $\sigma_1$ by dropping $k$.

Define a graph $H_2 = (V_2, E_2)$ where

$$V_2 = \{\sigma_2 \in \Sigma_2 \mid |L(\sigma_2)| = n\} \cup \{0^n\}$$

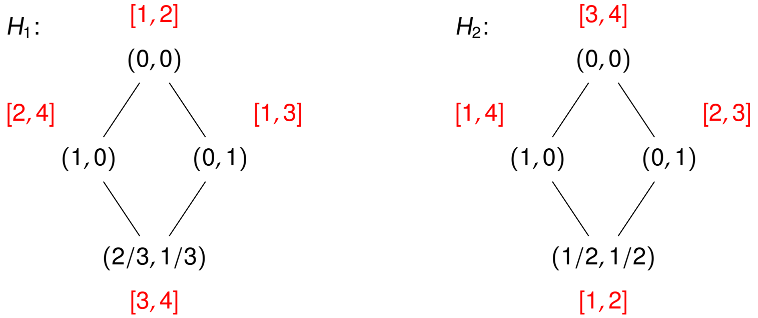
and $\{\sigma_2, \sigma_2'\} \in E_2$ iff $L(\sigma_2) \cap L(\sigma_2') = L(\sigma_2) \setminus \{\ell\}$ for some label $\ell$.

Note that $\sigma_2'$ is determined by $\sigma_2$ and $\ell$, we say that $\sigma_2'$ is obtained from $\sigma_2$ by dropping $\ell$.

Given $\sigma_i, \sigma_i' \in V_i$ and $k, \ell \in \{1, \dots, m+n\}$, we write $\sigma_i \xleftrightarrow{k,\ell} \sigma_i'$ if $L(\sigma_i) \cap L(\sigma_i') = L(\sigma_i) \setminus \{k\}$ and $L(\sigma_i) \cap L(\sigma_i') = L(\sigma_i') \setminus \{\ell\}$

# Running Example

| | 3 | 4 |
|---|---|---|
| 1 | 3, 1 | 2, 2 |
| 2 | 2, 3 | 3, 1 |




(Here, the red labels of nodes are not parts of the graphs.)

For example, $(0,0) \overset{2,3}{\longleftrightarrow} (0,1)$ and $(0,1) \overset{1,4}{\longleftrightarrow} (2/3,1/3)$ in $H_1$.

# Lemke-Howson (Idea)

The algorithm basically searches through $H_1 \times H_2 = (V_1 \times V_2, E)$ where $\left\{(\sigma_1, \sigma_2), (\sigma_1', \sigma_2')\right\} \in E$ iff either $\left\{\sigma_1, \sigma_1'\right\} \in E_1$, or $\left\{\sigma_2, \sigma_2'\right\} \in E_2$.

Given $i \in N$, we write

$$(\sigma_1, \sigma_2) \xrightarrow{k,\ell}_i \quad (\sigma_1', \sigma_2')$$

and say that *$k$ was dropped from $L(\sigma_i)$ and $\ell$ added to $L(\sigma_i)$* if

$$\sigma_i \xleftrightarrow{k,\ell} \sigma_i' \quad \text{and} \quad \sigma_{-i} = \sigma_{-i}'.$$

Observe that by Lemma 50, whenever a label $k$ is dropped from $L(\sigma_i)$, the resulting vertex of $H_1 \times H_2$ is uniquely determined.

Also, $|V| = |V_1||V_2| \leq \binom{m+n}{m}\binom{m+n}{n}$.

# Running Example

| | 3 | 4 |
|---|---|---|
| 1 | 3, 1 | 2, 2 |
| 2 | 2, 3 | 3, 1 |

The graph $H_1 \times H_2$ has 16 nodes.

Let us follow a path in $H_1 \times H_2$ starting in $((0,0),(0,0))$:

$$\begin{aligned} ((0,0),(0,0)) &\xrightarrow{2,3}_1 ((0,1),(0,0)) \\ &\xrightarrow{3,1}_2 ((0,1),(1,0)) \\ &\xrightarrow{1,4}_1 ((2/3,1/3),(1,0)) \\ &\xrightarrow{4,2}_2 ((2/3,1/3),(1/2,1/2)) \end{aligned}$$

This is one of the paths followed by Lemke-Howson:

- First, choose which label to drop from $L(\sigma_1)$ (here we drop 2 from $L(0,0)$), which adds exactly one new label (here 3)
- Then always drop the *duplicit* label, i.e. the one labeling both nodes, until no duplicit label is present (then we have a Nash equilibrium)

# Lemke-Howson (Idea)

Lemke-Howson algorithm works as follows:

- Start in $(\sigma_1, \sigma_2) = (0^m, 0^n)$.
- Pick a label $k \in \{1, \ldots, m\}$ and drop it from $L(\sigma_1)$.
  This adds a label, which then is the only element of $L(\sigma_1) \cap L(\sigma_2)$.
- loop
  - If $L(\sigma_1) \cap L(\sigma_2) = \emptyset$, then stop and return $(\sigma_1, \sigma_2)$.
  - Let $\{\ell\} = L(\sigma_1) \cap L(\sigma_2)$, drop $\ell$ from $L(\sigma_2)$.
    This adds exactly one label to $L(\sigma_2)$.
  - If $L(\sigma_1) \cap L(\sigma_2) = \emptyset$, then stop and return $(\sigma_1, \sigma_2)$.
  - Let $\{k\} = L(\sigma_1) \cap L(\sigma_2)$, drop $k$ from $L(\sigma_1)$.
    This adds exactly one label to $L(\sigma_1)$.

**Lemma 51**

*The algorithm proceeds through every vertex of $H_1 \times H_2$ at* most once.

Indeed, if $(\sigma_1, \sigma_2)$ is visited twice (with distinct predecessors), then either $\sigma_1$, or $\sigma_2$ would have (at least) two neighbors reachable by dropping the label $k \in L(\sigma_1) \cap L(\sigma_2)$, a contradiction with non-degeneracy.

Hence the algorithm stops after at most $\binom{m+n}{m}\binom{m+n}{n}$ iterations.

# Lemke-Howson Algorithm – Detailed Treatment

The previous description of the LH algorithm does not specify how to compute the graphs $H_1$ and $H_2$ and how to implement the dropping of labels.

In particular, it is not clear how to identify *fully* labeled strategies and "transitions" between them.

The complete algorithm relies on a reformulation which allows us to unify fully labeled strategies (i.e. vertices of $H_1$ and $H_2$) with vertices of certain convex polytopes.

The edges of $H_1$ and $H_2$ will correspond to edges of the polytopes.

This also gives a fully algebraic procedure for dropping labels.

# Convex Polytopes

- A *convex combination* of points $o_1, \ldots, o_i \in \mathbb{R}^k$ is a point $\lambda_1 o_1 + \cdots + \lambda_i o_i$ where $\lambda_i \geq 0$ for each $i$ and $\sum_{j=1}^{i} \lambda_j = 1$.
- A *convex polytope* determined by a set of points $o_1, \ldots, o_i$ is a set of all convex combinations of $o_1, \ldots, o_i$.
- A hyperplane $h$ is a *supporting hyperplane of a polytope $P$* if it has a non-empty intersection with $P$ and one of the closed half-spaces determined by $h$ contains $P$.
- A *face* of a polytope $P$ is an intersection of $P$ with one of its supporting hyperplanes.
- A *vertex* is a 0-dimensional face, an *edge* is a 1-dim. face.
- Two vertices are *neighbors* if they lie on the same edge (they are endpoints of the edge).

- A *polyhedron* is an intersection of finitely many closed half-spaces

  It is a set of solutions of a system of finitely many linear inequalities

- **Fact:** Each bounded polyhedron is a polytope, each polytope is a bounded polyhedron.

# Characterizing Nash Equilibria

Let us return back to Lemma 42:

$(\sigma_1, \sigma_2) = (\sigma(1), \ldots, \sigma(m+n))$ is a Nash equilibrium iff

- For all $\ell = m+1, \ldots, m+n : u_2(\sigma_1, \ell) \leq u_2(\sigma_1, \sigma_2)$ and either $\sigma(\ell) = 0$, or $u_2(\sigma_1, \ell) = u_2(\sigma_1, \sigma_2)$
- For all $k = 1, \ldots, m : u_1(k, \sigma_2) \leq u_1(\sigma_1, \sigma_2)$ and either $\sigma(k) = 0$, or $u_1(k, \sigma_2) = u_1(\sigma_1, \sigma_2)$

Now using the fact that

$$u_2(\sigma_1, \ell) = \sum_{k=1}^{m} \sigma(k) u_2(k, \ell)$$

and

$$u_1(k, \sigma_2) = \sum_{\ell=m+1}^{m+n} \sigma(\ell) u_1(k, \ell)$$

we obtain ...

# Reformulation

$(\sigma_1, \sigma_2) = (\sigma(1), \ldots, \sigma(m+n))$ is a Nash equilibrium iff

- For all $\ell = m+1, \ldots, m+n$,

$$\sum_{k=1}^{m} \sigma(k) \cdot u_2(k, \ell) \leq u_2(\sigma_1, \sigma_2) \tag{3}$$

  and either $\sigma(\ell) = 0$, or the ineq. (3) holds with equality.

- For all $k = 1, \ldots, m$,

$$\sum_{\ell=m+1}^{m+n} \sigma(\ell) \cdot u_1(k, \ell) \leq u_1(\sigma_1, \sigma_2) \tag{4}$$

  and either $\sigma(k) = 0$, or the ineq. (4) holds with equality.

Dividing (3) by $u_2(\sigma_1, \sigma_2)$ and (4) by $u_1(\sigma_1, \sigma_2)$ we get ...

# Reformulation

$(\sigma_1, \sigma_2) = (\sigma(1), \ldots, \sigma(m+n))$ is a Nash equilibrium iff

- For all $\ell = m+1, \ldots, m+n$,

$$\sum_{k=1}^{m} \frac{\sigma(k)}{u_2(\sigma_1, \sigma_2)} u_2(k, \ell) \leq 1 \tag{5}$$

  and either $\sigma(\ell) = 0$, or the ineq. (7) holds with equality.

- For all $k = 1, \ldots, m$,

$$\sum_{\ell=m+1}^{m+n} \frac{\sigma(\ell)}{u_1(\sigma_1, \sigma_2)} u_1(k, \ell) \leq 1 \tag{6}$$

  and either $\sigma(k) = 0$, or the ineq. (8) holds with equality.

Considering each $\sigma(k)/u_2(\sigma_1, \sigma_2)$ as an unknown value $x(k)$, and each $\sigma(\ell)/u_1(\sigma_1, \sigma_2)$ as an unknown value $y(\ell)$, we obtain ...

# Reformulation

... constraints in variables $x(1), \ldots, x(m)$ and $y(m+1), \ldots, y(m+n)$ :

- For all $\ell = m+1, \ldots, m+n$,

$$\sum_{k=1}^{m} x(k) \cdot u_2(k, \ell) \leq 1 \tag{7}$$

  and either $y(\ell) = 0$, or the ineq. (7) holds with equality.

- For all $k = 1, \ldots, m$,

$$\sum_{\ell=m+1}^{m+n} y(\ell) \cdot u_1(k, \ell) \leq 1 \tag{8}$$

  and either $x(k) = 0$, or the ineq. (8) holds with equality.

For all non-negative vectors $x \geq 0^m$ and $y \geq 0^n$ that satisfy the above contraints we have that $(\bar{x}, \bar{y})$ is a Nash equilibrium.
Here the strategy $\bar{x}$ is defined by $\bar{x}(k) := x(k) / \sum_{i=1}^{m} x(i)$, the strategy $\bar{y}$ is defined by $\bar{y}(\ell) := y(\ell) / \sum_{j=m+1}^{m+n} y(j)$
Given a Nash equilibrium $(\sigma_1, \sigma_2) = (\sigma(1), \ldots, \sigma(m+n))$, assigning $x(k) := \sigma(k)/u_1(\sigma_1, \sigma_2)$ for $k \in S_1$, and $y(\ell) := \sigma(\ell)/u_1(\sigma_1, \sigma_2)$ for $\ell \in S_2$ satisfies the above constraints.

# Reformulation

Let us extend the notion of expected payoff a bit.

Given $\ell = m+1, \ldots, m+n$ and $x = (x(1), \ldots, x(m)) \in [0, \infty)^m$ we define

$$u_2(x, \ell) = \sum_{k=1}^{m} x(k) \cdot u_2(k, \ell)$$

Given $k = 1, \ldots, m$ and $y = (y(m+1), \ldots, y(m+n)) \in [0, \infty)^n$ we define

$$u_1(k, y) = \sum_{\ell=m+1}^{m+n} y(\ell) \cdot u_1(k, \ell)$$

So the previous system of constraints can be rewritten succinctly:

- For all $\ell = m+1, \ldots, m+n$ we have that $u_2(x, \ell) \leq 1$ and either $y(\ell) = 0$, or $u_2(x, \ell) = 1$.
- For all $k = 1, \ldots, m$ we have that $u_1(k, y) \leq 1$, and either $x(k) = 0$, or $u_1(k, y) = 1$

# Geometric Formulation

Define

$$P := \{x \in \mathbb{R}^m \mid (\forall k \in S_1 : x(k) \geq 0) \wedge (\forall \ell \in S_2 : u_2(x, \ell) \leq 1)\}$$

$$Q := \{y \in \mathbb{R}^n \mid (\forall k \in S_1 : u_1(k, y) \leq 1) \wedge (\forall \ell \in S_2 : y(\ell) \geq 0)\}$$

$P$ and $Q$ are convex *polytopes*.

As payoffs are positive and linear in their arguments, $P$ and $Q$ are bounded polyhedra, which means that they are convex hulls of "corners", i.e., they are polytopes.

We label points of $P$ and $Q$ as follows:

- $L(x) = \{k \in S_1 \mid x(k) = 0\} \cup \{\ell \in S_2 \mid u_2(x, \ell) = 1\}$
- $L(y) = \{k \in S_1 \mid u_1(k, y) = 1\} \cup \{\ell \in S_2 \mid y(\ell) = 0\}$

**Proposition 4**

*For each point* $(x, y) \in P \times Q \setminus \{(0, 0)\}$ *such that* $L(x) \cup L(y) = \{1, \dots, m + n\}$ *we have that the corresponding strategy profile* $(\bar{x}, \bar{y})$ *is a Nash equilibrium. Each Nash equilibrium is obtained this way.*

# Geometric Formulation

**Without proof:** Non-degeneracy of $G$ implies that

- For all $x \in P$ we have $L(x) \leq m$.
- $x$ is a vertex of $P$ iff $|L(x)| = m$
  (That is, vertices of $P$ are exactly points incident on exactly $m$ faces)
- For two distinct vertices $x, x'$ we have $L(x) \neq L(x')$.
- Every vertex of $P$ is incident on exactly $m$ edges; in particular, for each $k \in L(x)$ there is a unique (neighboring) vertex $x'$ such that $L(x) \cap L(x') = L(x) \setminus \{k\}$.

Similar claims are true for $Q$ (just substitute $m$ with $n$ and $P$ with $Q$).

Define a graph $H_1 = (V_1, E_1)$ where $V_1$ is the set of all vertices $x$ of $P$ and $\{x, x'\} \in E_1$ iff $L(x) \cap L(x') = L(x) \setminus k$.

Define a graph $H_2 = (V_2, E_2)$ where $V_2$ is the set of all vertices $y$ of $Q$ and $\{y, y'\} \in E_2$ iff $L(y) \cap L(y') = L(y) \setminus k$.

The notions of dropping and adding labels from and to, resp., remain the same as before.

# Lemke-Howson (Algorithm)

Lemke-Howson algorithm works as follows:

- Start in $(x, y) := (0^m, 0^n) \in P \times Q$.
- Pick a label $k \in \{1, \ldots, m\}$ and drop it from $L(x)$.
  This adds a label, which then is the only element of $L(x) \cap L(y)$.
- loop
  - If $L(x) \cap L(y) = \emptyset$, then stop and return $(x, y)$.
  - Let $\{\ell\} = L(x) \cap L(y)$, drop $\ell$ from $L(y)$.
    This adds exactly one label to $L(\sigma_2)$.
  - If $L(x) \cap L(y) = \emptyset$, then stop and return $(x, y)$.
  - Let $\{k\} = L(x) \cap L(y)$, drop $k$ from $L(x)$.
    This adds exactly one label to $L(x)$.

**Lemma 52**

*The algorithm proceeds through every vertex of $H_1 \times H_2$ at* most once.

Hence the algorithm stops after at most $\binom{m+n}{m}\binom{m+n}{n}$ iterations.

# The Algebraic Procedure

How to effectively move between vertices of $H_1 \times H_2$ ?

That is how to compute the result of dropping a label?

We employ so called *tableau method* with an appropriate *pivoting*.

# Slack Variables Formulation

Recall our succinct characterization of Nash equilibria:

- For all $\ell = m+1, \ldots, m+n$ we have that $u_2(x,\ell) \leq 1$ and either $y(\ell) = 0$, or $u_2(x,\ell) = 1$.
- For all $k = 1, \ldots, m$ we have that $u_1(k,y) \leq 1$, and either $x(k) = 0$, or $u_1(k,y) = 1$

We turn this into a system o equations in variables $x(1), \ldots, x(m)$, $y(m+1), \ldots, y(m+n)$ and *slack variables* $r(1), \ldots, r(m)$, $z(m+1), \ldots, z(m+n)$ :

$$
\begin{array}{lll}
u_2(x,\ell) + z(\ell) = 1 & & \ell \in S_2 \\
u_1(k,y) + r(k) = 1 & & k \in S_1 \\
x(k) \geq 0 & y(\ell) \geq 0 & k \in S_1, \ell \in S_2 \\
r(k) \geq 0 & z(\ell) \geq 0 & k \in S_1, \ell \in S_2 \\
x(k) \cdot r(k) = 0 & y(\ell) \cdot z(\ell) = 0 & k \in S_1, \ell \in S_2
\end{array}
$$

Solving this is called *linear complementary problem (LCP).*

# Tableaux

The LM algorithm represents the current vertex of $H_1 \times H_2$ using a *tableau* defined as follows.

Define two sets of variables:
$\mathcal{M} := \{x(1), \ldots, x(m), z(m+1), \ldots, z(m+n)\}$
$\mathcal{N} := \{r(1), \ldots, r(m), y(m+1), \ldots, y(m+n)\}$
A *basis* is a pair of sets of variables $M \subseteq \mathcal{M}$ and $N \subseteq \mathcal{N}$ where $|M| = n$ and $|N| = m$.
Intuition: Labels correspond to variables that are *not* in the basis

A tableau $T$ for a given basis $(M, N)$:

$$P: \quad v = c_v - \sum_{v' \in \mathcal{M} \setminus M} a_{v'} \cdot v' \qquad v \in M$$

$$Q: \quad w = c_w - \sum_{w' \in \mathcal{N} \setminus N} a_{w'} \cdot w' \qquad w \in N$$

Here each $c_v, c_w \geq 0$ and $a_{v'}, a_{w'} \in \mathbb{R}$.

Note that the first part of the tableau corresponds to the polytope $P$, the second one to the polytope $Q$.

# Tableaux implementation of Lemke-Howson

A *basic solution* of a tableau $T$ is obtained by assigning zero to non-basic variables and computing the rest.

During a computation of the LM algorithm, the basic solutions will correspond to vertices of the two polytopes $P$ and $Q$.

Initial tableau:
$M = \{z(m+1), \ldots, z(m+n)\}$ and $N = \{r(1), \ldots, r(m)\}$

$$P: \quad z(\ell) = 1 - \sum_{k=1}^{m} x(k) \cdot u_2(k,\ell) \qquad \ell \in S_2$$

$$Q: \quad r(k) = 1 - \sum_{\ell=m+1}^{m+n} y(\ell) \cdot u_1(k,\ell) \qquad k \in S_1$$

Note that assigning 0 to all non-basic variables we obtain $x(k) = 0$ for $k = 1, \ldots, m$ and $y(\ell) = 0$ for $\ell = m+1, \ldots, m+n$.

So this particular tableau corresponds to $(0^m, 0^n)$.

Note that non-basic variables correspond precisely to labels of $(0^m, 0^n)$.

# Lemke-Howson – Pivoting

Given a tableau $T$ during a computation:

$$P: \quad v = c_v - \sum_{v' \in \mathcal{M} \smallsetminus M} a_{v'} \cdot v' \qquad\qquad v \in M$$

$$Q: \quad w = c_w - \sum_{w' \in \mathcal{N} \smallsetminus N} a_{w'} \cdot w' \qquad\qquad w \in N$$

Dropping a label corresponding to a variable $\bar{v} \in \mathcal{M} \smallsetminus M$ (i.e. dropping a label in $P$) is done by adding $\bar{v}$ to the basis as follows:

- Find an equation $v = c_v - \sum_{v' \in \mathcal{M} \smallsetminus M} a_{v'} \cdot v'$ , with *minimum* $c_v / a_{\bar{v}}$.
  Here $c_v \neq 0$, and we assume that if $a_{\bar{v}} = 0$, then $c_v / a_{\bar{v}} = \infty$
- $M := (M \smallsetminus \{v\}) \cup \{\bar{v}\}$
- Reorganize the equation so that $\bar{v}$ is on the left-hand side:

$$\bar{v} = \frac{c_v}{a_{\bar{v}}} - \sum_{v' \in \mathcal{M} \smallsetminus M, v' \neq v} \frac{a_{v'}}{a_{\bar{v}}} \cdot v' - \frac{v}{a_{\bar{v}}}$$

- Substitute the new expression for $v$ to all other equations.

Dropping labels in $Q$ works similarly.

# Lemke-Howson – Tableaux

The previous slide gives a procedure for computing one step of the LH algorithm.

The computation ends when:

- For each complementary pair $(x(k), r(k))$ one of the variables is in the basis and the other one is not
- For each complementary pair $(y(\ell), z(\ell))$ one of the variables is in the basis and the other one is not

# Lemke-Howson – Example

Initial tableau ($M = \{z(3), z(4)\}$, $N = \{r(1), r(2)\}$):

$$z(3) = 1 - x(1) \cdot 1 - x(2) \cdot 3 \quad (9)$$
$$z(4) = 1 - x(1) \cdot 2 - x(2) \cdot 1 \quad (10)$$

$$r(1) = 1 - y(3) \cdot 3 - y(4) \cdot 2 \quad (11)$$
$$r(2) = 1 - y(3) \cdot 2 - y(4) \cdot 3 \quad (12)$$

Drop the label 2 from $P$: The minimum ratio $1/3$ is in (9).

$$x(2) = 1/3 - (1/3) \cdot x(1) - (1/3) \cdot z(3) \quad (13)$$
$$z(4) = 2/3 - (5/3) \cdot x(1) - (1/3) \cdot z(3) \quad (14)$$

$$r(1) = 1 - y(3) \cdot 3 - y(4) \cdot 2 \quad (15)$$
$$r(2) = 1 - y(3) \cdot 2 - y(4) \cdot 3 \quad (16)$$

Here $M = \{x(2), z(4)\}$, $N = \{r(1), r(2)\}$.

Drop the label 3 from $Q$: The minimum ratio $1/3$ is in (15).

# Lemke-Howson – Example (Cont.)

$$\begin{aligned} x(2) &= 1/3 - (1/3)\cdot x(1) - (1/3)\cdot z(3) & (17) \\ z(4) &= 2/3 - (5/3)\cdot x(1) - (1/3)\cdot z(3) & (18) \end{aligned}$$

$$\begin{aligned} y(3) &= 1/3 - (2/3)\cdot y(4) - (1/3)\cdot r(1) & (19) \\ r(2) &= 1/3 - (5/3)\cdot y(4) - (1/3)\cdot r(1) & (20) \end{aligned}$$

Here $M = \{x(2), z(4)\}$, $N = \{y(3), r(2)\}$.

Drop the label 1: The minimum ratio $(2/3)/(5/3) = 2/5$ is in (18).

$$\begin{aligned} x(2) &= 1/5 - (4/15)\cdot z(3) - (1/5)\cdot z(4) & (21) \\ x(1) &= 2/5 - (1/5)\cdot z(3) - (3/5)\cdot z(4) & (22) \\ y(3) &= 1/3 - (2/3)\cdot y(4) - (1/3)\cdot r(1) & (23) \\ r(2) &= 1/3 - (5/3)\cdot y(4) - (1/3)\cdot r(1) & (24) \end{aligned}$$

Here $M = \{x(2), x(1)\}$, $N = \{y(3), r(2)\}$.

Drop the label 4: The minimum ratio $1/5$ is in (24).

# Lemke-Howson – Example (Cont.)

$$x(2) = 1/5 - (4/15) \cdot z(3) - (1/5) \cdot z(4) \quad (25)$$
$$x(1) = 2/5 - (1/5) \cdot z(3) - (3/5) \cdot z(4) \quad (26)$$

$$y(3) = 1/5 - (1/5) \cdot r(1) - (6/15) \cdot r(2) \quad (27)$$
$$y(4) = 1/5 - (1/5) \cdot r(1) - (3/5) \cdot r(2) \quad (28)$$

Here $M = \{x(2), x(1)\}$, $N = \{y(3), y(4)\}$ and thus

- $x(1) \in M$ but $r(1) \notin N$
- $x(2) \in M$ but $r(2) \notin N$
- $y(3) \in N$ but $z(3) \notin M$
- $y(4) \in N$ but $z(4) \notin M$

So the algorithm stops.

Assign $z(3) = z(4) = r(1) = r(2) = 0$ and obtain the following Nash equilibrium:

$$x(1) = 2/5, \quad x(2) = 1/5, \quad y(3) = 1/5, \quad y(4) = 1/5$$