IA008: Computational Logic

2. First-Order Logic

Achim Blumensath blumens@fi.muni.cz

Faculty of Informatics, Masaryk University, Brno

Basic Concepts

First-Order Logic

Syntax

- Variables $x, y, z, \ldots$
- Terms $x, f(t_0, \ldots, t_n)$
- Relations $R(t_0, \ldots, t_n)$ and equality $t_0 = t_1$
- Operators $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- Quantifiers $\exists x \varphi$, $\forall x \varphi$

Examples

$$\varphi := \forall x \exists y [f(y) = x],$$
$$\psi := \forall x \forall y \forall z [x \leq y \wedge y \leq z \rightarrow x \leq z].$$

Prenex Normal Form

$$Q_0 x_0 \cdots Q_n x_n \psi(\bar{x}), \qquad \psi \text{ quantifier-free}$$

Skolemisation

Eliminate **existential quantifiers:**

replace $\forall \bar{x} \exists y \varphi(\bar{x}, y)$ by $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$$\forall x \exists y \exists z [y > x \wedge z < x]$$

Skolemisation

Eliminate **existential quantifiers:**

replace $\forall\bar{x}\exists y\varphi(\bar{x}, y)$ by $\forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$$\forall x\exists y\exists z[y > x \land z < x] \qquad \forall x[f(x) > x \land g(x) < x]$$

Skolemisation

Eliminate **existential quantifiers:**

replace $\forall \bar{x} \exists y \varphi(\bar{x}, y)$ by $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$\forall x \exists y \exists z [y > x \wedge z < x]$ $\qquad$ $\forall x [f(x) > x \wedge g(x) < x]$

$\exists x \forall y [y + 1 \neq x]$

Skolemisation

Eliminate **existential quantifiers:**

replace $\forall \bar{x} \exists y \varphi(\bar{x}, y)$ by $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$\forall x \exists y \exists z [y > x \land z < x]$	$\forall x [f(x) > x \land g(x) < x]$
$\exists x \forall y [y + 1 \neq x]$	$\forall y [y + 1 \neq c]$

Skolemisation

Eliminate **existential quantifiers:**

replace $\forall\bar{x}\exists y\varphi(\bar{x}, y)$ by $\forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$\forall x\exists y\exists z[y > x \wedge z < x]$	$\forall x[f(x) > x \wedge g(x) < x]$
$\exists x\forall y[y + 1 \neq x]$	$\forall y[y + 1 \neq c]$
$\exists x\forall y\exists z\forall u\exists v[R(x, y, z, u, v)]$	

Skolemisation

Eliminate **existential quantifiers:**

replace $\forall\bar{x}\exists y\varphi(\bar{x}, y)$ by $\forall\bar{x}\varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$\forall x\exists y\exists z[y > x \wedge z < x]$	$\forall x[f(x) > x \wedge g(x) < x]$
$\exists x\forall y[y + 1 \neq x]$	$\forall y[y + 1 \neq c]$
$\exists x\forall y\exists z\forall u\exists v[R(x, y, z, u, v)]$	$\forall y\forall u[R(c, y, f(y), u, g(y, z))]$

Skolemisation

Eliminate **existential quantifiers:**

replace $\forall \bar{x} \exists y \varphi(\bar{x}, y)$ by $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$$\forall x \exists y \exists z [y > x \wedge z < x] \qquad \forall x [f(x) > x \wedge g(x) < x]$$
$$\exists x \forall y [y + 1 \neq x] \qquad \forall y [y + 1 \neq c]$$
$$\exists x \forall y \exists z \forall u \exists v [R(x, y, z, u, v)] \qquad \forall y \forall u [R(c, y, f(y), u, g(y, z))]$$

Theorem

Let φ_s be a Skolemisation of φ. Then φ_s is satisfiable iff φ is satisfiable.

Theorem of Herbrand

Theorem of Herbrand

A formula $\exists \bar{x} \varphi(\bar{x})$ is valid if, and only if, there are terms $\bar{t}_0, \ldots, \bar{t}_n$ such that the disjunction $\bigvee_{i \leq n} \varphi(\bar{t}_i)$ is valid.

Corollary

A formula $\forall \bar{x} \varphi(\bar{x})$ is unsatisfiable if, and only if, there are terms $\bar{t}_0, \ldots, \bar{t}_n$ such that the conjunction $\bigwedge_{i \leq n} \varphi(\bar{t}_i)$ is unsatisfiable.

Substitution

Definition

A **substitution** σ is a function that replaces in a formula every free variable by a term (and renames bound variables if necessary).
Instead of $\sigma(\varphi)$ we also write $\varphi[x \mapsto s, y \mapsto t]$ if $\sigma(x) = s$ and $\sigma(y) = t$.

Examples

$$(x = f(y))[x \mapsto g(x),\ y \mapsto c] \quad = \quad g(x) = f(c)$$

$$\exists z(x = z + z)[x \mapsto z] \quad = \quad \exists u(z = u + u)$$

Unification

Definition

A **unifier** of two terms $s(\bar{x})$ and $t(\bar{x})$ is a pair of substitution σ, τ such that $\sigma(s) = \tau(t)$.
A unifier σ, τ is **most general** if every other unifier σ', τ' can be written as $\sigma' = \rho \circ \sigma$ and $\tau' = \upsilon \circ \tau$, for some ρ, υ.

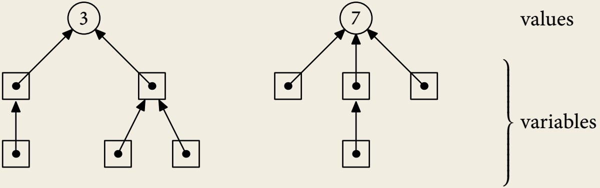
Examples

$s = f(x, g(x))$	$t = f(c, y)$	$x \mapsto c,\ y \mapsto g(c)$
$s = f(x, g(x))$	$t = f(x, y)$	$x \mapsto x,\ y \mapsto g(x)$
		$x \mapsto g(x),\ y \mapsto g(g(x))$
$s = f(x)$	$t = g(x)$	unification not possible

Unification Algorithm

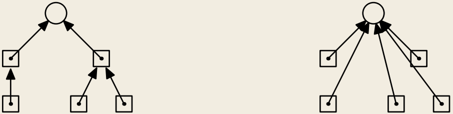
unify(s, t)
    if s is a variable x **then**
        set x to t
    else if t is a variable x **then**
        set x to s
    else $s = f(\bar{u})$ and $t = g(\bar{v})$
        if $f = g$ **then**
            forall i unify(u_i, v_i)
        else
            fail

Union-Find-Algorithm



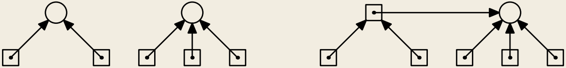
find : *variable* $\rightarrow$ *value*

- follows pointers to the root and creates shortcuts



union : (*variable* $\times$ *variable*) $\rightarrow$ *unit*

- links roots by a pointer



Resolution

Clauses

Definitions

- **literal** $R(\bar{t})$ or $\neg R(\bar{t})$
- **clause** set of literals $\{P(\bar{s}), R(\bar{t}), \neg S(\bar{u})\}$

Clauses

Definitions

- **literal** $R(\bar{t})$ or $\neg R(\bar{t})$
- **clause** set of literals $\{P(\bar{s}), R(\bar{t}), \neg S(\bar{u})\}$

Example

CNF $\varphi := \forall x \forall y \big[R(x,y) \lor \neg R(x, f(x))\big] \land \forall y \big[\neg R(f(y), y) \lor P(y)\big]$

(no existential quantifiers)

clauses $\big\{R(x,y)\ \neg R(x, f(x))\big\}, \ \big\{\neg R(f(y), y), P(y)\big\}$

Resolution

Resolution Step

Consider two clauses

$$C = \{P(\bar{s}), R_0(\bar{t}_0), \dots, R_m(\bar{t}_m)\}$$
$$C' = \{\neg P(\bar{s}'), S_0(\bar{u}_0), \dots, S_n(\bar{u}_n)\}$$

where $\bar{s}$ and $\bar{s}'$ have no common variables, and let σ be the most general unifier of $\bar{s}$ and $\bar{s}'$. The **resolvent** of C and C' is the clause

$$\{R_0(\sigma(\bar{t}_0)), \dots, R_m(\sigma(\bar{t}_m)), S_0(\sigma(\bar{u}_0)), \dots, S_n(\sigma(\bar{u}_n))\}.$$

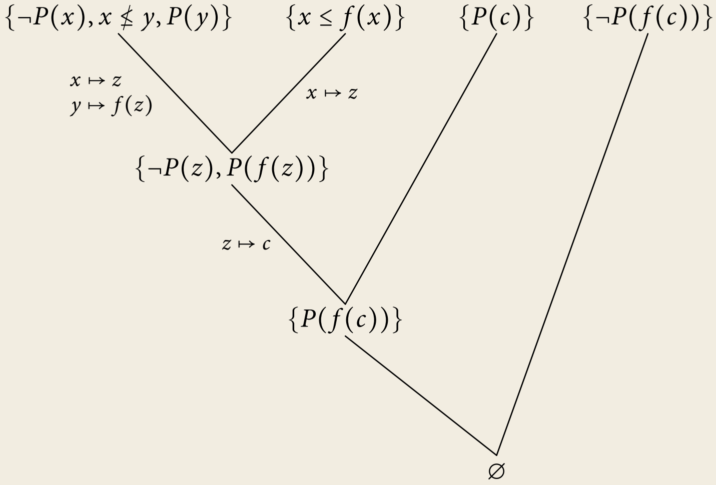
Lemma

Let C be the resolvent of two clauses in Φ. Then

$$\Phi \vDash \Phi \cup \{C\}.$$

Example

$$\varphi = \forall x \forall y[P(x) \land x \leq y \rightarrow P(y)] \land \forall x[x \leq f(x)] \land Pc \land \neg P(f(c))$$



The Resolution Method

Theorem

The resolution method for first-order logic (without equality) is **sound** and **complete.**

Theorem

Satisfiability for first-order logic is **undecidable.**

Proof

Turing machine $\mathcal{M} = \langle Q, \Sigma, \Delta, a_0, F_+, F_- \rangle$

Q	set of states
Σ	tape alphabet
Δ	set of transitions $\langle p, a, b, m, q \rangle \in Q \times \Sigma \times \Sigma \times \{-1, 0, 1\} \times Q$
q_0	initial state
F_+	accepting states
F_-	rejecting states

Proof

Turing machine $\mathcal{M} = \langle Q, \Sigma, \Delta, a_0, F_+, F_- \rangle$

Q	set of states
Σ	tape alphabet
Δ	set of transitions $\langle p, a, b, m, q \rangle \in Q \times \Sigma \times \Sigma \times \{-1, 0, 1\} \times Q$
q_0	initial state
F_+	accepting states
F_-	rejecting states

Encoding in FO

$S_q(t)$	**state** q at time t
$h(t)$	**head** in field $h(t)$ at time t
$W_a(t, k)$	**letter** a in field k at time t
s	**successor** function $s(n) = n + 1$

$$\varphi_w := \mathrm{ADM} \wedge \mathrm{INIT} \wedge \mathrm{TRANS} \wedge \mathrm{ACC}$$

Proof

$S_q(t)$	**state** q at time t
$h(t)$	**head** in field $h(t)$ at time t
$W_a(t,k)$	**letter** a in field k at time t
s	**successor** function $s(n) = n + 1$

Admissibility formula

$$\text{ADM} := \forall t \bigwedge_{p \neq q} \neg [S_p(t) \wedge S_q(t)] \qquad \text{unique state}$$

$$\wedge\ \forall t \forall k \bigwedge_{a \neq b} \neg [W_a(t,k) \wedge W_b(t,k)] \qquad \text{unique letter}$$

Proof

$S_q(t)$	**state** q at time t
$h(t)$	**head** in field $h(t)$ at time t
$W_a(t,k)$	**letter** a in field k at time t
s	**successor** function $s(n) = n + 1$

Initialisation formula for input: $a_0 \dots a_{n-1}$

$$\begin{aligned} \mathrm{INIT} := {} & S_{q_0}(0) && \text{initial state} \\ & \wedge h(0) = 0 && \text{initial head position} \\ & \wedge \bigwedge_{k<n} W_{a_k}(0, \underline{k}) \wedge \forall k[k \geq \underline{n} \to W_{\square}(0,k)] && \text{initial tape content} \end{aligned}$$

(here $\underline{k} := s(s(\cdots s(0)))$)

Acceptance formula

$$\mathrm{ACC} := \exists t \bigvee_{q \in F_+} S_q(t) \qquad \text{accepting state}$$

Proof

$S_q(t)$	**state** q at time t
$h(t)$	**head** in field $h(t)$ at time t
$W_a(t,k)$	**letter** a in field k at time t
s	**successor** function $s(n) = n + 1$

Transition formula

$$\begin{aligned}\mathrm{TRANS} := \forall t \bigvee_{\langle p,a,b,m,q\rangle\in\Delta} \Big[& S_p(t) \wedge W_a(t,h(t)) \wedge S_q(s(t)) \wedge \\ & h(s(t)) = h(t) + m \wedge W_b(s(t),h(t))\Big] \\ \wedge\ \forall t \forall k \bigwedge_{a\in\Sigma} \Big[& k \neq h(t) \rightarrow \big[W_a(t,k) \leftrightarrow W_a(s(t),k)\big]\Big]\end{aligned}$$

where

$$h(s(t)) = h(t) + m := \begin{cases} h(s(t)) = s(h(t)) & \text{if } m = 1, \\ h(s(t)) = h(t) & \text{if } m = 0, \\ s(h(s(t))) = h(t) & \text{if } m = -1. \end{cases}$$

Linear Resolution and Horn Formulae

Horn formulae

A **Horn formulae** is a formula in CNF where each clause contains at most one positive literal.

Theorem

A set of Horn clauses is unsatisfiable if, and only if, one can use linear resolution to derive the empty clause from it.

SLD Resolution

Linear resolution where the clauses are **sequences** instead of sets and we always resolve the **leftmost literal** of the current clause.