IA008: Computational Logic

7. Many-Valued Logics

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Basic Concepts

Many-valued logics: Motivation

To some sentences we cannot – or do not want to – assign a truth value since

- they make **presuppositions** that are not fulfilled
  John regrets beating his wife.
  John does not regret beating his wife.
- they refer to **non-existing** objects
  The king of Paris has a pet lion.
- they are too **vague**
  The next supermarket is far away.
- we have **insufficient information**
  The favourite colour of Odysseus was blue.
- we cannot determine their truth
  The Goldbach conjecture holds.

This leads to logics with **truth values** other than 'true' and 'false'.

3-valued logic

truth values 'false' $\bot$, 'uncertain' u, and 'true' $\top$.

A	$\neg A$
$\bot$	$\top$
u	u
$\top$	$\bot$

$\wedge$	$\bot$	u	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
u	$\bot$	u	u
$\top$	$\bot$	u	$\top$

$\vee$	$\bot$	u	$\top$
$\bot$	$\bot$	u	$\top$
u	u	u	$\top$
$\top$	$\top$	$\top$	$\top$

Kleene K3

$\to$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	u	$\boldsymbol{u}$	$\top$
$\top$	$\bot$	u	$\top$

Łukasiewicz L3

$\to$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	u	$\boldsymbol{\top}$	$\top$
$\top$	$\bot$	u	$\top$

Example

A	B	$A \wedge (A \to B)$	$A \wedge (A \to B) \to B$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	u	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
u	$\bot$	u	u
u	u	u	$u/\top$
u	$\top$	u	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\top$	u	u	$u/\top$
$\top$	$\top$	$\top$	$\top$

Fuzzy logic

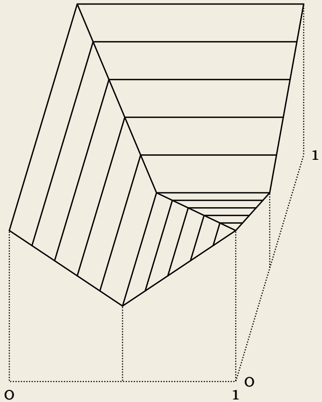
Truth values: $v \in [0, 1]$ measuring **how true** a statement is.
0 means 'false' and 1 means 'true'.

Several possible semantics:

$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$
$1 - A$	$A \cdot B$	$1 - (1 - A)(1 - B)$	$1 - A(1 - B)$
$1 - A$	$\min(A, B)$	$\max(A, B)$	$\max(1 - A, B)$
$1 - A$	$\max(A + B - 1, 0)$	$\min(A + B, 1)$	$\min(1 - A + B, 1)$

Example

$$A \wedge (A \to B) \to B = \max(1 - \min(A, \max(1 - A, B)), B)$$



Tableaux for L3

statements: $t \leq \varphi$, $\varphi \leq t$, $t \nleq \varphi$, or $\varphi \nleq t$, for $t \in \{\bot, u, \top\}$

Construction

A **tableau** for a formula φ is constructed as follows:

- start with $1 \nleq \varphi$
- choose a branch of the tree
- choose a statement σ on the branch
- choose a rule with head σ
- add it at the bottom of the branch
- repeat until every branch contains one of the following **contradictions**

$\bot \nleq \varphi$	$s \leq t$ with $s \nleq t$	$s \leq \varphi$ and $t \nleq \varphi$ with $s \leq t$
$\varphi \nleq \top$	$s \nleq t$ with $s \leq t$	$\varphi \leq s$ and $\varphi \nleq t$ with $t \leq s$

where $s, t \in \{\bot, u, \top\}$ and φ is a formula

Tableaux Rules

$$\dfrac{t \not\leq \varphi}{\varphi \leq s} \qquad \dfrac{t \leq \varphi}{\varphi \not\leq s} \qquad \dfrac{\varphi \not\leq t}{s \leq \varphi} \qquad \dfrac{\varphi \leq t}{s \not\leq \varphi}$$

$s < t$ maximal

$$\dfrac{t \leq \neg\varphi}{\varphi \leq \neg t} \qquad \dfrac{t \not\leq \neg\varphi}{\varphi \not\leq \neg t}$$

$$\dfrac{t \leq \varphi \wedge \psi}{\begin{array}{c} t \leq \varphi \\ t \leq \psi \end{array}} \qquad \dfrac{t \not\leq \varphi \wedge \psi}{t \not\leq \varphi \mid t \not\leq \psi} \qquad \dfrac{\varphi \vee \psi \leq t}{\begin{array}{c} \varphi \leq t \\ \psi \leq t \end{array}} \qquad \dfrac{\varphi \vee \psi \not\leq t}{\varphi \not\leq t \mid \psi \not\leq t}$$

$t \neq \bot$

$$\dfrac{\top \not\leq \varphi \to \psi}{\begin{array}{c} u \leq \varphi \\ u \not\leq \psi \end{array} \mid \begin{array}{c} \top \leq \varphi \\ \top \not\leq \psi \end{array}} \qquad \dfrac{u \not\leq \varphi \to \psi}{\begin{array}{c} u \leq \varphi \\ u \not\leq \psi \end{array}} \qquad \dfrac{\top \leq \varphi \to \psi}{\top \not\leq \varphi \mid \top \leq \psi} \qquad \dfrac{\top \leq \varphi \to \psi}{u \not\leq \varphi \mid u \leq \psi}$$

$$\dfrac{u \leq \varphi \to \psi}{u \not\leq \varphi \mid u \leq \psi}$$

Example

$$u \nleq [(u \to A) \wedge (\top \to (A \to B))] \to B$$

$$u \leq (u \to A) \wedge (\top \to (A \to B))$$

$$u \nleq B$$

$$u \leq u \to A$$

$$u \leq \top \to (A \to B)$$

$$u \nleq u \qquad u \leq A$$

$$u \nleq \top \qquad u \leq A \to B$$

$$u \nleq A \qquad u \leq B$$

Intuitionistic Logic

The constructivists view

- We are not interested in **truth** but in **provability.**
- To prove the **existence** of an object is to give a concrete example.

$$\text{prove } \exists x \varphi(x) \quad \Leftrightarrow \quad \text{find } t \text{ with } \varphi(t)$$

- To prove a **disjunction** is to prove one of the choices.

$$\text{prove } \varphi \vee \psi \quad \Leftrightarrow \quad \text{prove } \varphi \text{ or prove } \psi$$

Goal

A variant of first-order logic that captures these ideas.

Boolean algebras

In **classical logic** the **truth values** form a **boolean algebra** with operations

$$\wedge,\ \vee,\ \neg,\ \top,\ \bot$$

Properties of negation:

$$x \wedge \neg x = \bot \qquad x \vee \neg x = \top$$

Heyting algebras

In **intuitionistic logic** the **truth values** form instead a **Heyting algebra** with operations

$$\wedge,\ \vee,\ \rightarrow,\ \top,\ \bot$$

Properties of implication:

$$z \leq x \rightarrow y \quad \text{iff} \quad z \wedge x \leq y$$

(that is $x \rightarrow y$ is the largest element satisfying $(x \rightarrow y) \wedge x \leq y$)

$$x \wedge (x \rightarrow y) = x \wedge y \qquad x \rightarrow x = \top$$

$$y \wedge (x \rightarrow x) = y \qquad x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)\,.$$

Heyting algebras

In **intuitionistic logic** the **truth values** form instead a **Heyting algebra** with operations

$$\wedge,\ \vee,\ \rightarrow,\ \top,\ \bot$$

Properties of implication:

$$z \le x \rightarrow y \quad \text{iff} \quad z \wedge x \le y$$

(that is $x \rightarrow y$ is the largest element satisfying $(x \rightarrow y) \wedge x \le y$)

$$x \wedge (x \rightarrow y) = x \wedge y \qquad x \rightarrow x = \top$$

$$y \wedge (x \rightarrow x) = y \qquad x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)\,.$$

Negation $\neg x := x \rightarrow \bot$

satisfies $x \wedge \neg x = \bot$, but not $x \vee \neg x = \top$

Forcing Frames

Definition

Transition system $\mathfrak{S} = \langle S, \leq, (P_i)_{i \in I}, s_0 \rangle$ with one edge relation $\leq$ that forms a **partial order**:

- **reflexive** $s \leq s$
- **transitive** $s \leq t \leq u$ implies $s \leq u$
- **anti-symmetric** $s \leq t$ and $t \leq s$ implies $s = t$

The forcing relation

$\mathfrak{S}$ forcing frame, $s \in S$ state, φ formula

$$
\begin{array}{lll}
s \Vdash P_i & :\text{iff} & t \in P_i \quad \text{for all } t \geq s \\
s \Vdash \varphi \wedge \psi & :\text{iff} & s \Vdash \varphi \text{ and } s \Vdash \psi \\
s \Vdash \varphi \vee \psi & :\text{iff} & s \Vdash \varphi \text{ or } s \Vdash \psi \\
s \Vdash \neg\varphi & :\text{iff} & t \nVdash \varphi \quad \text{for all } t \geq s \\
s \Vdash \varphi \to \psi & :\text{iff} & t \Vdash \varphi \text{ implies } t \Vdash \psi \quad \text{for all } t \geq s
\end{array}
$$

The **truth value** of φ in $\mathfrak{S}$ is

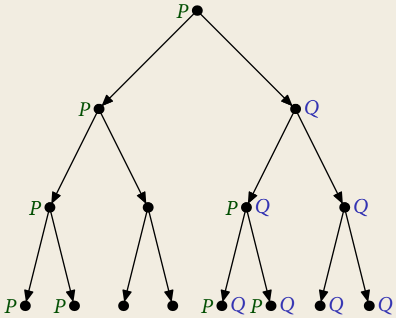
$$[\![\varphi]\!]_{\mathfrak{S}} := \{\, s \in S \mid s \Vdash \varphi \,\},$$

which is **upwards-closed** with respect to $\leq$.

Intuition

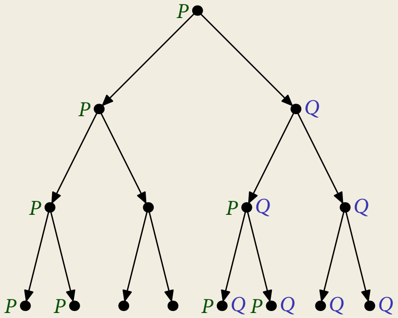
Intuitionistic logic speaks about the **limit behaviour** of φ for large s.

Example



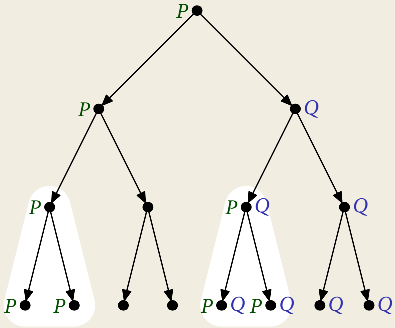
Example

$\varphi := P$



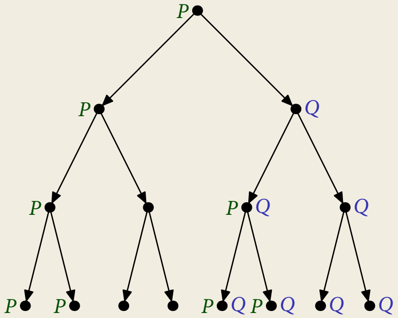
Example

$\varphi := P$



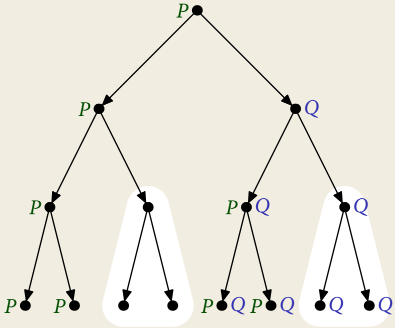
Example

$\varphi := \neg P$



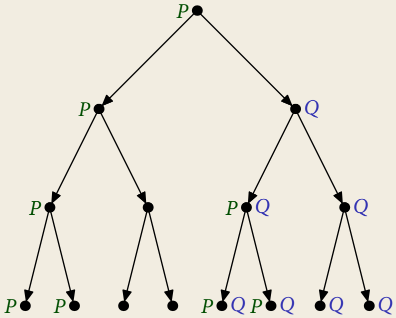
Example

$\varphi := \neg P$



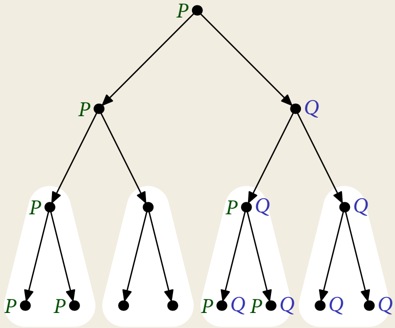
Example

$\varphi := P \vee \neg P$



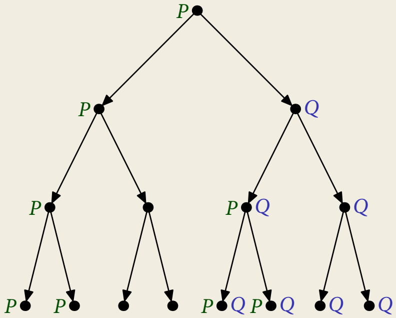
Example

$\varphi := P \vee \neg P$



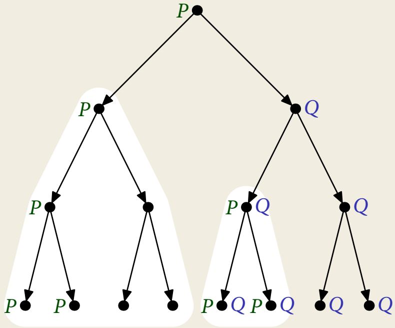
Example

$\varphi := Q \to P$



Example

$\varphi := Q \rightarrow P$



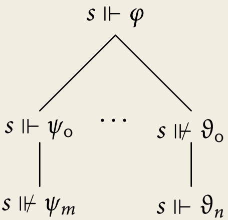
Tableaux for Intuitionistic Logic

Statements

$s \Vdash \varphi \qquad s \nVdash \varphi \qquad s \leq t$

s, t state labels, φ a formula

Rules



$$s \Vdash \varphi$$
$$t \Vdash \varphi$$
φ atomic, $t \geq s$ arbitrary

$$s \nVdash \varphi$$
φ atomic

$$s \Vdash \neg\varphi$$
$$t \nVdash \varphi$$
$t \geq s$ arbitrary

$$s \nVdash \neg\varphi$$
$$s \leq t$$
$$t \Vdash \varphi$$
t new

$$s \Vdash \varphi \wedge \psi$$
$$s \Vdash \varphi$$
$$s \Vdash \psi$$

$$s \nVdash \varphi \wedge \psi$$
$$s \nVdash \varphi \qquad\qquad s \nVdash \psi$$

$$s \Vdash \varphi \to \psi$$
$$t \nVdash \varphi \qquad\qquad t \Vdash \psi$$
$t \geq s$ arbitrary

$$s \nVdash \varphi \to \psi$$
$$s \leq t$$
$$t \Vdash \varphi$$
$$t \nVdash \psi$$
t new

$$s \Vdash \varphi \vee \psi$$
$$s \Vdash \varphi \qquad\qquad s \Vdash \psi$$

$$s \nVdash \varphi \vee \psi$$
$$s \nVdash \varphi$$
$$s \nVdash \psi$$

$$s \Vdash \exists x\varphi$$
$$s \Vdash \varphi(c)$$
c new

$$s \nVdash \exists x\varphi$$
$$s \nVdash \varphi(c)$$
c arbitrary

$$s \Vdash \forall x\varphi$$
$$t \Vdash \varphi(c)$$
c, t arbitrary with $s \leq t$

$$s \nVdash \forall x\varphi$$
$$s \leq t$$
$$t \nVdash \varphi(c)$$
c, t new

('c arbitrary' means either new or appearing somewhere on the same branch.)

$$s \nVdash A \to (B \to A)$$

$$s \leq t$$

$$t \Vdash A$$

$$t \nVdash B \to A$$

$$t \leq u$$

$$u \Vdash B$$

$$u \nVdash A$$

$$u \Vdash B$$

$$s \nVdash \exists x(\varphi \vee \psi) \to (\exists x\varphi \vee \exists x\psi)$$

$$s \leq t$$

$$t \Vdash \exists x(\varphi \vee \psi)$$

$$t \nVdash \exists x\varphi \vee \exists x\psi$$

$$t \Vdash \varphi(c) \vee \psi(c)u$$

$$t \nVdash \exists x\varphi$$

$$t \nVdash \exists x\psi$$

$$t \nVdash \varphi(c)$$

$$t \nVdash \psi(c)$$

$$t \Vdash \varphi(c) \qquad\qquad t \Vdash \psi(c)$$

$$s \nVdash \forall x(\varphi \wedge \psi) \to (\forall x\varphi \wedge \forall x\psi)$$

$$s \leq t$$

$$t \Vdash \forall x(\varphi \wedge \psi)$$

$$t \nVdash \forall x\varphi \wedge \forall x\psi$$

$$\begin{array}{cc}
t \nVdash \forall x\varphi & t \nVdash \forall x\psi \\
t \leq u & t \leq u' \\
u \nVdash \varphi(c) & u' \nVdash \psi(d) \\
u \Vdash \varphi(c) \wedge \psi(c) & u' \Vdash \varphi(d) \wedge \psi(d) \\
u \Vdash \varphi(c) & u' \Vdash \varphi(d) \\
u \Vdash \psi(c) & u' \Vdash \psi(d)
\end{array}$$