Recurrent Neural Networks - LSTM

RNN

 \blacktriangleright Input: $\vec{x} = (x_1, ..., x_M)$

$$
\triangleright \text{ Hidden:} \\ \vec{h} = (h_1, \dots, h_H)
$$

$$
\begin{array}{ll}\n\blacktriangleright & \text{Output:} \\
\vec{y} = (y_1, \ldots, y_N)\n\end{array}
$$

RNN example

Activation function:

$$
\sigma(\xi) = \begin{cases} 1 & \xi \ge 0 \\ 0 & \xi < 0 \end{cases}
$$

y 1 0 1
\n**h** (0,0)
$$
\begin{pmatrix} 1,1 \\ 0,0 \end{pmatrix}
$$
 (1,0) (0,1) ...
\n**x** $\begin{pmatrix} 0,0 \\ 0,0 \end{pmatrix}$ (1,0) (1,1)

$$
-1 \cdot 1 + O \cdot 1 + 1 + (-1) \cdot 0 = 0
$$

0 \cdot 1 + (-2) \cdot 1 + (-1) \cdot 1 + 1 \cdot 0 = -2

RNN example

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$$
\sigma(\xi) = \begin{cases} 1 & \xi \ge 0 \\ 0 & \xi < 0 \end{cases}
$$

y
\n**h**
$$
\vec{h}_0 = (0,0)
$$
 $\vec{h}_1 = (1,1)$ $\vec{h}_2 = (1,0)$ $\vec{h}_3 = (0,1)$...
\n**x**
\n**x**
\n**y**
\n**y**
\n**h**
\n $\vec{h}_0 = (0,0)$ $\vec{h}_1 = (1,1)$ $\vec{h}_2 = (1,0)$ $\vec{h}_3 = (0,1)$...
\n $\vec{x}_1 = (0,0)$ $\vec{x}_2 = (1,0)$ $\vec{x}_3 = (1,1)$

RNN example

y
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$$
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\n**x**
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\n**y**
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\n**y**
\n**y**
\n**x**
\n**y**
\n**y**
\n**x**
\n**y**
\n**y**

- \blacktriangleright M inputs: $\vec{x} = (x_1, \ldots, x_M)$
- \blacktriangleright H hidden neurons: $\vec{h} = (h_1, \ldots, h_H)$
- \triangleright N output neurons: $\vec{y} = (y_1, \ldots, y_N)$
- Weights:
	- \blacktriangleright $U_{kk'}$ from input $x_{k'}$ to hidden h_k
	- \blacktriangleright $W_{kk'}$ from hidden $h_{k'}$ to hidden h_k
	- \blacktriangleright $V_{kk'}$ from hidden $h_{k'}$ to output y_k

RNN – formally

► Input sequence:
$$
\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T
$$

 $\vec{x}_t = (x_{t1}, \ldots, x_{tM})$

RNN – formally

► Input sequence:
$$
\mathbf{x} = \vec{x}_1, ..., \vec{x}_T
$$

$$
\vec{x}_t = (x_{t1}, \ldots, x_{tM})
$$

► Hidden sequence:
$$
\mathbf{h} = \vec{h}_0, \vec{h}_1, \ldots, \vec{h}_T
$$

$$
\vec{h}_t = (h_{t1}, \ldots, h_{tH})
$$

We have $\vec{h}_0 = (0, \ldots, 0)$ and

$$
\vec{h}_{tk} = \sigma \left(\sum_{k'=1}^{M} U_{kk'} x_{tk'} + \sum_{k'=1}^{H} W_{kk'} h_{(t-1)k'} \right)
$$

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D Output sequence: $\mathbf{y} = \vec{y}_1, \dots, \vec{y}_T$

$$
\vec{y}_t = (y_{t1}, \dots, y_{tN})
$$
\nwhere $y_{tk} = \sigma \left(\sum_{k'=1}^H V_{kk'} h_{tk'} \right)$.

RNN – in matrix form

Input sequence: $\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$

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RNN – in matrix form

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$$

and

$$
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$$

D Output sequence: $\mathbf{v} = \vec{v}_1, \dots, \vec{v}_T$ where

 $y_t = \sigma(Vh_t)$

- \blacktriangleright \vec{h}_t is the memory of the network, captures what happened in all previous steps (with decaying quality).
- \blacktriangleright RNN **shares weights** U, V, W along the sequence. Note the similarity to convolutional networks where the weights were shared spatially over images, here they are shared temporally over sequences.
- **RNN can deal with sequences of variable length.** Compare with MLP which accepts only fixed-dimension vectors on input.

Training set

$$
\mathcal{T} = \left\{(\bm{x}_1, \bm{d}_1), \ldots, (\bm{x}_p, \bm{y}_p)\right\}
$$

here

P each $\mathbf{x}_{\ell} = \vec{x}_{\ell 1}, \ldots, \vec{x}_{\ell T_{\ell}}$ is an input sequence,

• each $\mathbf{d}_{\ell} = \vec{d}_{\ell 1}, \ldots, \vec{d}_{\ell T_{\ell}}$ is an expected output sequence. Here each $\vec{x}_{rt} = (x_{rt1}, \dots, x_{rtM})$ is an input vector and each $\vec{d}_{\ell t} = (d_{\ell t1}, \dots, d_{\ell tN})$ is an expected output vector.

In what follows I will consider a training set with a **single element** (x, d) . I.e. drop the index ℓ and have

$$
\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T \text{ where } \vec{x}_t = (x_{t1}, \dots, x_{tM})
$$

\n
$$
\mathbf{d} = \vec{d}_1, \dots, \vec{d}_T \text{ where } \vec{d}_t = (d_{t1}, \dots, d_{tN})
$$

The squared error of (**x**, **d**) is defined by

$$
E_{(\mathbf{x},\mathbf{d})} = \sum_{t=1}^{T} \sum_{k=1}^{N} \frac{1}{2} (y_{tk} - d_{tk})^2
$$

Recall that we have a sequence of network outputs ${\bf v} = \vec{v}_1, \ldots, \vec{v}_T$ and thus ${\bf v}_{tk}$ is the k-th component of \vec{v}_t

Consider a single training example (**x**, **d**).

The algorithm computes a sequence of weight matrices as follows:

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- In the step $\ell + 1$ (here $\ell = 0, 1, 2, \ldots$) compute "new" weights $\bm{\mathsf{U}}^{(\ell+1)}$, $\bm{\mathsf{V}}^{(\ell+1)}$, $\bm{\mathsf{W}}^{(\ell+1)}$ from the "old" weights $U^{(\ell)}$, $V^{(\ell)}$, $W^{(\ell)}$ as follows:

$$
U_{kk'}^{(\ell+1)} = U_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x}, \mathbf{d})}}{\delta U_{kk'}}
$$

$$
V_{kk'}^{(\ell+1)} = V_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x}, \mathbf{d})}}{\delta V_{kk'}}
$$

$$
W_{kk'}^{(\ell+1)} = W_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x}, \mathbf{d})}}{\delta W_{kk'}}
$$

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$$

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$$

The above is THE learning algorithm that modifies weights!

Backpropagation

Computes the derivatives of E**, no weights are modified!**

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$$
\frac{\delta E(\mathbf{x}, \mathbf{d})}{\delta U_{kk'}} = \sum_{t=1}^{T} \frac{\delta E(\mathbf{x}, \mathbf{d})}{\delta h_{tk}} \cdot \sigma' \cdot x_{tk'}
$$
\n
$$
k' = 1, ..., M
$$
\n
$$
\frac{\delta E(\mathbf{x}, \mathbf{d})}{\delta V_{kk'}} = \sum_{t=1}^{T} \frac{\delta E(\mathbf{x}, \mathbf{d})}{\delta y_{tk}} \cdot \sigma' \cdot h_{tk'}
$$
\n
$$
k' = 1, ..., H
$$
\n
$$
\frac{\delta E(\mathbf{x}, \mathbf{d})}{\delta W_{kk'}} = \sum_{t=1}^{T} \frac{\delta E(\mathbf{x}, \mathbf{d})}{\delta h_{tk}} \cdot \sigma' \cdot h_{(t-1)k'}
$$
\n
$$
k' = 1, ..., H
$$

$$
\frac{\delta E_{(\textbf{x},\textbf{d})}}{\delta h_{tk}} = \sum_{k'=1}^{N} \frac{\delta E_{(\textbf{x},\textbf{d})}}{\delta y_{tk'}} \cdot \sigma' \cdot V_{k'k} + \sum_{k'=1}^{H} \frac{\delta E_{(\textbf{x},\textbf{d})}}{\delta h_{(t+1)k'}} \cdot \sigma' \cdot W_{k'k}
$$

► Unless $\sum_{k'=1}^{H} \sigma' \cdot W_{k'k} \approx 1$, the gradient either vanishes, or explodes.

- \triangleright For a large T (long-term dependency), the gradient "deeper" in the past tends to be too small (large).
- \blacktriangleright A solution: LSTM

LSTM

$$
(2, 4) \circ (3, 1) = (2.3, 4.1) \circ (6, 4)
$$

$$
\vec{h}_t = \vec{o}_t \circ \sigma_h(\vec{C}_t)
$$
\noutput
\n
$$
\vec{C}_t = \vec{f}_t \circ \vec{C}_{t-1} + \vec{i}_t \circ \tilde{C}_t
$$
\nmemory
\n
$$
\tilde{C}_t = \sigma_h(W_C \cdot \vec{h}_{t-1} + U_C \cdot \vec{x}_t)
$$
\nnew memory contents
\n
$$
\vec{o}_t = \sigma_g(W_0 \cdot \vec{h}_{t-1} + U_0 \cdot \vec{x}_t)
$$
\noutput gate
\n
$$
\vec{f}_t = \sigma_g(W_t \cdot \vec{h}_{t-1} + U_t \cdot \vec{x}_t)
$$
\nforget gate
\n
$$
\vec{f}_t = \sigma_g(W_i \cdot \vec{h}_{t-1} + U_i \cdot \vec{x}_t)
$$
\ninput gate
\n
$$
\vec{f}_t = \sigma_g(W_i \cdot \vec{h}_{t-1} + U_i \cdot \vec{x}_t)
$$
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\n
$$
\vec{f}_t = \sigma_g(W_i \cdot \vec{h}_{t-1} + U_i \cdot \vec{x}_t)
$$
\ninput gate
\n
$$
\vec{f}_t = \vec{f}_t \cdot \vec{f}_t
$$
\nis the computer-wise product of vectors

- \triangleright σ_h hyperbolic tangents (applied component-wise)
- \triangleright σ_g logistic sigmoid (aplied component-wise)

RNN vs LSTM

$$
\vec{h}_t = \vec{\sigma}_t \circ \sigma_h(\vec{C}_t)
$$
\n
$$
\Rightarrow \vec{C}_t = \vec{f}_t \circ \vec{C}_{t-1} + \vec{i}_t \circ \tilde{C}_t
$$
\n
$$
\tilde{C}_t = \sigma_h(W_C \cdot \vec{h}_{t-1} + U_C \cdot \vec{x}_t)
$$
\n
$$
\vec{\sigma}_t = \sigma_g(W_0 \cdot \vec{h}_{t-1} + U_0 \cdot \vec{x}_t)
$$
\n
$$
\vec{f}_t = \sigma_g(W_f \cdot \vec{h}_{t-1} + U_f \cdot \vec{x}_t)
$$
\n
$$
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$$

LSTM

$$
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$$
\n
$$
\vec{C}_t = \vec{f}_t \circ \vec{C}_{t-1} + \vec{i}_t \circ \tilde{C}_t
$$
\n
$$
\tilde{C}_t = \sigma_h(W_C \cdot \vec{h}_{t-1} + U_C \cdot \vec{x}_t)
$$
\n
$$
\vec{\sigma}_t = \sigma_g(W_0 \cdot \vec{h}_{t-1} + U_0 \cdot \vec{x}_t)
$$
\n
$$
\Rightarrow \vec{f}_t = \sigma_g(W_f \cdot \vec{h}_{t-1} + U_f \cdot \vec{x}_t)
$$

$$
\vec{l}_t = \sigma_g(W_i \cdot \vec{h}_{t-1} + U_i \cdot \vec{x}_t)
$$

LSTM

$$
\begin{array}{c}\n\vec{f}_{\mu} \circ \vec{C}_{\mu^{-1}} \left(\vec{S}_{1} \vec{C}_{1} \right) & \left(\vec{S}_{2} \vec{C}_{1} \right) \\
\vec{C}_{\mu} \circ \left(\vec{C}_{1} \vec{C}_{1} \right) & \vec{C}_{\mu} \\
\vec{C}_{\mu} \circ \left(\vec{C}_{1} \vec{C}_{1} \vec{C}_{1} \right) & \vec{C}_{2} \\
\vec{C}_{\mu} \circ \left(\vec{C}_{1} \vec{C}_{1} \vec{C}_{1} \vec{C}_{1} \right) & \vec{C}_{2} \\
\vec{C}_{\mu} \circ \left(\vec{C}_{1} \vec{C}_{1} \vec{C}_{1} \vec{C}_{1} \right) & \vec{C}_{1} \\
\vec{C}_{2} \circ \left(\vec{C}_{2} \vec{C}_{1} \vec{C}_{1} \vec{C}_{1} \right) & \vec{C}_{2} \\
\vec{C}_{3} \circ \left(\vec{C}_{2} \vec{C}_{1} \vec{C}_{1} \vec{C}_{1} \right) & \vec{C}_{3} \\
\vec{C}_{4} \circ \vec{C}_{1} & \vec{C}_{1} \\
\vec{C}_{2} \circ \vec{C}_{2} & \vec{C}_{2} \\
\vec{C}_{2} \circ \vec{C}_{2} & \vec{C}_{2} \\
\vec{C}_{3} \circ \vec{C}_{2} & \vec{C}_{3} \\
\vec{C}_{4} \circ \vec{C}_{1} & \vec{C}_{1} \\
\vec{C}_{1} \circ \vec{C}_{1} & \vec{C}_{1} \\
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\vec{C}_{2} \circ \vec{C}_{2} & \vec{C}_{2} \\
\vec{C}_{3} \circ \vec{C}_{2} & \vec{C}_{3} \\
\vec
$$

$$
\vec{h}_t = \vec{\sigma}_t \circ \sigma_h(\vec{C}_t)
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\Rightarrow \vec{C}_t = \vec{f}_t \circ \vec{C}_{t-1} + \vec{i}_t \circ \tilde{C}_t
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$$
\n
$$
\Rightarrow \vec{i}_t = \sigma_g(W_i \cdot \vec{h}_{t-1} + U_i \cdot \vec{x}_t)
$$

LSTM

$$
\frac{\partial}{\partial t} = \left(1, 2, 3\right) \quad \left\{\n\begin{array}{c}\n\frac{1}{\partial t} \\
\frac{1}{\partial t} \\
\frac
$$

Source: https://colah.github.io/posts/2015-08-Understanding-LSTMs/

 $\vec{h}_t = \sigma_g(W_i\cdot \vec{h}_{t-1} + U_i\cdot \vec{x_t})$

- \triangleright LSTM (almost) solves the vanishing gradient problem w.r.t. the "internal" state of the network.
- \blacktriangleright Learns to control its own memory (via forget gate).
- \blacktriangleright Revolution in machine translation and text processing.

Convolutions & LSTM in action – cancer research

The problem: Predict 5-year survival probability from an image of a small region of tumour tissue (1 mm diameter).

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Data:

- \triangleright Training set: 420 patients of Helsinki University Centre Hospital, diagnosed with colorectal cancer, underwent primary surgery.
- \blacktriangleright Test set: 182 patients
- \blacktriangleright Follow-up time and outcome known for each patient.

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Human expert comparison:

- \blacktriangleright Histological grade assessed at the time of diagnosis.
- \triangleright Visual Risk Score: Three pathologists classified to high/low-risk categories (by majority vote).

Source: D. Bychkov et al. Deep learning based tissue analysis predicts outcome in colorectal cancer. Scientific Reports, Nature, 2018. **233**

Data & workflow

- Input images: 3500 px \times 3500 px
	- \triangleright Cut into tiles: 224 px \times 224 px \Rightarrow 256 tiles
- \blacktriangleright Each tile pased to a convolutional network (CNN)
	- \triangleright Ouptut of CNN: 4096 dimensional vector.
- ▶ A "string" of 256 vectors (each of the dimension 4096) pased into a LSTM.
- \blacktriangleright LSTM outputs the probability of 5-year survival.

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The authors also tried to substitute the LSTM on top of CNN with

- \blacktriangleright logistic regression
- \blacktriangleright naive Bayes
- \blacktriangleright support vector machines

CNN architecture – VGG-16

(Pre)trained on ImageNet (cats, dogs, chairs, etc.)

\blacktriangleright LSTM has three layers (264, 128, 64 cells)

LSTM – training

- \triangleright L1 regularization (0.005) at each hidden layer of LSTM i.e. 0.005 times the sum of absolute values of weights added to the error
- \blacktriangleright L2 regularization (0.005) at each hidden layer of LSTM i.e. 0.005 times the sum of squared values of weights added to the error
- \triangleright Dropout 5% at the input and the last hidden layers of LSTM
- ▶ Datasets:
	- \blacktriangleright Training: 220 samples,
	- \blacktriangleright Validation 60 samples,
	- \blacktriangleright Test 140 samples.

Source: D. Bychkov et al. Deep learning based tissue analysis predicts outcome in colorectal cancer. Scientific Reports, Nature, 2018.