Recurrent Neural Networks - LSTM



• Input: $\vec{x} = (x_1, \dots, x_M)$

• Hidden:
$$\vec{h} = (h_1, \dots, h_H)$$

• Output:
$$\vec{y} = (y_1, \dots, y_N)$$

RNN example



Activation function:

$$\sigma(\xi) = \begin{cases} 1 & \xi \ge 0 \\ 0 & \xi < 0 \end{cases}$$

y101h
$$(0,0)$$
 $(1,1)$ $(1,0)$ $(0,1)$...x $(0,0)$ $(1,0)$ $(1,1)$

$$-1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + (-1) \cdot 0 = 0$$

0 \cdot 1 + (-2) \cdot 1 + (-1) \cdot 1 + 1 \cdot 0 = -2

RNN example



Activation function:

$$\sigma(\xi) = \begin{cases} 1 & \xi \ge 0 \\ 0 & \xi < 0 \end{cases}$$

RNN example



- *M* inputs: $\vec{x} = (x_1, \dots, x_M)$
- *H* hidden neurons: $\vec{h} = (h_1, \dots, h_H)$
- N output neurons: $\vec{y} = (y_1, \dots, y_N)$
- Weights:
 - $U_{kk'}$ from input $x_{k'}$ to hidden h_k
 - $W_{kk'}$ from hidden $h_{k'}$ to hidden h_k
 - $V_{kk'}$ from hidden $h_{k'}$ to output y_k



RNN – formally

• Input sequence:
$$\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$$

 $\vec{x}_t = (x_{t1}, \ldots, x_{tM})$

RNN – formally

• Input sequence:
$$\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$$

$$\vec{x}_t = (x_{t1}, \ldots, x_{tM})$$

• Hidden sequence:
$$\mathbf{h} = \vec{h}_0, \vec{h}_1, \dots, \vec{h}_T$$

$$\vec{h}_t = (h_{t1}, \ldots, h_{tH})$$

We have $\vec{h}_0 = (0, \dots, 0)$ and

$$\vec{h}_{tk} = \sigma \left(\sum_{k'=1}^{M} U_{kk'} x_{tk'} + \sum_{k'=1}^{H} W_{kk'} h_{(t-1)k'} \right)$$

RNN – formally

• Input sequence:
$$\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$$

$$\vec{x}_t = (x_{t1}, \ldots, x_{tM})$$

• Hidden sequence:
$$\mathbf{h} = \vec{h}_0, \vec{h}_1, \dots, \vec{h}_T$$

$$\vec{h}_t = (h_{t1}, \ldots, h_{tH})$$

We have $ec{h_0} = (0, \dots, 0)$ and

$$\vec{h}_{tk} = \sigma \left(\sum_{k'=1}^{M} U_{kk'} x_{tk'} + \sum_{k'=1}^{H} W_{kk'} h_{(t-1)k'} \right)$$

• Output sequence: $\mathbf{y} = \vec{y}_1, \dots, \vec{y}_T$

$$ec{\mathbf{y}_t} = (\mathbf{y}_{t1}, \dots, \mathbf{y}_{tN})$$

where $\mathbf{y}_{tk} = \sigma \left(\sum_{k'=1}^{H} \mathbf{V}_{kk'} h_{tk'} \right)$.

RNN – in matrix form

• Input sequence: $\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$

RNN – in matrix form

- Input sequence: $\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$
- Hidden sequence: $\mathbf{h} = \vec{h}_0, \vec{h}_1, \dots, \vec{h}_T$ where

$$\vec{h}_0=(0,\ldots,0)$$

and

$$\vec{h}_t = \sigma(U\vec{x}_t + W\vec{h}_{t-1})$$

RNN – in matrix form

- Input sequence: $\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$
- Hidden sequence: $\mathbf{h} = \vec{h}_0, \vec{h}_1, \dots, \vec{h}_T$ where

$$\vec{h}_0=(0,\ldots,0)$$

and

$$\vec{h}_t = \sigma(U\vec{x}_t + W\vec{h}_{t-1})$$

• Output sequence: $\mathbf{y} = \vec{y}_1, \dots, \vec{y}_T$ where

$$\mathbf{y}_t = \sigma(\mathbf{V}\mathbf{h}_t)$$

- \vec{h}_t is the memory of the network, captures what happened in all previous steps (with decaying quality).
- RNN shares weights U, V, W along the sequence. Note the similarity to convolutional networks where the weights were shared spatially over images, here they are shared temporally over sequences.
- RNN can deal with sequences of variable length. Compare with MLP which accepts only fixed-dimension vectors on input.

Training set

$$\mathcal{T} = \left\{ (\mathbf{x}_1, \mathbf{d}_1), \dots, (\mathbf{x}_p, \mathbf{y}_p) \right\}$$

here

• each $\mathbf{x}_{\ell} = \vec{x}_{\ell 1}, \dots, \vec{x}_{\ell T_{\ell}}$ is an input sequence,

• each $\mathbf{d}_{\ell} = \vec{d}_{\ell 1}, \dots, \vec{d}_{\ell T_{\ell}}$ is an expected output sequence. Here each $\vec{x}_{\ell t} = (x_{\ell t 1}, \dots, x_{\ell t M})$ is an input vector and each $\vec{d}_{\ell t} = (d_{\ell t 1}, \dots, d_{\ell t N})$ is an expected output vector. In what follows I will consider a training set with a **single** element (\mathbf{x}, \mathbf{d}) . I.e. drop the index ℓ and have

•
$$\mathbf{x} = \vec{x}_1, \dots, \vec{x}_T$$
 where $\vec{x}_t = (x_{t1}, \dots, x_{tM})$
• $\mathbf{d} = \vec{d}_1, \dots, \vec{d}_T$ where $\vec{d}_t = (d_{t1}, \dots, d_{tN})$

The squared error of (\mathbf{x}, \mathbf{d}) is defined by

$$E_{(\mathbf{x},\mathbf{d})} = \sum_{t=1}^{T} \sum_{k=1}^{N} \frac{1}{2} (y_{tk} - d_{tk})^2$$

Recall that we have a sequence of network outputs $\mathbf{y} = \vec{y}_1, \dots, \vec{y}_T$ and thus y_{tk} is the *k*-th component of \vec{y}_t

Consider a single training example (**x**, **d**).

The algorithm computes a sequence of weight matrices as follows:

Consider a single training example (\mathbf{x}, \mathbf{d}) .

The algorithm computes a sequence of weight matrices as follows:

Initialize all weights randomly close to 0.

Consider a single training example (\mathbf{x}, \mathbf{d}) .

The algorithm computes a sequence of weight matrices as follows:

- Initialize all weights randomly close to 0.
- In the step ℓ + 1 (here ℓ = 0, 1, 2, ...) compute "new" weights U^(ℓ+1), V^(ℓ+1), W^(ℓ+1) from the "old" weights U^(ℓ), V^(ℓ), W^(ℓ) as follows:

$$U_{kk'}^{(\ell+1)} = U_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta U_{kk'}}$$
$$V_{kk'}^{(\ell+1)} = V_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta V_{kk'}}$$
$$W_{kk'}^{(\ell+1)} = W_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta W_{kk'}}$$

Consider a single training example (\mathbf{x}, \mathbf{d}) .

The algorithm computes a sequence of weight matrices as follows:

- Initialize all weights randomly close to 0.
- In the step ℓ + 1 (here ℓ = 0, 1, 2, ...) compute "new" weights U^(ℓ+1), V^(ℓ+1), W^(ℓ+1) from the "old" weights U^(ℓ), V^(ℓ), W^(ℓ) as follows:

$$U_{kk'}^{(\ell+1)} = U_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta U_{kk'}}$$
$$V_{kk'}^{(\ell+1)} = V_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta V_{kk'}}$$
$$W_{kk'}^{(\ell+1)} = W_{kk'}^{(\ell)} - \varepsilon(\ell) \cdot \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta W_{kk'}}$$

The above is THE learning algorithm that modifies weights!

Backpropagation

Computes the derivatives of *E*, no weights are modified!

Backpropagation

Computes the derivatives of *E*, no weights are modified!

$$\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta U_{kk'}} = \sum_{t=1}^{T} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta h_{tk}} \cdot \sigma' \cdot \mathbf{x}_{tk'} \qquad \qquad k' = 1, \dots, M$$
$$\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta V_{kk'}} = \sum_{t=1}^{T} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta y_{tk}} \cdot \sigma' \cdot h_{tk'} \qquad \qquad k' = 1, \dots, H$$
$$\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta W_{kk'}} = \sum_{t=1}^{T} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta h_{tk}} \cdot \sigma' \cdot h_{(t-1)k'} \qquad \qquad k' = 1, \dots, H$$

Backpropagation $M_{L} = \delta(Vh_{V})$ JN-1 $\mathcal{J}_{L+1} = \sigma(V_{L+1})$ Computes the derivatives of *E*, no weights are modified! $\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta U_{\mathbf{k}\mathbf{k}'}} = \sum_{i=1}^{T} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta h_{t\mathbf{k}}} \cdot \sigma' \cdot \mathbf{x}_{t\mathbf{k}'}$ $k'=1,\ldots,M$ $\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta V_{kk'}} = \sum_{t=1}^{T} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta y_{tk}} \cdot \sigma' \cdot h_{tk'}$ $\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta W_{kk'}} = \sum_{t=1}^{T} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta h_{tk}} \cdot \sigma' \cdot h_{(t-1)k'}$ V k' = 1, ..., HXr $k'=1,\ldots,H$ Backpropagation: $\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta \mathbf{y}_{tk}} = \mathbf{y}_{tk} - \mathbf{d}_{tk}$ (assuming squared error) XL+1 $\frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta h_{tk}} = \sum_{i=1}^{N} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta y_{tk'}} \cdot \sigma' \cdot V_{k'k} + \sum_{i=1}^{H} \frac{\delta E_{(\mathbf{x},\mathbf{d})}}{\delta h_{(t+1)k'}} \cdot \sigma' \cdot W_{k'k}$ XN

$$\frac{\delta \boldsymbol{E}_{(\mathbf{x},\mathbf{d})}}{\delta h_{tk}} = \sum_{k'=1}^{N} \frac{\delta \boldsymbol{E}_{(\mathbf{x},\mathbf{d})}}{\delta y_{tk'}} \cdot \sigma' \cdot \boldsymbol{V}_{k'k} + \sum_{k'=1}^{H} \frac{\delta \boldsymbol{E}_{(\mathbf{x},\mathbf{d})}}{\delta h_{(t+1)k'}} \cdot \sigma' \cdot \boldsymbol{W}_{k'k}$$

► Unless $\sum_{k'=1}^{H} \sigma' \cdot W_{k'k} \approx 1$, the gradient either vanishes, or explodes.

- For a large T (long-term dependency), the gradient "deeper" in the past tends to be too small (large).
- A solution: LSTM

LSTM

$$(2,3)o(3,1) = (2\cdot3,3\cdot1)(6,5)$$

$$\vec{h}_{t} = \vec{o}_{t} \circ \sigma_{h}(\vec{C}_{t}) \qquad \text{output}$$

$$\vec{C}_{t} = \vec{f}_{t} \circ \vec{C}_{t-1} + \vec{i}_{t} \circ \tilde{C}_{t} \qquad \text{memory}$$

$$\tilde{C}_{t} = \sigma_{h}(W_{C} \cdot \vec{h}_{t-1} + U_{C} \cdot \vec{x}_{t}) \qquad \text{new memory contents}$$

$$\vec{o}_{t} = \sigma_{g}(W_{o} \cdot \vec{h}_{t-1} + U_{o} \cdot \vec{x}_{t}) \qquad \text{output gate}$$

$$\vec{f}_{t} = \sigma_{g}(W_{f} \cdot \vec{h}_{t-1} + U_{f} \cdot \vec{x}_{t}) \qquad \text{forget gate}$$

$$\vec{i}_{t} = \sigma_{g}(W_{i} \cdot \vec{h}_{t-1} + U_{i} \cdot \vec{x}_{t}) \qquad \text{input gate}$$

$$\circ \text{ is the component-wise product of vectors}$$

$$\cdot \text{ is the matrix-vector product}$$

$$\sigma_{h} \text{ hyperbolic tangents (applied component-wise)}$$

σ_g logistic sigmoid (aplied component-wise)

RNN vs LSTM





$$\vec{h}_{t} = \vec{o}_{t} \circ \sigma_{h}(\vec{C}_{t})$$

$$\Rightarrow \vec{C}_{t} = \vec{f}_{t} \circ \vec{C}_{t-1} + \vec{i}_{t} \circ \tilde{C}_{t}$$

$$\tilde{C}_{t} = \sigma_{h}(W_{C} \cdot \vec{h}_{t-1} + U_{C} \cdot \vec{x}_{t})$$

$$\vec{o}_{t} = \sigma_{g}(W_{o} \cdot \vec{h}_{t-1} + U_{o} \cdot \vec{x}_{t})$$

$$\vec{f}_{t} = \sigma_{g}(W_{f} \cdot \vec{h}_{t-1} + U_{f} \cdot \vec{x}_{t})$$

$$\vec{i}_{t} = \sigma_{g}(W_{i} \cdot \vec{h}_{t-1} + U_{i} \cdot \vec{x}_{t})$$

LSTM



$$\vec{h}_{t} = \vec{o}_{t} \circ \sigma_{h}(\vec{C}_{t})$$
$$\vec{C}_{t} = \vec{f}_{t} \circ \vec{C}_{t-1} + \vec{i}_{t} \circ \tilde{C}_{t}$$
$$\tilde{C}_{t} = \sigma_{h}(W_{C} \cdot \vec{h}_{t-1} + U_{C} \cdot \vec{x}_{t})$$
$$\vec{o}_{t} = \sigma_{g}(W_{o} \cdot \vec{h}_{t-1} + U_{o} \cdot \vec{x}_{t})$$

$$\vec{o}_t = \sigma_g(W_o \cdot \vec{h}_{t-1} + U_o \cdot \vec{x}_t)$$

$$\Rightarrow \vec{f}_t = \sigma_g(W_f \cdot \vec{h}_{t-1} + U_f \cdot \vec{x}_t)$$

$$\vec{i}_t = \sigma_g(W_i \cdot \vec{h}_{t-1} + U_i \cdot \vec{x}_t)$$

LSTM

$$\begin{cases} \mathcal{L}^{\circ} \mathcal{C}_{t} = (5, 4) \\ \mathcal{C}_{t} = (2, 4, 6) \\ \mathcal{C}_{t} = (2, 6) \\$$



$$\vec{h}_{t} = \vec{o}_{t} \circ \sigma_{h}(\vec{C}_{t})$$

$$\Rightarrow \vec{C}_{t} = \vec{f}_{t} \circ \vec{C}_{t-1} + \vec{i}_{t} \circ \tilde{C}_{t}$$

$$\Rightarrow \tilde{C}_{t} = \sigma_{h}(W_{C} \cdot \vec{h}_{t-1} + U_{C} \cdot \vec{x}_{t})$$

$$\vec{o}_{t} = \sigma_{g}(W_{o} \cdot \vec{h}_{t-1} + U_{o} \cdot \vec{x}_{t})$$

$$\Rightarrow \vec{f}_{t} = \sigma_{g}(W_{f} \cdot \vec{h}_{t-1} + U_{f} \cdot \vec{x}_{t})$$

$$\Rightarrow \vec{i}_{t} = \sigma_{g}(W_{i} \cdot \vec{h}_{t-1} + U_{i} \cdot \vec{x}_{t})$$

LSTM

$$\begin{array}{c}
\tilde{C}_{\mathcal{N}} = (1, 2, 3) \\
\tilde{\sigma}_{\mathcal{L}} = (0, \frac{1}{2}, 1) \\
\overset{h_{t}}{\longrightarrow} \\
\overset{h_{t-1}}{\longrightarrow} \\
\overset{h_{t-1}}{\longrightarrow} \\
\overset{h_{t-1}}{\longrightarrow} \\
\overset{h_{t-1}}{\longrightarrow} \\
\overset{h_{t}}{\longrightarrow} \\
\overset{h_{t}}{\longrightarrow}$$

- LSTM (almost) solves the vanishing gradient problem w.r.t. the "internal" state of the network.
- Learns to control its own memory (via forget gate).
- Revolution in machine translation and text processing.

Convolutions & LSTM in action – cancer research



The problem: Predict 5-year survival probability from an image of a small region of tumour tissue (1 mm diameter).

The problem: Predict 5-year survival probability from an image of a small region of tumour tissue (1 mm diameter).

Input: Digitized haematoxylin-eosin-stained tumour tissue microarray samples. **Output:** Estimated survival probability.



The problem: Predict 5-year survival probability from an image of a small region of tumour tissue (1 mm diameter).

Input: Digitized haematoxylin-eosin-stained tumour tissue microarray samples. **Output:** Estimated survival probability.



Data:

- Training set: 420 patients of Helsinki University Centre Hospital, diagnosed with colorectal cancer, underwent primary surgery.
- Test set: 182 patients
- Follow-up time and outcome known for each patient.

The problem: Predict 5-year survival probability from an image of a small region of tumour tissue (1 mm diameter).

Input: Digitized haematoxylin-eosin-stained tumour tissue microarray samples. **Output:** Estimated survival probability.



Data:

- Training set: 420 patients of Helsinki University Centre Hospital, diagnosed with colorectal cancer, underwent primary surgery.
- Test set: 182 patients
- Follow-up time and outcome known for each patient.

Human expert comparison:

- Histological grade assessed at the time of diagnosis.
- Visual Risk Score: Three pathologists classified to high/low-risk categories (by majority vote).

Source: D. Bychkov et al. Deep learning based tissue analysis predicts outcome in colorectal cancer. Scientific Reports, Nature, 2018.





Data & workflow

- Input images: 3500 px × 3500 px
 - Cut into tiles: 224 px \times 224 px \Rightarrow 256 tiles
- Each tile pased to a convolutional network (CNN)

Ouptut of CNN: 4096 dimensional vector.

- A "string" of 256 vectors (each of the dimension 4096) pased into a LSTM.
- LSTM outputs the probability of 5-year survival.

Data & workflow

- Input images: 3500 px × 3500 px
 - Cut into tiles: 224 px \times 224 px \Rightarrow 256 tiles
- Each tile pased to a convolutional network (CNN)
 - Ouptut of CNN: 4096 dimensional vector.
- A "string" of 256 vectors (each of the dimension 4096) pased into a LSTM.
- LSTM outputs the probability of 5-year survival.

The authors also tried to substitute the LSTM on top of CNN with

- logistic regression
- naive Bayes
- support vector machines

CNN architecture – VGG-16



(Pre)trained on ImageNet (cats, dogs, chairs, etc.)

LSTM has three layers (264, 128, 64 cells)



LSTM – training

- L1 regularization (0.005) at each hidden layer of LSTM i.e. 0.005 times the sum of absolute values of weights added to the error
- L2 regularization (0.005) at each hidden layer of LSTM
 i.e. 0.005 times the sum of squared values of weights added to the error
- Dropout 5% at the input and the last hidden layers of LSTM
- Datasets:
 - Training: 220 samples,
 - Validation 60 samples,
 - Test 140 samples.



Source: D. Bychkov et al. Deep learning based tissue analysis predicts outcome in colorectal cancer. Scientific Reports, Nature, 2018.