Integral and Discrete Transforms in Image Processing

Fourier Transform & Spherical Harmonics

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Outline

Let

$$\varphi_\omega(x) = e^{2\pi i x\omega} = \cos 2\pi x\omega + i \sin 2\pi x\omega$$

be a orthonormal basis, where $x \in \mathbb{R}$ defines the index and $\omega \in \mathbb{R}$ defines the frequency (speed of oscilation). We can perform a projection of any function f onto the basis $\varphi_\omega(x)$ as follows:

$$f \cdot \varphi_\omega = \int_{-\infty}^{\infty} f(x)\overline{\varphi_\omega(x)}dx = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\omega}dx$$

Given 1D integrable function f and a basis $(\varphi_\omega, \omega \in \mathbb{R})$, let us define:

- forward 1D continuous Fourier transform

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\omega} dx$$

- inverse 1D continuous Fourier transform

$$f(x) \equiv \int_{-\infty}^{\infty} F(\omega) e^{2\pi i x\omega} d\omega$$

Fourier Transform Properties

Oddness and evenness

1. Each function $f(x)$ is sum of its odd and even part:

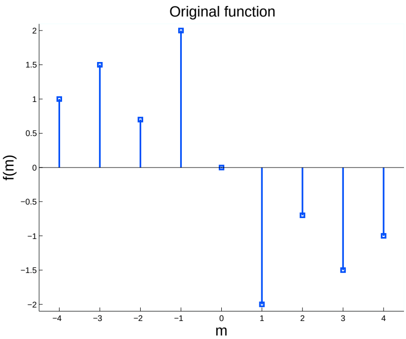
$$\begin{aligned} E(x) &= \frac{1}{2}[f(x) + f(-x)] \\ O(x) &= \frac{1}{2}[f(x) - f(-x)] \\ f(x) &= E(x) + O(x) \end{aligned}$$

2. $\sin(x)$ is an odd function
   $\cos(x)$ is an even function
3. Any FT basis function is composed of sine and cosine waves

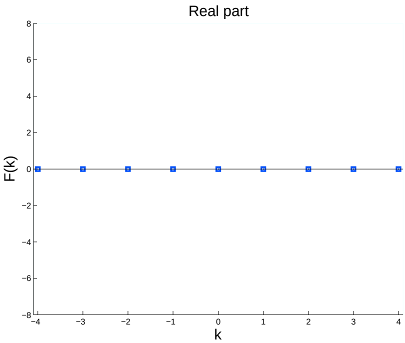
Corollary:

- FT of even function misses imaginary part (sine waves)
- FT of odd function misses real part (cosine waves)

Fourier Transform Properties
Oddness

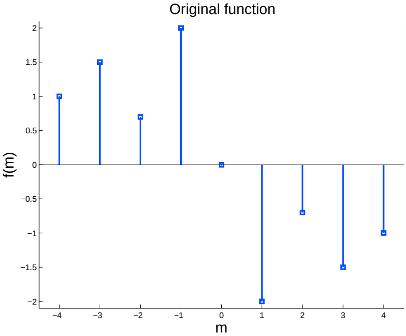


original function (odd)

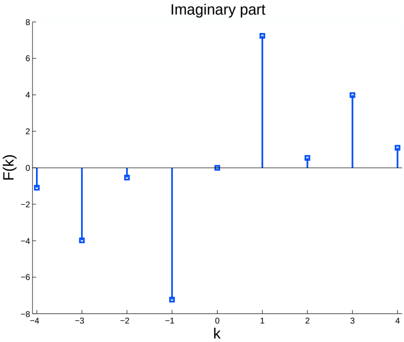


its transform pair

Fourier Transform Properties
Oddness



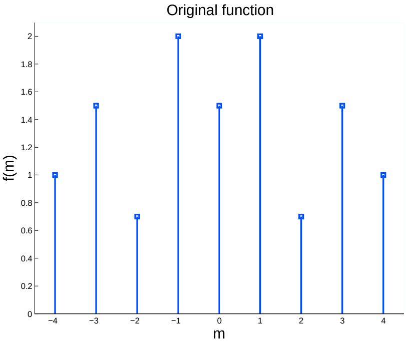
original function (odd)



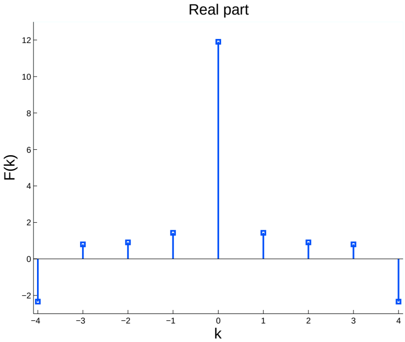
its transform pair

Fourier Transform Properties

Evenness



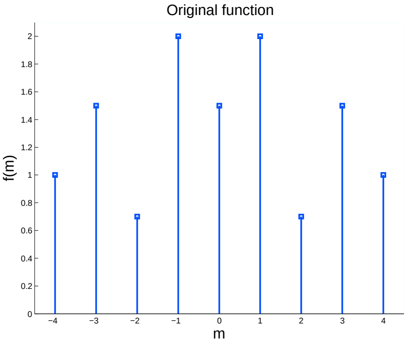
original function (even)



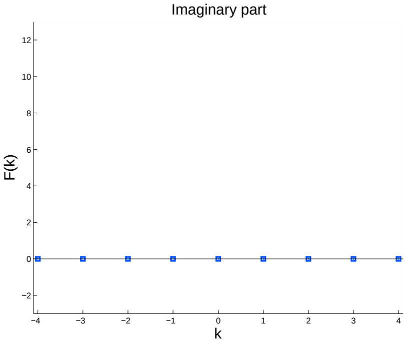
its transform pair

Fourier Transform Properties

Evenness



original function (even)



its transform pair

Fourier Transform Propertis

Scaling

Statement:

$$f(ax) \supset \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

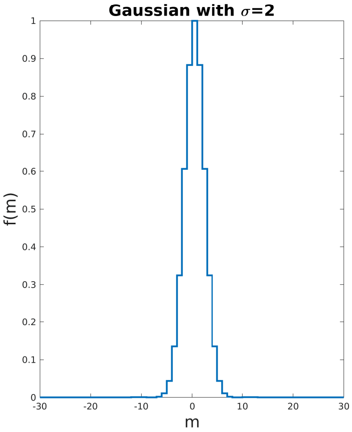
Proof:

$$\begin{aligned}
\mathcal{F}\left[f(ax)\right](\omega) &= \int_{-\infty}^{\infty} f(ax) e^{-2\pi i x \omega} dx \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} f(ax) e^{-2\pi i (ax)(\omega/a)} d(ax) \\
&= \frac{1}{|a|} F\left(\frac{\omega}{a}\right)
\end{aligned}$$

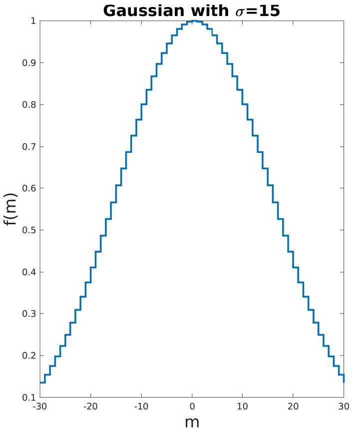
□

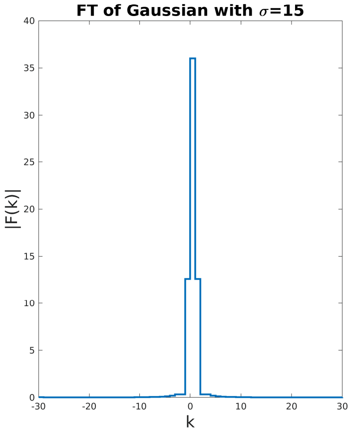
Notice: Stretch in time domain corresponds to shrinkage in Fourier domain and vice versa.

Fourier Transform Properties
Scaling/Reciprocity



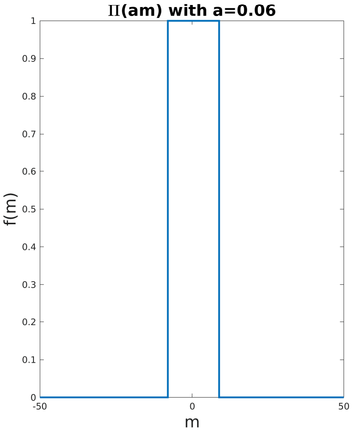
Fourier Transform Properties
Scaling/Reciprocity

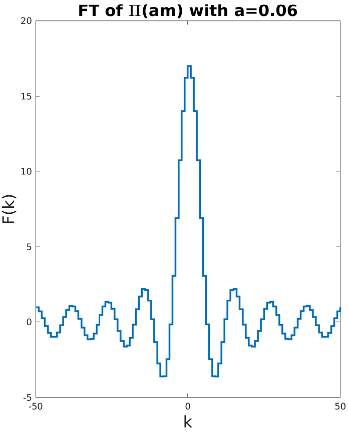




Fourier Transform Properties

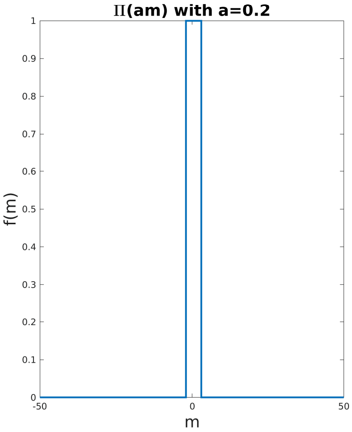
Scaling/Reciprocity

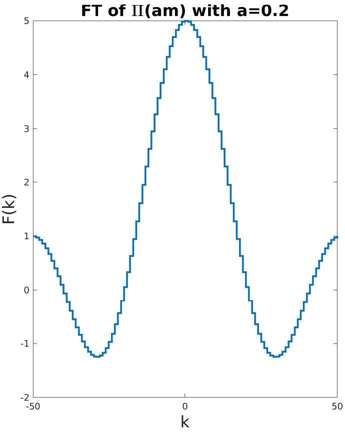




Fourier Transform Properties

Scaling/Reciprocity





Fourier Transform Properties
Shift

Statement:

$$f(x-a) \supset e^{-2\pi i a\omega} F(\omega)$$

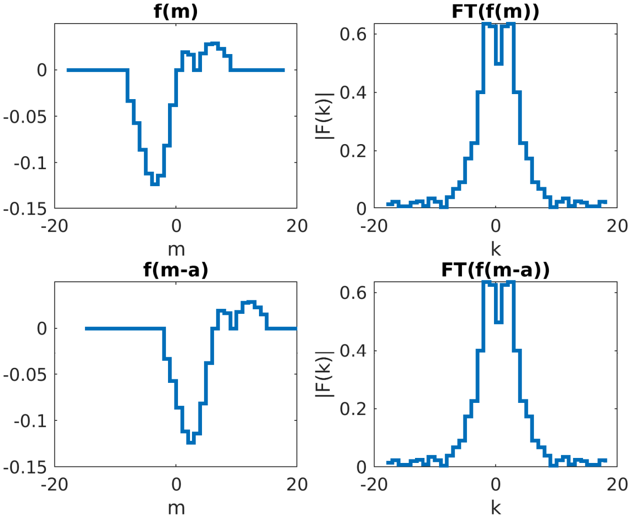
Proof:

$$\begin{aligned}\mathcal{F}\left[f(x-a)\right](\omega) &= \int_{-\infty}^{\infty} f(x-a)e^{-2\pi i x\omega}dx \\ &= \int_{-\infty}^{\infty} f(x-a)e^{-2\pi i (x-a)\omega}e^{-2\pi i a\omega}d(x-a) \\ &= e^{-2\pi i a\omega}F(\omega)\end{aligned}$$

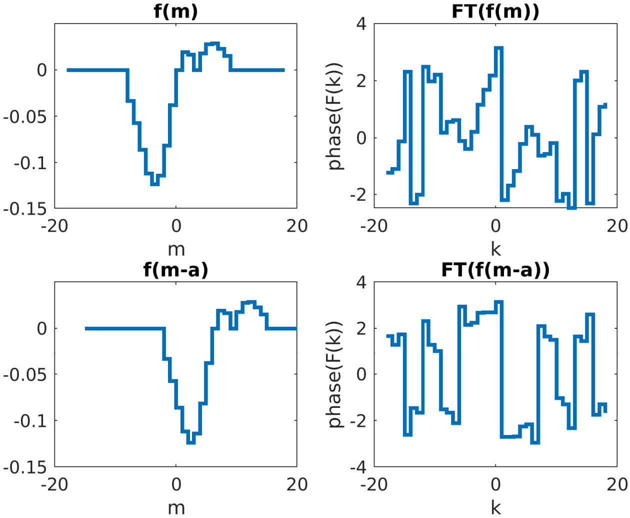
□

Notice: Shift affects only phase. The higher the frequency ω is the more the corresponding cosine wave is affected.

Fourier Transform Properties
Shift



Fourier Transform Properties
Shift



Statement:

$$F(-\omega) = \overline{F(\omega)}$$

Proof:

$$\begin{aligned} F(-\omega) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x(-\omega)} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{2\pi i x \omega} \\ &= \overline{F(\omega)} \quad \text{iff} \quad f : \mathbb{R} \to \mathbb{R} \end{aligned}$$

□

Statement:

$$\alpha f(x) + \beta g(x) \supset \alpha F(\omega) + \beta G(\omega)$$

Proof:

$$\begin{aligned}
\mathcal{F}\left[\alpha f(x) + \beta g(x)\right](\omega) &= \int_{-\infty}^{\infty} \left[\alpha f(x) + \beta g(x)\right] e^{-2\pi i x \omega} dx \\
&= \alpha \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \omega} dx + \beta \int_{-\infty}^{\infty} g(x) e^{-2\pi i x \omega} dx \\
&= \alpha F(\omega) + \beta G(\omega)
\end{aligned}$$

□

Fourier Transform Properties

Convolution theorem

Statement:

$$f(x) * g(x) \supset F(\omega)G(\omega)$$
$$f(x)g(x) \supset F(\omega) * G(\omega)$$

Proof:

$$\begin{aligned}
\mathcal{F}\left[f(x) * g(x)\right](\omega) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x')g(x-x')dx'\right] e^{-2\pi i x\omega} dx \\
&= \int_{-\infty}^{\infty} f(x') \left[\int_{-\infty}^{\infty} g(x-x')e^{-2\pi i x\omega} dx\right] dx' \\
&= \int_{-\infty}^{\infty} f(x')e^{-2\pi i x'\omega} G(\omega)dx' = F(\omega)G(\omega)
\end{aligned}$$

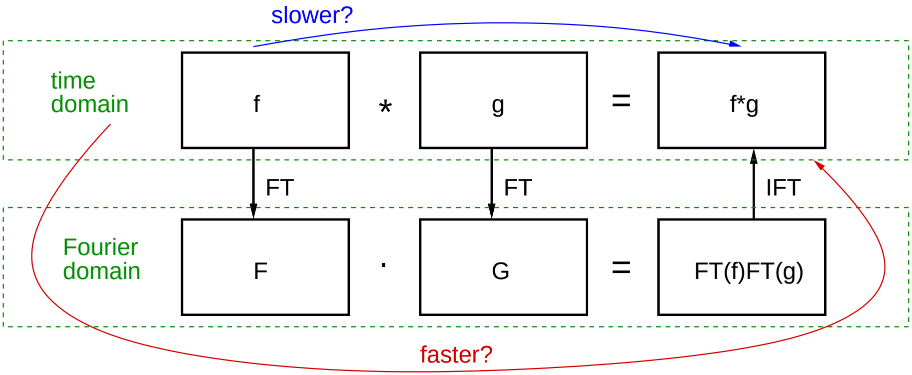
□

Fourier Transform Properties

Convolution theorem

The meaning:

- The convolution in time domain corresponds to point-wise multiplication in the Fourier domain, and vice versa.



Fourier Transform Properties

Rayleigh's energy theorem (Parseval's theorem)

Statement:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 dx &= \int_{-\infty}^{\infty} f(x)\overline{f(x)} dx \\
&= \int_{-\infty}^{\infty} f(x)\overline{f(x)} e^{-2\pi i x \omega'} dx & \omega' = 0 \\
&= F(\omega') * \overline{F(-\omega')} & \omega' = 0 \\
&= \int_{-\infty}^{\infty} F(\omega')\overline{F(\omega - \omega')} d\omega & \omega' = 0 \\
&= \int_{-\infty}^{\infty} F(\omega)\overline{F(\omega)} d\omega & \square
\end{aligned}$$

Notice: Rayleigh's energy theorem – the integral of the square of a function is equal to the integral of the square of its transform.

Given a signal f, $|f| = N$, let

$$\varphi_k(m) = \frac{1}{\sqrt{N}} e^{\frac{2\pi imk}{N}} = \frac{1}{\sqrt{N}} \left(\cos \frac{2\pi mk}{N} + i \sin \frac{2\pi mk}{N} \right)$$

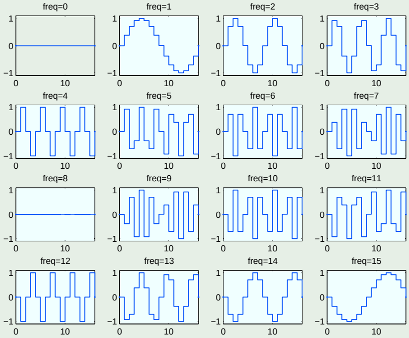
be an orthonormal basis, where $m \in \{0, \ldots, N-1\}$ defines the index (position) and $k \in \{0, \ldots, N-1\}$ defines the frequency (speed of oscilation). We can perform a projection of a signal f onto the basis $\varphi_k(m)$ as follows:

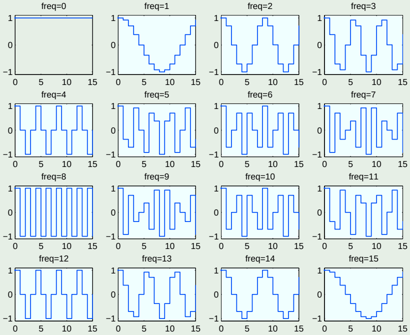
$$f \cdot \varphi_k = \sum_{m=0}^{N-1} f(m) \overline{\varphi_k(m)} = \sum_{m=0}^{N-1} f(m) \frac{1}{\sqrt{N}} e^{-\frac{2\pi ikm}{N}}$$

Discrete Fourier Transform

Motivation

An example of 16 sampled basis functions for $N = 16$:





Properties

- periodical:

$$\varphi_{k+N}(m) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i m(k+N)}{N}} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i m k}{N}} = \varphi_k(m)$$

- symmetrical:

$$\begin{aligned}\varphi_{N-k}(m) &= \frac{1}{\sqrt{N}} e^{\frac{2\pi i m(N-k)}{N}} \\ &= \frac{1}{\sqrt{N}} e^{-\frac{2\pi i m k}{N}} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i(-m)k}{N}} = \varphi_k(-m)\end{aligned}$$

- orthonormal:

$$\varphi_k(m) \cdot \varphi_l(m) = \sum_{m=0}^{N-1} \frac{1}{\sqrt{N}} e^{\frac{2\pi m k i}{N}} \overline{\frac{1}{\sqrt{N}} e^{\frac{2\pi m l i}{N}}} = \frac{1}{N} \sum_{m=0}^{N-1} e^{\frac{2\pi m(k-l)i}{N}} = \delta_{k,l}$$

Given 1D discrete function f of N samples and a basis $(\varphi_k(m), k = \{0, \ldots, N-1\})$, let us define:

- forward 1D discrete Fourier transform:

$$F(k) \equiv f \cdot \varphi_k = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi imk}{N}}$$

- inverse 1D discrete Fourier transform:

$$f(m) \equiv F \cdot \overline{\varphi_m} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} F(k) e^{\frac{2\pi imk}{N}}$$

$$f \cdot \varphi_k = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) \overline{e^{\frac{2\pi imk}{N}}} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi imk}{N}}$$

Discrete Fourier Transform

Matrix notation

If

$$\overline{\varphi_k(m)} = \frac{1}{\sqrt{N}} e^{\frac{-2\pi imk}{N}} = \frac{1}{\sqrt{N}} \left(e^{-\frac{2\pi i}{N}} \right)^{mk} = \frac{1}{\sqrt{N}} \psi^{mk}$$

then

$$A = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \psi^1 & \psi^2 & \dots & \psi^{N-1} \\ 1 & \psi^2 & \psi^{2\cdot 2} & \dots & \psi^{(N-1)2} \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi^{N-1} & \psi^{2(N-1)} & \dots & \psi^{(N-1)(N-1)} \end{bmatrix}$$

and

$$F = Af \quad \Rightarrow \quad f = A^{-1}F = \overline{A}^T F$$

Notice: $\psi = e^{-\frac{2\pi i}{N}}$ is called the N^{th} root of unity.

Discrete Fourier Transform

The obvious question

When we apply FT, we usually say "let us decompose our signal into the sine waves …" Why do we use another (so complicated) basis?

Basis function is a "sine wave"

- we avoid complex numbers
- more intuitive if basis function is a simple "sine wave"
- sine waves without phase shift do not generate the whole space
- possible basis function: $\sin(km - \alpha)$
- α hidden in the sine function spoils the linearity; matrix multiplication cannot be used

Basis function is $\varphi_k(m)$

- we have to use complex numbers
- $\varphi_k(m)$ functions generate the whole space (form basis)
- this basis is orthonormal
- transform is linear, i.e. realized via matrix multiplication

If you perform inverse FT

$$f(m) = F \cdot \overline{\varphi_m} = \sum_{k=0}^{N-1} F(k)\overline{\overline{\varphi_k(m)}} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} F(k) e^{i\frac{2\pi km}{N}}$$

you literally compose the original signal f by combining the individual basis functions.

- Each basis function $\varphi_k(m) = e^{i\frac{2\pi mk}{N}}$ defines only the frequency.
- Fourier coefficient $F(k) = |z_k|\,(\cos\alpha_k + i\sin\alpha_k) = |z_k|e^{i\alpha_k}$ modifies the corresponding basis function $\varphi_k(m)$ by scaling it and shifting it.

$$\begin{aligned} F(k)\overline{\overline{\varphi_m(k)}} &= |z_k|e^{i\alpha_k}e^{i\frac{2\pi mk}{N}} = |z_k|e^{i\alpha_k + i\frac{2\pi mk}{N}} \\ &= |z_k|\left\{\cos\left(\frac{2\pi k}{N}m + \alpha_k\right) + i\sin\left(\frac{2\pi k}{N}m + \alpha_k\right)\right\} \end{aligned}$$

Discrete Fourier Transform

The meaning of Fourier coefficients

- $F(0)$ matches the lowest frequency in the signal f and corresponds to the "mean" of f:

$$F(0) \equiv \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m 0}{N}} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m)$$

- $F(0)$ is usually called DC term
  DC ... "direct current" (current of zero frequency)

- $F(1) \ldots F(N-1)$ are called AC terms
  AC ... "alternating current"

- $F(\frac{N-1}{2})$ matches the highest frequency in the signal f

Exercise: Why does the component $F(\frac{N-1}{2})$ correspond to the highest frequency?

Discrete Fourier Transform Properties

Some properties can be adopted from continuous Fourier transform

- Evenness & Oddness . . . valid
- Shift . . . valid
- Linearity . . . valid
- Rayleigth theorem . . . valid
- Hermitian symmetry of real signal . . . valid
- Scaling . . . invalid
- Convolution theorem . . . modification required

Some new properties are introduced

- Periodicity
- Stretch
- Rearrangement

Let f and g be 1D discrete periodic signals of length N, then:

$$f * g \quad = \quad IDFT\left[DFT(f) \cdot DFT(g)\right]\sqrt{N}$$

Proof:

$$\begin{aligned}
f(m) * g(m) &= \sum_{k=0}^{N-1} f(k) g(m-k) \\
&= \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} F(n) e^{2\pi ikn/N}\right] \left[\frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} G(l) e^{2\pi i(m-k)l/N}\right] \\
&= \frac{1}{N} \sum_{n=0}^{N-1} F(n) \sum_{l=0}^{N-1} G(l) \sum_{k=0}^{N-1} e^{2\pi ikn/N} e^{2\pi i(m-k)l/N} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} F(n) \sum_{l=0}^{N-1} G(l) e^{2\pi iml/N} \sum_{k=0}^{N-1} e^{2\pi ik(n-l)/N} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} F(n) \sum_{l=0}^{N-1} G(l) e^{2\pi iml/N} \delta(n-l) N = \sum_{n=0}^{N-1} \left[F(n) \cdot G(n)\right] e^{2\pi imn/N} \quad \square
\end{aligned}$$

If $f(m)$ is a 1D function of length N, $p \in \mathbb{N}$, and $stretch_p\{f\} = \{g\}$, where

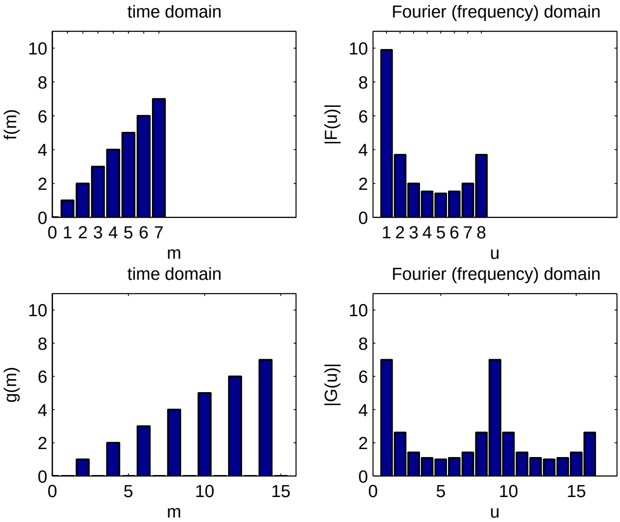
$$g(n) = \begin{cases} f(n/p) & n = 0, p, 2p, \ldots, (N-1)p \\ 0 & \text{otherwise} \end{cases}$$

then

$$G(k) = \frac{1}{\sqrt{p}} \begin{cases} F(k) & k = 0, \ldots, N-1 \\ F(k-N) & k = N, \ldots, 2N-1 \\ \vdots & \vdots \\ F(k-(p-1)N) & k = (p-1)N, \ldots, pN-1 \end{cases}$$

Notice: Stretch by a factor p in the time domain results in p-fold repetition of $F(k)$ in the frequency domain.

An example of stretch for $p = 2$



Let f be a 1D discrete signal, where $|f| = N$, and let $a \in \mathbb{N}$ be number that satisfies the condition $gcd(a, N) = 1$. The rearrangement of signal f determines the rearangement of Fourier coefficients in the following way:

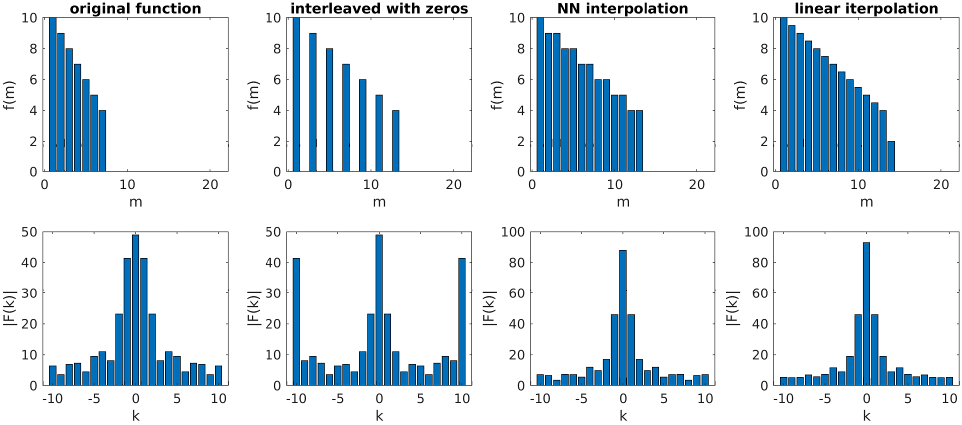
$$\begin{array}{rcl} F(k) & = & \mathcal{F}(f(m)) \\ & \Downarrow & \\ F(\langle bk \rangle_N) & = & \mathcal{F}(f(\langle am \rangle_N)) \end{array}$$

where

- $\langle m \rangle_N = m \bmod N$
- $b = \langle a^{\mathrm{Euler(N)}-1} \rangle_N$

Discrete Fourier Transform Properties

Why scaling does not work



Discrete Fourier Transform Properties

Periodicity

Statement:

$$F(k+N) = F(k)$$

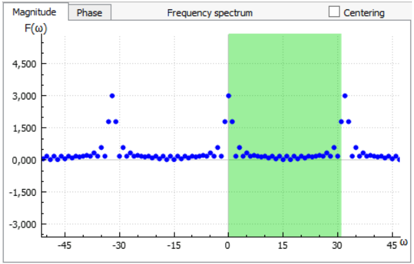
Proof:

$$\begin{aligned}
F(k+N) &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m(k+N)}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m k}{N}} e^{-\frac{2\pi i m N}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi i m k}{N}} \\
&= F(k)
\end{aligned}$$

□

Discrete Fourier Transform Properties

Periodicity and Symmetry ⇒ Domain centering



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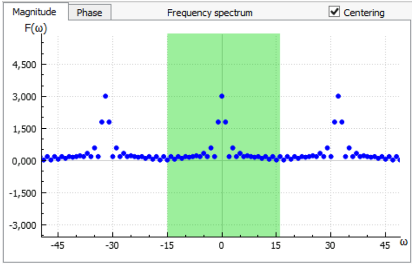


Fast Fourier Transform

Idea: N-point signal ($N = 2^m, m \in \mathbb{N}$) is decomposed into two $N/2$-point signals:

- one with all odd samples
- one with all even samples

Example:

1. input signal: $[\ f_0\ f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ ] \supset [\ ?\ ?\ ?\ ?\ ?\ ?\ ?\ ?\ ]$
2. $2\times$ simpler DFT: $[\ f_0\ f_2\ f_4\ f_6\ ] \supset [\ A\ B\ C\ D\ ]$
   $2\times$ simpler DFT: $[\ f_1\ f_3\ f_5\ f_7\ ] \supset [\ P\ Q\ R\ S\ ]$
3. stretch: $[\ f_0\ 0\ f_2\ 0\ f_4\ 0\ f_6\ 0\ ] \supset \frac{1}{\sqrt{2}}\ [\ A\ B\ C\ D\ A\ B\ C\ D\ ]$
4. stretch: $[\ f_1\ 0\ f_3\ 0\ f_5\ 0\ f_7\ 0\ ] \supset \frac{1}{\sqrt{2}}\ [\ P\ Q\ R\ S\ P\ Q\ R\ S\ ]$
   shift: $[\ 0\ f_1\ 0\ f_3\ 0\ f_5\ 0\ f_7\ ] \supset \frac{1}{\sqrt{2}}\ [\ P\ \psi Q\ \psi^2 R\ \psi^3 S\ \psi^4 P\ \psi^5 Q\ \psi^6 R\ \psi^7 S\ ]$
5. linearity: $[\ f_0\ f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ ] = [\ f_0\ 0\ f_2\ 0\ f_4\ 0\ f_6\ 0\ ] + [\ 0\ f_1\ 0\ f_3\ 0\ f_5\ 0\ f_7\ ]$

$$\Downarrow$$

$$[\ f_0\ f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ ] \supset \frac{1}{\sqrt{2}}\ [\ (A+P)\ \ (B+\psi Q)\ \ (C+\psi^2 R)\ \ldots]$$

Fast Fourier Transform

Derivation:

$$
\begin{aligned}
F(k) &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) e^{-\frac{2\pi imk}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m) e^{-\frac{2\pi i(2m)k}{N}} + \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m+1) e^{-\frac{2\pi i(2m+1)k}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m) e^{-\frac{2\pi imk}{N/2}} + e^{-\frac{2\pi ik}{N}} \frac{1}{\sqrt{N}} \sum_{m=0}^{N/2-1} f(2m+1) e^{-\frac{2\pi imk}{N/2}} \\
&= F^e(k) + \psi^k F^o(k)
\end{aligned}
$$

Notice: $\psi = e^{-\frac{2\pi i}{N}}$

Fast Fourier Transform

Idea: While it is possible, repeat the division.

$$
\begin{aligned}
F(k) &\rightarrow F^{e}(k), F^{o}(k) \\
&\rightarrow F^{ee}(k), F^{eo}(k), F^{oe}(k), F^{oo}(k) \\
&\rightarrow F^{eee}(k), F^{eeo}(k), F^{eoe}(k), F^{eoo}(k), F^{oee}(k), F^{oeo}(k), \\
&\quad\; F^{ooe}(k), F^{ooo}(k) \\
&\rightarrow \ldots
\end{aligned}
$$

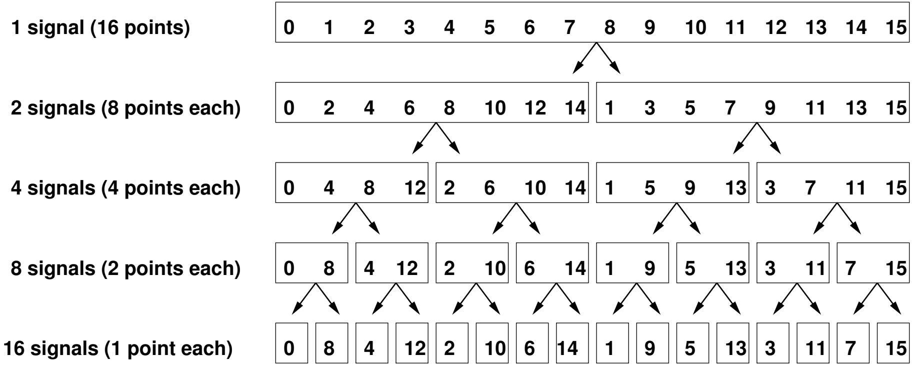
After $\log_2(n)$ divisions we have $F^{eeeoe \ldots eoeooee}(k)$ which is just one point long signal in Fourier domain.

You should know that: $\{X\} \supset \{X\}$

Exercise: What is the complexity of FFT?

Fast Fourier Transform

One more illustration



Given 2D discrete function f of (M, N) samples and two bases $(\varphi_k, k = \{0, \ldots, M-1\})$ and $(\varphi_l, l = \{0, \ldots, N-1\})$, let us define:

- forward 2D discrete Fourier transform:

$$F(k,l) \equiv \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) e^{-2\pi i \left(\frac{mk}{M} + \frac{nl}{N}\right)}$$

- inverse 2D discrete Fourier transform:

$$f(m,n) \equiv \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F(k,l) e^{2\pi i \left(\frac{mk}{M} + \frac{nl}{N}\right)}$$

Some properties are adopted from lower-dimensional case

- Shift
- Periodicity
- Convolution theorem
- Stretch

Some new properties are introducted

- Separability
- Rotation

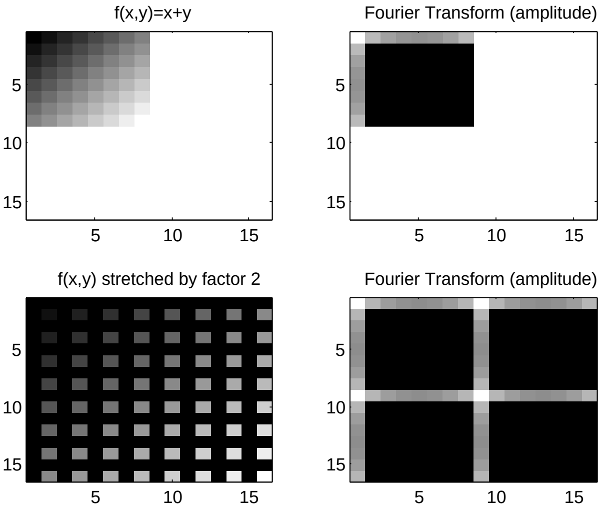
Separability

The evaluation of 2D-(D)FT can be decomposed into two simpler tasks:

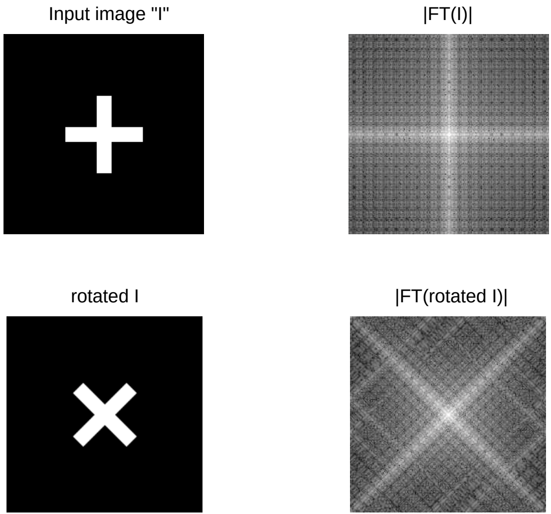
$$\begin{aligned}
F(k,l) &= \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) e^{-2\pi i\left(\frac{mk}{M}+\frac{nl}{N}\right)} \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left\{ \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f(m,n) e^{-\frac{2\pi imk}{M}} \right\} e^{-\frac{2\pi inl}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} F(k,n) e^{-\frac{2\pi inl}{N}}
\end{aligned}$$

Discrete Fourier Transform in 2D

Stretch



Discrete Fourier Transform in 2D
Rotation



Let us introduce the polar coordinates:

$$
\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi \\
\omega_x &= R\cos\psi \\
\omega_y &= R\sin\psi
\end{aligned}
$$

Then

$$
\begin{aligned}
f(x,y) &\rightarrow f(r,\phi) \\
F(\omega_x,\omega_y) &\rightarrow F(R,\psi)
\end{aligned}
$$

It is now clear to see that:

$$f(r, \phi + \phi_0) \supset F(R, \psi + \phi_0)$$

Conclusion: Rotating $f(x, y)$ by an angle ϕ_0 rotates $F(\omega_x, \omega_y)$ by the same angle, and vice versa.

How does the time/input domain influences the transform selection

- Discrete-time unit step sequence:

$$u_1(n) = \begin{cases} 1 & \text{if} \quad n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

  suitable transform: DFT in 1D
- Discrete-time unit step function:

$$u_2(m, n) = u_1(m) \cdot u_1(n)$$

  suitable transform: DFT in 2D
- Discrete-time regularly sampled unit sphere:
  suitable transform: SHT (Spherical Harmonics Transform)

Notice: Keep in mind, that Fourier transform requires the input domain to be regularly sampled!

Let us define a new basis $Y_\ell^m(\theta, \varphi)$ on a unit sphere as follows:

$$Y_\ell^m(\theta, \varphi) = k_{\ell,m} P_\ell^m(\cos\theta) e^{im\varphi}$$

where

- ℓ and m are respectively the degree and order of the harmonic
- $k_{\ell,m}$ is the normalization coefficient:

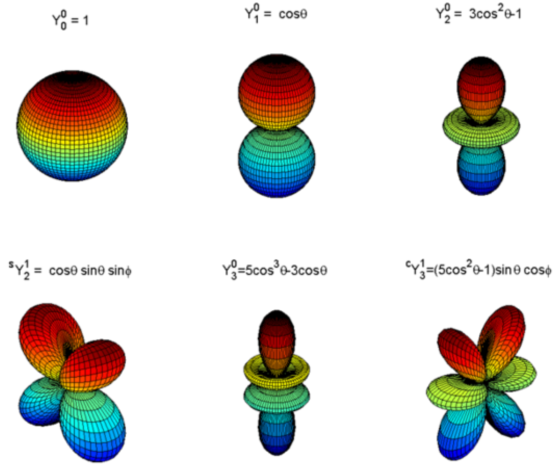
$$k_{\ell,m} = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}$$

- θ and φ are respectively the azimuth and the elevation
- $P_\ell^m(x)$ is the *associated Legendre polynomial*:

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2} \, \frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

Notice: Associated Legendre polynomials are mutualy orthogonal polynomials for fixed m or ℓ.

An example of some items (their magnitudes only) from the Y_ℓ^m basis



Using this Y_ℓ^m basis, any spherical scalar function $f(\theta, \varphi)$ can be expressed as follows:

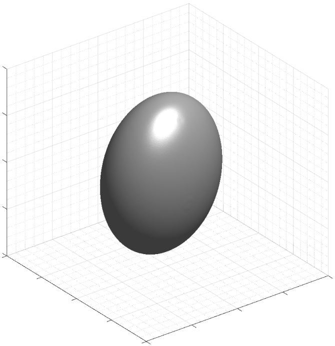
$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} H(\ell, m) Y_\ell^m(\theta, \varphi),$$

where $H(\ell, m)$ is a harmonic coefficient, given by:

$$H(\ell, m) = \int_0^{\pi} \int_0^{2\pi} f(\theta, \varphi) \overline{Y_\ell^m(\theta, \varphi)} \sin\theta \, d\varphi \, d\theta$$

Notice: Compare the terms *Harmonic coefficient* and *Fourier coefficient*.

3D cell shape modeling



$\ell = 1$

$$H(0,0) = \begin{bmatrix} 8.892 \\ 8.282 \\ 9.269 \end{bmatrix}$$

$$H(1,-1) = \begin{bmatrix} 0.609 - 1.354i \\ 1.228 + 1.264i \\ -1.797 + 0.119i \end{bmatrix}$$

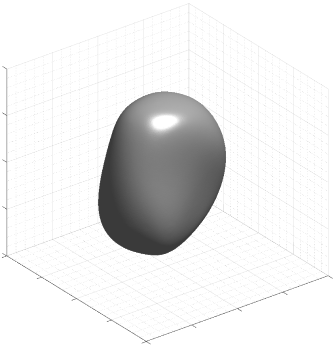
$$H(1,0) = \begin{bmatrix} 1.251 \\ 2.045 \\ 3.078 \end{bmatrix}$$

$$H(1,1) = \begin{bmatrix} -0.609 - 1.354i \\ -1.228 + 1.264i \\ 1.797 + 0.119i \end{bmatrix}$$

courtesy: images prepared by D. Wiesner

Notice: For more details how to get regular sampling over a sphere, see `http://lishenlab.com/spharm/SPHARM-MAT.pdf`

3D cell shape modeling



$\ell = 5$

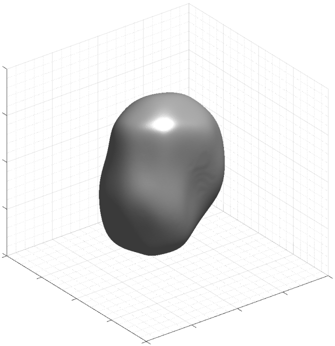
courtesy: images prepared by D. Wiesner

Notice: For more details how to get regular sampling over a sphere, see `http://lishenlab.com/spharm/SPHARM-MAT.pdf`

Spherical Harmonics Transform

Application

3D cell shape modeling

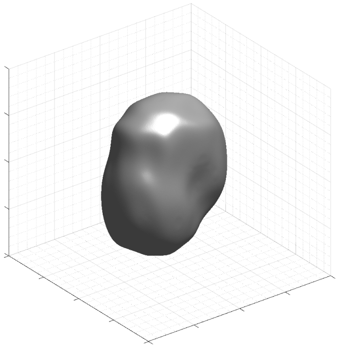


$\ell = 9$

courtesy: images prepared by D. Wiesner

Notice: For more details how to get regular sampling over a sphere, see `http://lishenlab.com/spharm/SPHARM-MAT.pdf`

3D cell shape modeling



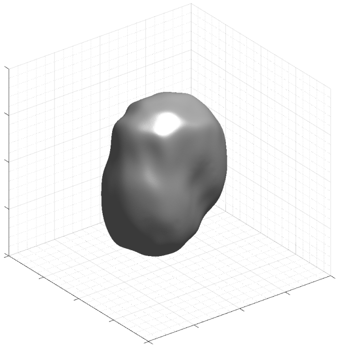
$\ell = 13$

courtesy: images prepared by D. Wiesner

Notice: For more details how to get regular sampling over a sphere, see `http://lishenlab.com/spharm/SPHARM-MAT.pdf`

Spherical Harmonics Transform
Application

3D cell shape modeling



$\ell = 17$

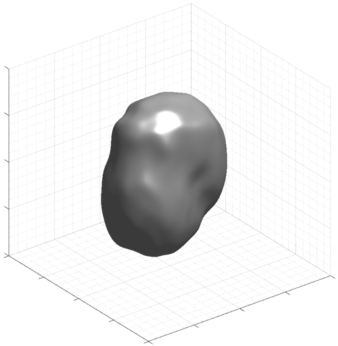
courtesy: images prepared by D. Wiesner

Notice: For more details how to get regular sampling over a sphere, see `http://lishenlab.com/spharm/SPHARM-MAT.pdf`

Spherical Harmonics Transform

Application

3D cell shape modeling



$\ell = 21$

courtesy: images prepared by D. Wiesner

Notice: For more details how to get regular sampling over a sphere, see
`http://lishenlab.com/spharm/SPHARM-MAT.pdf`

Let f be a real-valued signal. The *Hilbert transform* of f is defined as a convolution:

$$\mathcal{H}(f) = f * h,$$

where $h(t) = \frac{1}{\pi t}$ is called a Hilbert kernel.

Alternatively, we can define Hilbert transform via frequency domain:

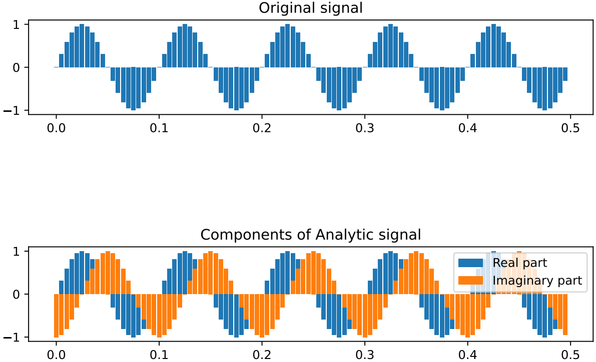
$$\mathcal{F}(\mathcal{H}(f))(\omega) = -i\, sign(\omega)\mathcal{F}(f)(\omega),$$

where

$$sign(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Hilbert transform is further used to form so called *Analytic Signal*:

$$f_A = f + i\,\mathcal{H}(f)$$



Analytic signal has the following properties:

- $\mathrm{real}(f_A) = f$
- $\mathrm{imag}(f_A) = \mathcal{H}(f)$
- f_A is a complex signal, i.e. can be interpreted in polar form:

$$f_A(x) = A(x)e^{i\varphi(x)}$$

where

- $A(x) = \sqrt{f(x)^2 + \mathcal{H}^2(f)(x)}$ . . . amplitude/envelope
- $\varphi(x) = \arctan\left(\frac{\mathcal{H}(f)(x)}{f(x)}\right)$ . . . phase

Positive versus Negative frequencies

Fourier coefficent $F(k)$ corresponds to:

- positive frequency ... $b_k(m) = e^{\frac{2\pi imk}{N}}$ if $0 \leq k \leq N/2$
- negative frequency ... $b_{-k}(m) = e^{-\frac{2\pi imk}{N}}$ otherwise

Real-valued signal

Fourier transform produces:

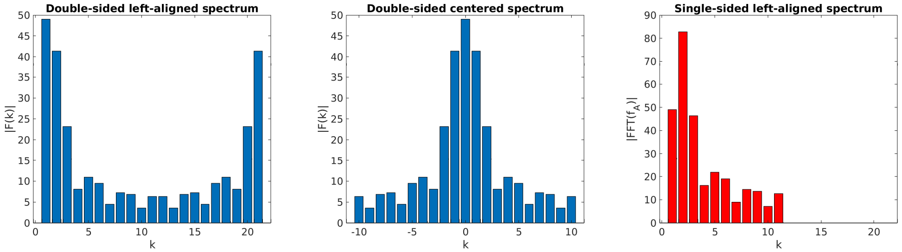
- double-sided spectrum with redundant items (Fourier coefficients)
- pairs of identical items with positive and negative frequencies

Analytic signal

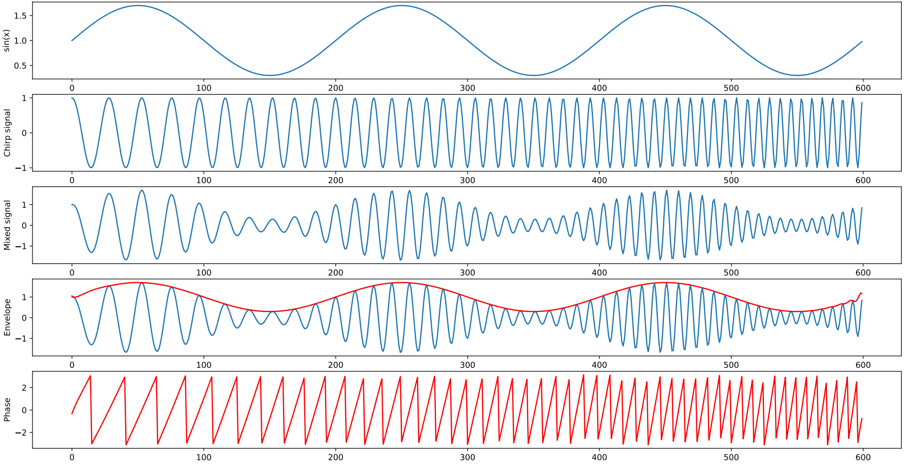
Fourier transform produces:

- single-sided spectrum (no redundancies)
- spectrum contains only positive frequencies

Double-sided versus single-sided spectrum



Signal decomposition



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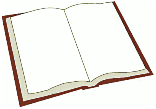

You should know the answers . . .

- Express the discrete Fourier transform as a matrix multiplication. Derive this matrix.
- How many Fourier basis functions do we need if we transform the signal of length N into the frequency domain?
- Formulate the forward discrete Fourier transform. Explain all the variables and constants.
- What does DC and AC terms mean?
- Explain the meaning of one particular Fourier coefficient in inverse Fourier transform.
- What is the product of the projection of an even function into a sin wave?
- Why are the wide functions in time domain transformed into their narrow counterparts in frequency domain and vice versa?
- Derive FFT for a signal of length $3^m, m \in \mathbb{N}$.