Integral and Discrete Transforms in Image Processing

Singular Value Decomposition
and
Independent Component Analysis

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Outline

1. Singular Value Decomposition

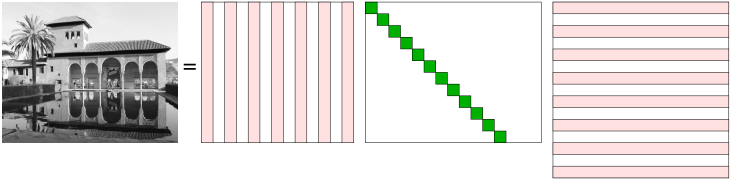
2. Independent Component Analysis (ICA)

Singular Value Decomposition (SVD)

Motivation

Can we decompose the image into more & less important parts?

- Each image is understood as a matrix.
- Can we factorize such matrix into:
  - *2 transform matrices* and
  - *1 matrix with (singular) values on its diagonal*?



Singular Value Decomposition (SVD)

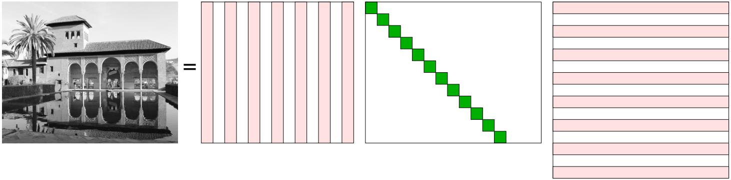
Definition

Any matrix $A \in \mathbb{R}^{m \times n}$ can be rewritten in the form

$$A = U \Sigma V^T$$

where

- $U \in \mathbb{R}^{m \times m}$ ... orthonormal matrix
- $\Sigma \in \mathbb{R}^{m \times n}$ ... sparse matrix with nonzero values on diagonal only
- $V \in \mathbb{R}^{n \times n}$ ... orthonormal matrix



Provided $A = U\Sigma V^T$ and U, V are supposed to be orthonormal, and Σ diagonal one, we can derive:

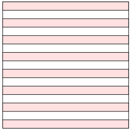
$$
\begin{aligned}
A^T A &= \left(U\Sigma V^T\right)^T \left(U\Sigma V^T\right) \\
&= V\Sigma U^T U\Sigma V^T \quad /U^T U = I/ \\
&= V\Sigma^2 V^T
\end{aligned}
$$

Eigen decomposition brings:

- V ... matrix of eigenvectors of $A^T A$
- Σ^2 ... diagonal matrix with eigenvalues (ordered by importance)

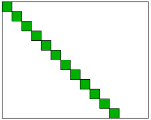
Singular Value Decomposition (SVD)

Background



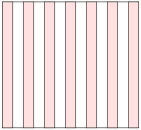
V^T

Rows of matrix V^T are composed of normalized eigenvectors of $A^T A$.



Σ

Diagonal matrix Σ is formed by square roots of eigenvalues of $A^T A$.



U

Matrix U can be simply derived $(V^{-1} = V^T)$:

$A = U \Sigma V^T \Rightarrow U = A \Sigma^{-1} {V^T}^{-1} = A \Sigma^{-1} V$

Singular Value Decomposition (SVD)

An example

Let $A \in \mathbb{R}^{2\times 3}$ be a matrix defined as follows:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

First, we need to form $A^T A$, which is positive definite, i.e. all the eigenvalues are real and positive:

$$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 8 & 11 \\ 8 & 5 & 6 \\ 11 & 6 & 17 \end{bmatrix}.$$

Singular Value Decomposition (SVD)

An example (cont'd)

The eigenvalues and normalized eigenvectors of $A^T A$ are, respectively:

$$\begin{aligned} \lambda_1 &= 30 \\ \lambda_2 &= 5 \\ \lambda_3 &= 0 \end{aligned} \qquad \begin{aligned} \mathbf{e}_1 &= \left[-\frac{17}{5\sqrt{30}}, -\frac{10}{5\sqrt{30}}, \frac{19}{5\sqrt{30}}\right] \\ \mathbf{e}_2 &= \left[-\frac{6}{5\sqrt{5}}, -\frac{5}{5\sqrt{5}}, \frac{8}{5\sqrt{5}}\right] \\ \mathbf{e}_3 &= \left[\frac{7}{5\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{1}{5\sqrt{6}}\right] \end{aligned}$$

Hence

$$\begin{aligned} V^T &= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \\ \Sigma &= \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix} \end{aligned}$$

Notice: Zeros eigenvalues are omitted.

Singular Value Decomposition (SVD)

An example (cont'd)

Last step is to get matrix U:

$$U = A\Sigma^{-1}V$$

Due to diagonal nature of matrix Σ, we can do:

$$\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A\mathbf{e_i}$$

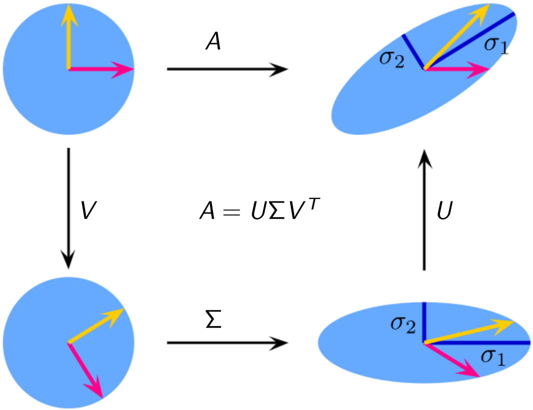
Therefore

$$U = [\mathbf{u}_1 \mathbf{u}_2] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

Finally:

$$A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} -\frac{17}{5\sqrt{30}} & -\frac{10}{5\sqrt{30}} & \frac{19}{5\sqrt{30}} \\ -\frac{6}{5\sqrt{5}} & -\frac{5}{5\sqrt{5}} & \frac{8}{5\sqrt{5}} \\ \frac{7}{5\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{5\sqrt{6}} \end{bmatrix}$$

Matrix A can be understood as a composition of several basic linear transforms:



source: wikipedia.org

Let A be a flattened database of faces:

$$A = \begin{bmatrix} \text{face 1} & \text{face 2} & \text{face 3} & \dots & \text{face n} \end{bmatrix}$$

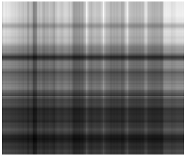
The SVD of A into $U \Sigma V^T$ gives us the *eigenfaces* as columns in U, ordered by importance.

$\Rightarrow$

0^{th} eigenvector

An arbitrary image (stored in matrix A) can be

1. decomposed $\left(A \rightarrow U\Sigma V^T\right)$
2. some of singular values from Σ can be eliminated
3. composed $\left(A \leftarrow U\Sigma V^T\right)$



$n = 1$

$n = 10$

$n = 30$

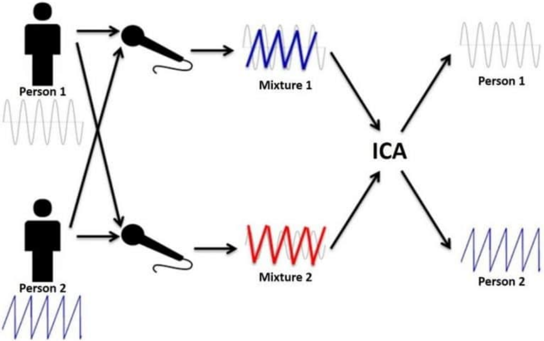
$n = 60$

$n = 100$

original image ($n = 512$)

n . . . number of singular values

Cocktail party problem



source: `https://vocal.com/blind-signal-separation/independent-component-analysis/`

Latent components:

- Person 1 . . . signal $s_1(t)$
- Person 2 . . . signal $s_2(t)$

Measured components:

- Mixture 1 . . . signal $x_1(t)$
- Mixture 2 . . . signal $x_2(t)$

Definition

Let

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad S = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}$$

then $X = AS$, where A is an *unknown* mixing matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} x_1(t) = a_{11}s_1(t) + a_{12}s_2(t) \\ x_2(t) = a_{21}s_1(t) + a_{22}s_2(t) \end{array}$$

We search for an unmixing matrix $W \approx A^{-1}$: $S = WX$

ICA can be used only if

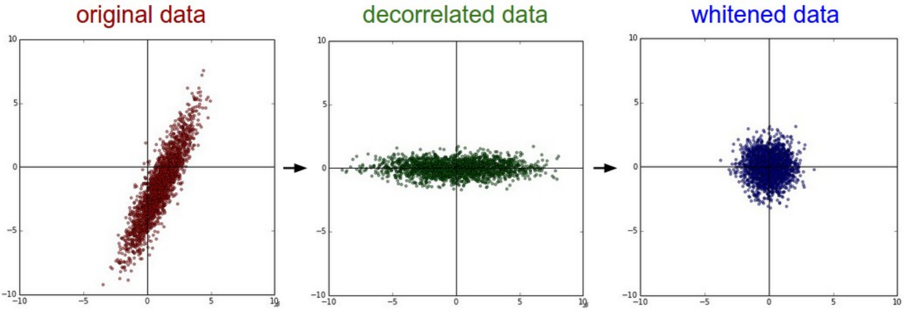
- s_1, s_2 are independent
- s_1, s_2 do not follow Gaussian distribution

1. Preprocessing the data matrix X
   whitening the input (measured) data
2. FastICA algorithm
   rotation of the data to find the projection that optimizes non-Gaussianity of the source data $\Rightarrow$ getting the projection matrix W
3. Estimating the original data
   $S = WX$

Notice: FastICA is one among many methods (InfoMAX, Projection pursuit, kurtosis, maximum likelihood) that solve the problem of ICA.

The objective is to normalize the data

1. Center the data (shifting by mean)
2. Uncorrelate the data (apply PCA)
3. Equalize the deviations (divide the *i*-th component by square root of its eigenvalue $\sqrt{\sigma_i}$



source: `https://pantelis.github.io/cs677`

Independent Component Analysis (ICA)

FastICA algorithm

Inputs:

- c . . . amount of latent components
- $X \in \mathbb{R}^{n \times m}$ . . . matrix representing the whitened measured signals
- $g(u) = tanh(u)$

Outputs:

- $W \in \mathbb{R}^{n \times c}$ . . . unmixing matrix $W \approx A^{-1}$
- $S \in \mathbb{R}^{c \times m}$ . . . matrix of estimated original components

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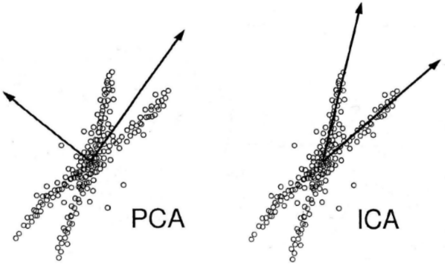
for $p = 1 \ldots c$ **do**
    $\mathbf{w}_p \leftarrow$ random vector of length n
    while $\mathbf{w}_p$ converges **do**
        $\mathbf{w}_p \leftarrow E\{\mathbf{x}g(\mathbf{w}_p^T\mathbf{x})\} - E\{g'(\mathbf{w}_p^T\mathbf{x})\}\mathbf{w}_p$ ▷ Newton iteration
        $\mathbf{w}_p \leftarrow \mathbf{w}_p - \sum_{j=1}^{p-1}(\mathbf{w}_p^T\mathbf{w}_j)\mathbf{w}_j$ ▷ orthogonalization
        $\mathbf{w}_p \leftarrow \frac{\mathbf{w}_p}{||\mathbf{w}_p||}$ ▷ normalization
    end while
end for
$W \leftarrow [\mathbf{w}_1, \ldots, \mathbf{w}_c]$ ▷ build unmixing matrix
$S \leftarrow W^T X$ ▷ estimate original components

Conclusion

Let's inspect the mutual relationship of:

- Principal component analysis (PCA) . . . decorrelation
- Singular Value Decomposition (SVD) . . . generalization of PCA
- Independent Component Analysis (ICA) . . . independence



source: www.tu-chemnitz.de

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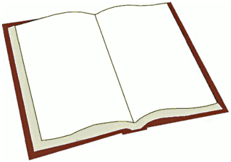
- Hyvärinen A, Oja E. Independent component analysis: algorithms and applications. Neural Networks: the Official Journal of the International Neural Network Society. 2000 May-Jun;13(4-5):411-430.
- YouTube lectures by Steven L. Brunton (`https://youtu.be/gXbThCXjZFM`)



You should know the answers . . .

- Explain the purpose of using SVD.
- What are the main properties of matrices U and V?
- What is does *singular value* mean?
- Can PCA fail to find proper components? Support your claim.
- Give an example how ICA can be applied to image/signal processing
- What does *whitening* mean?