# Integral and Discrete Transforms in Image Processing

## Z-transform

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# Outline





# Z-Transform

Definition

(bilateral) Z-Transform

$$F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

(unilateral) Z-Transform

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Notice: Here $z \in \mathbb{C}$, i.e. for $z = e^{i\omega}$ we get a special case of Z-transform, which is DFT.



# Z-Transform

Definition

Forward transform

- converts discrete series into continuous signal (Z-plane)

$$F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

Inverse transform

- If $F(z)$ is a polynomial, i.e. $F(z) = \sum_{k=-\infty}^{\infty} c_k z^{-k}$, then

$$f(n) = \mathcal{Z}^{-1}(F(z)) = \sum_{k=-\infty}^{\infty} c_k \delta(n-k)$$

Notice: Forward transform applied to discrete signals or linear filters always creates a polynomial with one variable $z$.

## Z-Transform

Some examples

- Z-transform of a constant signal/averaging filter (left aligned):

$$f(n) = [\underline{1}, 1, 1, 1, 1]$$
$$F(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

- Z-transform of a constant signal/averaging filter (centered):

$$f(n) = [1, 1, \underline{1}, 1, 1]$$
$$F(z) = z^{+2} + z^{+1} + 1 + z^{-1} + z^{-2}$$

- Z-transform of Sobel filter:

$$f(n) = [-1, \underline{0}, 1]$$
$$F(z) = -z^{+1} + z^{-1}$$

- Z-transform of any random filter:

$$f(n) = [1, \underline{3}, 1, 4, 2]$$
$$F(z) = z^{+1} + 3 + z^{-1} + 4z^{-2} + 2z^{-3}$$

Notice: With Z-transform, we always pay attention to origin!



## Z-Transform

An inverse transfom

Inverse transform for linear filters is an easy step:

- Signal of length 5:

$$F(z) = 3z^3 + z - 3z^{-1}$$
$$f(n) = \mathcal{Z}^{-1}(F(z)) = [3, 0, 1, \underline{0}, -3]$$

- Gaussian filter in Z-domain:

$$F(z) = \frac{1}{4}z + \frac{1}{2} + \frac{1}{4}z^{-1}$$
$$f(n) = \mathcal{Z}^{-1}(F(z)) = \frac{1}{4}[1, \underline{2}, 1]$$

- Step function in Z-domain

$$F(z) = z^{-1} + z^{-2} + z^{-3} + z^{-4}$$
$$f(n) = \mathcal{Z}^{-1}(F(z)) = [0, 0, 0, 0, \underline{0}, 1, 1, 1, 1, 1]$$

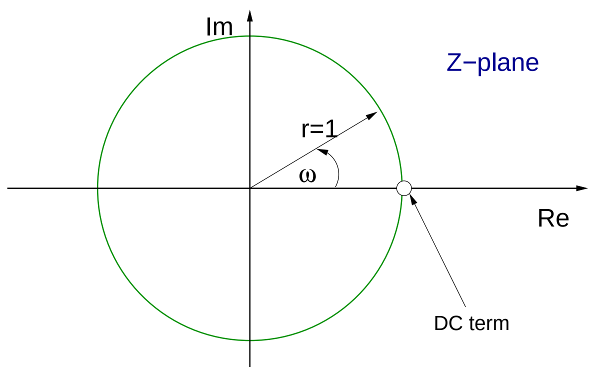


## Z-Transform

Relationship to other transforms

Important notes:

- green circle ($z = e^{i\varphi} \Rightarrow |z| = 1$) reduces Z-transform simply to discrete Fourier transform
- DC (direct current) term is DC term from DFT






## Z-Transform – Properties

Delay/Shift

$$\mathcal{Z}\{f(n-k)\} = F(z)z^{-k}$$

An example:

$$f(n) = [\underline{1}, 2, 3, 4]$$
$$g(n) = [1, \underline{2}, 3, 4]$$
$$h(n) = [1, 2, \underline{3}, 4]$$

$$F(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$
$$G(z) = z^1 + 2 + 3z^{-1} + 4z^{-2}$$
$$H(z) = z^2 + 2z + 3 + 4z^{-1}$$

$$F(z) = G(z)z^{-1}$$
$$F(z) = H(z)z^{-2}$$
$$G(z) = H(z)z^{-1}$$

# Z-Transform – Properties

## Convolution theorem

$$\mathcal{Z}\{f(n) * g(n)\} = F(z) \cdot G(z)$$

Corollary: Convolution in time domain becomes a polynomial multiplication in Z-domain:

$$\mathcal{Z}([\underline{3}, 5] * [\underline{-1}, 1]) = \left(3 + 5z^{-1}\right) \cdot \left(-1 + z^{-1}\right)$$

An example:

$$\begin{aligned}
[\underline{3}, 5] * [\underline{-1}, 1] &= [\underline{-3}, -2, 5] \quad /\mathcal{Z}(\cdot) \\
\mathcal{Z}([3, 5] * [\underline{-1}, 1]) &= \mathcal{Z}([\underline{-3}, -2, 5]) \\
\mathcal{Z}([3, 5]) \cdot \mathcal{Z}([\underline{-1}, 1]) &= \mathcal{Z}([\underline{-3}, -2, 5]) \\
(3 + 5z^{-1}) \cdot (-1 + z^{-1}) &= -3 - 2z^{-1} + 5z^{-2} \\
-3 - 2z^{-1} + 5z^{-2} &= -3 - 2z^{-1} + 5z^{-2}
\end{aligned}$$



# Z-Transform – Properties

## Convolution theorem - cont'd

| time domain | Z-domain |
|---|---|
| separation of convolved signals | polynomial factorization |

An example (Gaussian decomposition):

$$\begin{aligned}
[1, \underline{2}, 1] &= \mathcal{Z}^{-1}\left[z + 2 + z^{-1}\right] \\
&= \mathcal{Z}^{-1}\left[\left(1 + z^{-1}\right) \cdot (z + 1)\right] \\
&= \mathcal{Z}^{-1}\left[1 + z^{-1}\right] * \mathcal{Z}^{-1}[z + 1] \\
&= [\underline{1}, 1] * [1, \underline{1}]
\end{aligned}$$



# Z-Transform – Properties

## Linearity

$$\mathcal{Z}\{af(n) + bg(n)\} = aF(z) + bG(z)$$

An example:

$$\begin{aligned}
f'(n) &= f(n) * [1, -1, \underline{2}] \\
f'(n) &= 2f(n) - f(n+1) + f(n+2) \quad /\mathcal{Z}(\cdot) \\
\mathcal{Z}(f'(z)) &= \mathcal{Z}(2f(n)) - \mathcal{Z}(f(n+1)) + \mathcal{Z}(f(n+2)) \\
F'(z) &= 2F(z) - F(z)z^{1} + F(z)z^{2} \\
F'(z) &= F(z)[2 - z^{1} + z^{2}]
\end{aligned}$$

Notice: Keep in mind that: $\mathcal{Z}([1, -1, \underline{2}]) = z^2 - z^1 + 2$



# Z-Transform – Properties

## Symmetry

$$\mathcal{Z}\{f(-n)\} = F\left(\frac{1}{z}\right)$$

See the relationship betwenn time-revesed signals:

$$\begin{aligned}
f(n) &= [\underline{1}, 2, 3, 4, 5, 6] \\
F(z) &= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 6z^{-5} \\
&\Downarrow \\
f(-n) &= [6, 5, 4, 3, 2, \underline{1}] \\
F(z^{-1}) &= 1 + 2z^{1} + 3z^{2} + 4z^{3} + 5z^{4} + 6z^{5}
\end{aligned}$$

# Z-Transform – Properties

Signal upsampling

Let $L$ be a positive integer and

$$h(n) = \begin{cases} f(n/L) & n/L \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$H(z) = F(z^L)$$

Derivation:

$$\begin{aligned} H(z) &= \sum_{n=\dots,-L,0,L,\dots} f(n/L) z^{-n} \\ &= \sum_{m=\dots,-1,0,1,\dots} f(m) z^{-mL} \\ &= \sum_{m=\dots,-1,0,1,\dots} f(m) \left(z^L\right)^{-m} \\ &= F(z^L) \end{aligned}$$



# Z-Transform – Properties

Signal decomposition with downsampling

$$F(z) = F_{even}(z^2) + z^{-1} F_{odd}(z^2)$$

An example:

$$\begin{aligned} f(n) &= [\underline{a}, b, c, d, e, f] & F(z) &= a + bz^{-1} + cz^{-2} + dz^{-3} + ez^{-4} + fz^{-5} \\ f_{even}(n) &= [\underline{a}, c, e] & F_{even}(z) &= a + cz^{-1} + ez^{-2} \\ f_{odd}(n) &= [\underline{b}, d, f] & F_{odd}(z) &= b + dz^{-1} + fz^{-2} \end{aligned}$$

$$\begin{aligned} F_{even}(z^2) &= a + cz^{-2} + ez^{-4} \\ F_{odd}(z^2) &= b + dz^{-2} + fz^{-4} \\ F_{even}(z^2) + z^{-1} F_{odd}(z^2) &= a + bz^{-1} + cz^{-2} + dz^{-3} + ez^{-4} + fz^{-5} \end{aligned}$$



# Z-Transform – Properties

Signal decomposition without downsampling

$$F(z) = \frac{1}{2}\left[F(z) + F(-z)\right] + \frac{1}{2}\left[F(z) - F(-z)\right]$$

where

$$F_{even}(z^2) = \frac{1}{2}\left[F(z) + F(-z)\right] \quad \dots \quad \text{even component}$$

$$F_{odd}(z^2) = \frac{1}{2}\left[F(z) - F(-z)\right] \quad \dots \quad \text{odd component}$$

An example:

$$\begin{aligned} f &= [\underline{a}, b, c, d, e, f] \quad /\mathcal{Z}(\cdot) \\ F(z) &= a + bz^{-1} + cz^{-2} + dz^{-3} + ez^{-4} + fz^{-5} \end{aligned}$$

$$\begin{aligned} \text{EVEN} = \frac{1}{2}\left[F(z) + F(-z)\right] &= a + cz^{-2} + ez^{-4} \\ &= \mathcal{Z}^{-1}[\underline{a}, 0, c, 0, e, 0] \\ \text{ODD} = \frac{1}{2}\left[F(z) - F(-z)\right] &= bz^{-1} + dz^{-3} + fz^{-5} \\ &= \mathcal{Z}^{-1}[\underline{0}, b, 0, d, 0, f] \end{aligned}$$



# Z-Transform

Transfer function

**Definition**

Let $f$ and $h$ be input and output signals, respectively, and let $g$ be a linear filter such that $h = f * g$, then $\mathcal{Z}(g)$ is called a *transfer function*.

An example:

$$\begin{aligned} h &= f * [1, \underline{2}, 1] \quad /\mathcal{Z}(\cdot) \\ H(z) &= F(z) \cdot \left(z + 2 + z^{-1}\right) \\ \frac{H(z)}{F(z)} &= z + 2 + z^{-1} \\ G(z) = \frac{H(z)}{F(z)} &= z + 2 + z^{-1} \end{aligned}$$

Notice: Transfer function expresses the frequency response of selected linear filter. In optics, $G(z)$, when assigning $z = e^{i\varphi}$, is called a *optical transfer function (OTF)* and its time-domain counter part is a point spread function (PSF).

# Z-Transform

Applications

The areas that rely on Z-transform

- Linear recursive filters
- Optimization of wavelet transform
- Understanding and frequency analysis of linear filters
- ...



# Bibliography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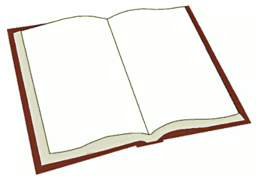



# You should know the answers ...

- Describe the relationship between Fourier transform and Z-transform.
- What is the Z-transform of signal $[1, 2, 4, 3, 20, -5, 23]$?
- What is the Z-transform of Gaussian filter?
- How is the time delay expressed with Z-transform?
- Is it possible to perform inverse Z-transform for a linear filter?
- How is the signal upsampling expressed with Z-transform?
- What is a transfer function?
- What does the polynomial $z - \frac{1}{z}$ mean?