Integral and Discrete Transforms in Image Processing

Subband Coding & Fast Wavelet Transform

David Svoboda

email: svoboda@fi.muni.cz
Centre for Biomedical Image Analysis
Faculty of Informatics, Masaryk University, Brno, CZ

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Outline

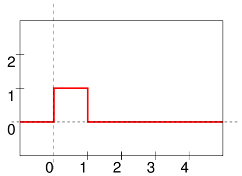
1D Discrete Wavelet Transform (DWT)

Revision

Haar basis generated from φ & ψ:

Scaling function φ

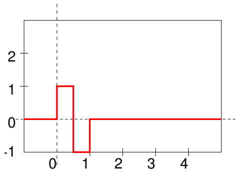
$$\varphi(x) = \begin{cases} 1 & \text{iff} \quad 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\varphi_{j,k}(x) = 2^{j/2}\varphi\left(2^j \frac{x}{M} - k\right)$$

Wavelet function ψ

$$\psi(x) = \begin{cases} 1 & \text{iff} \quad 0 \le x < 0.5 \\ -1 & \text{iff} \quad 0.5 \le x < 1 \\ 0 & \text{elsewhere} \end{cases}$$



$$\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j \frac{x}{M} - k\right)$$

- j . . . scale
- k . . . shift
- M . . . signal length

1D Discrete Wavelet Transform (DWT)

Revision

Forward

$$A_{j_0}(k) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f(m)\varphi_{j_0,k}(m)$$

$$D_j(k) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f(m)\psi_{j,k}(m)$$

Inverse

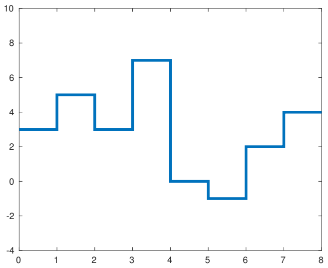
$$f(m) = \frac{1}{\sqrt{M}} \sum_{k=0}^{2^{j_0}-1} A_{j_0}(k)\varphi_{j_0,k}(m) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{J-1} \sum_{k=0}^{2^j-1} D_j(k)\psi_{j,k}(m)$$

- φ, ψ . . . orthogonal scaling and wavelet function, respectively
- $A_{j_0}(k)$ . . . scaling coefficients (approximations)
- $D_j(k)$ . . . wavelet coefficients (details)
- $M = 2^J$ . . . number of samples in function f
- $j \in \{j_0, ..., J-1\}$ . . . level of detail, where $j_0 \ge 0$
- $k \in \{0, 1, \dots, 2^j - 1\}$

1D Discrete Wavelet Transform (DWT)

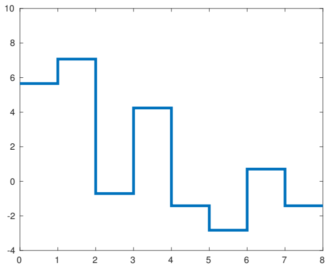
Revision

$$Haar = \frac{1}{\sqrt{8}} \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \begin{matrix} = \varphi_{2,0} \\ = \varphi_{2,1} \\ = \varphi_{2,2} \\ = \varphi_{2,3} \\ = \psi_{2,0} \\ = \psi_{2,1} \\ = \psi_{2,2} \\ = \psi_{2,3} \end{matrix}$$



f = [3 5 3 7 0 -1 2 4]

$\Rightarrow$

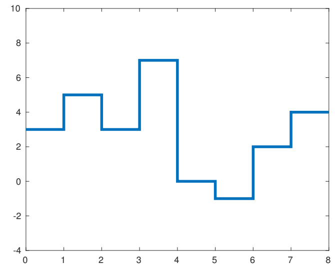


Haar * f^T

1D Discrete Wavelet Transform (DWT)

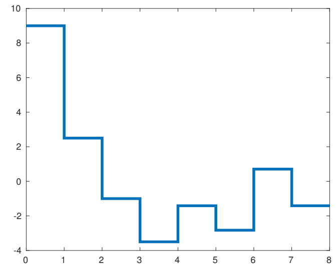
Revision

$$Haar = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \begin{matrix} = \varphi_{1,0} \\ = \varphi_{1,1} \\ = \psi_{1,0} \\ = \psi_{1,1} \\ = \psi_{2,0} \\ = \psi_{2,1} \\ = \psi_{2,2} \\ = \psi_{2,3} \end{matrix}$$



f = [3 5 3 7 0 -1 2 4]

$\Rightarrow$

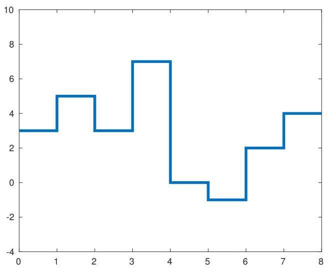


Haar * f^T

1D Discrete Wavelet Transform (DWT)

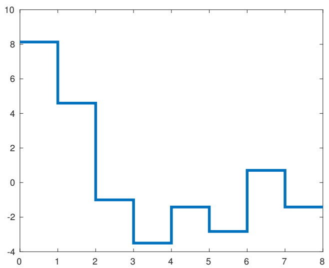
Revision

$$Haar = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \begin{matrix} = \varphi_{0,0} \\ = \psi_{0,0} \\ = \psi_{1,0} \\ = \psi_{1,1} \\ = \psi_{2,0} \\ = \psi_{2,1} \\ = \psi_{2,2} \\ = \psi_{2,3} \end{matrix}$$



f = [3 5 3 7 0 -1 2 4]

$\Rightarrow$



Haar * f^T

1D Discrete Wavelet Transform (DWT)

Issues

Obstacles when computing DWT:

- issue: M (signal length) is not power of 2
- possible solution: transform matrix need not be square $\Rightarrow$ technical solution but rather complicated

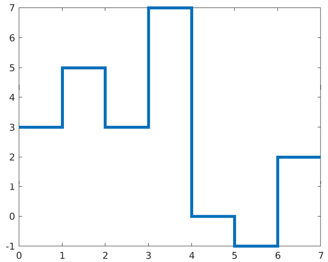
- issue: transform matrix is rather sparse $\Rightarrow$ a lot useless multiplications
- possible solution: skipping the zero positions

1D Discrete Wavelet Transform (DWT)

Issue – Signal Length

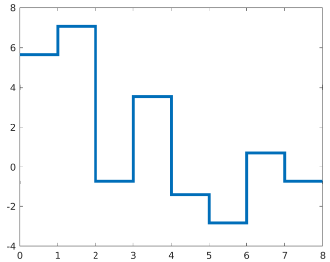
Haar transform matrix for the signal of length $M = 7$ and $j \in \{2\}$

$$Haar = \frac{1}{\sqrt{8}} \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$



f = [3 5 3 7 0 -1 2]

$\Rightarrow$



Haar * f^T

1D Discrete Wavelet Transform (DWT)

Issue – Matrix Sparsity

0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249
0.249	0.249	0.249	0.249	0.249	0.249	0.249	0.249	-0.249	-0.249	-0.249	-0.249	-0.249	-0.249	-0.249	-0.249
0.353	0.353	0.353	0.353	-0.353	-0.353	-0.353	-0.353	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.353	0.353	0.353	0.353	-0.353	-0.353	-0.353	-0.353
0.499	0.499	-0.499	-0.499	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.499	0.499	-0.499	-0.499	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.499	0.499	-0.499	-0.499	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0.499	0.499	-0.499	-0.499
0.707	-0.707	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.707	-0.707	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.707	-0.707	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0.707	-0.707	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.707	-0.707	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.707	-0.707	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0.707	-0.707	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.707	-0.707

Haar matrix for $M = 16$ and $j_0 = 0$

1D Discrete Wavelet Transform (DWT)

Issue – Matrix Sparsity

Questions to be answered

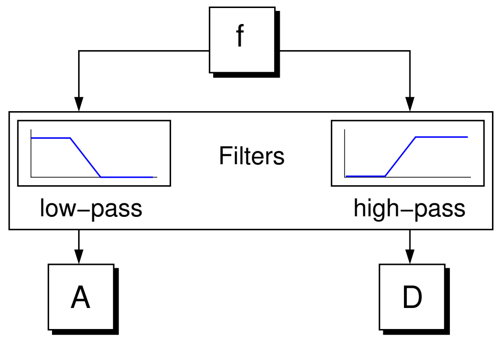
- What is the relationship of convolution and matrix multiplication?
- Can we subtitute each of them with one another?
- Can we express subsampling by a matrix multiplication?

Subband Coding

Signal Analysis

Any signal f can be decomposed into two parts:

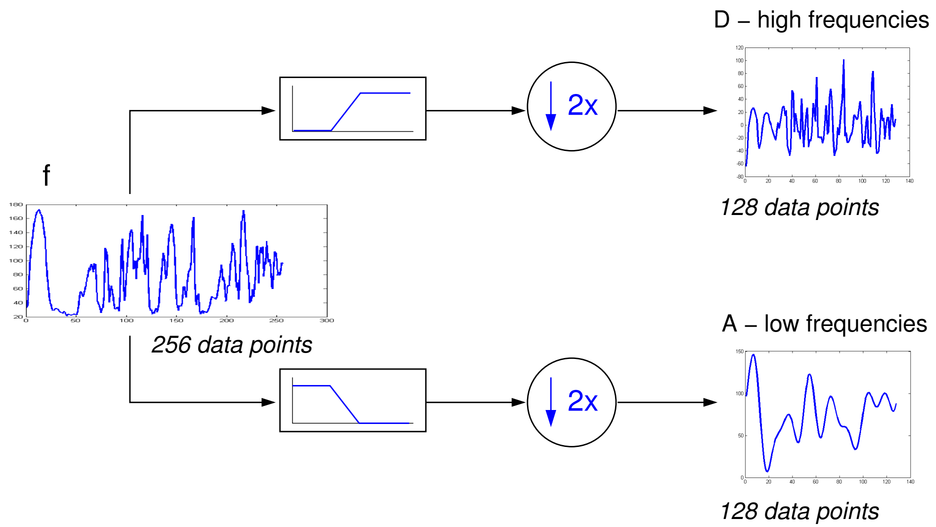
- approximation (A) . . . obtained by low-pass filtering of the original signal
- detail (D) . . . obtained by high-pass filtering of the original signal



Subband Coding

Signal Analysis

The filtered signal must be downsampled ($\downarrow 2\times$) to avoid data redundancy.

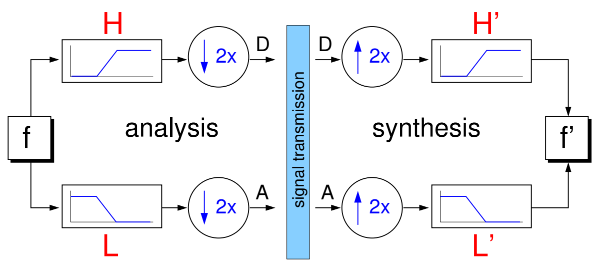


Subband Coding

Signal Analysis and Synthesis

The decomposed signal may be reconstructed:

- detail (D) is upsampled ($\uparrow 2\times$) and then high-pass filtered
- approximation (A) is upsampled ($\uparrow 2\times$) and then low-pass filtered
- results are added $\rightarrow f'$



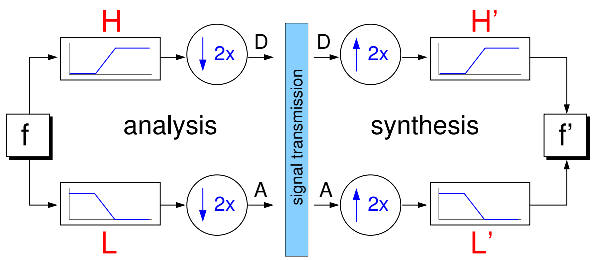
Notice: We would like to have $f = f'$

Subband Coding

Signal Analysis and Synthesis

Filter banks

- H . . . high-pass analysis filter
- L . . . low-pass analysis filter
- H' . . . high-pass synthesis filter
- L' . . . low-pass synthesis filter



Subband Coding

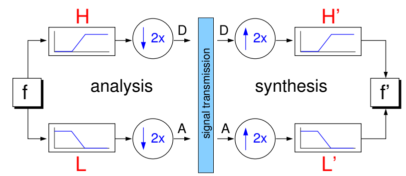
Signal Analysis and Synthesis

Filter banks

If $f = f'$ then the filters L, L', H, H' are called perfect reconstruction filters and they must fulfill one of the following conditions:

$$\begin{aligned} H'(n) &= (-1)^n L(n) \\ L'(n) &= (-1)^{n+1} H(n) \end{aligned}$$

$$\begin{aligned} H'(n) &= (-1)^{n+1} L(n) \\ L'(n) &= (-1)^n H(n) \end{aligned}$$

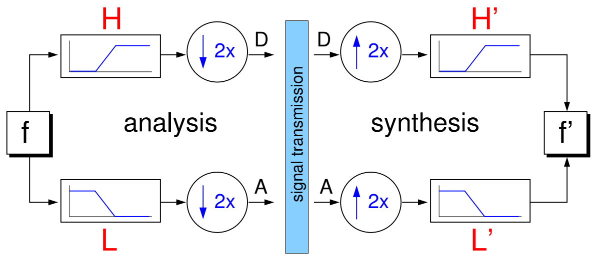


Subband Coding

Signal Analysis and Synthesis

Filter banks

- H and L' are mutually cross-modulated
- H' and L are mutually cross-modulated
- H, H', L, L' are called quadrature mirror filters (QMF)



Subband Coding

Signal Analysis and Synthesis

Filter banks

Biorthogonal filters

We need to define two filters H and L. The remaining H' and L' are derived by cross-modulation.

Orthogonal filters

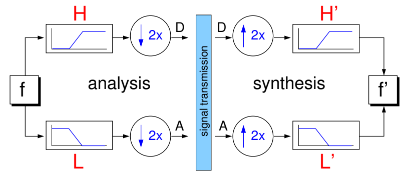
We define only one filter H'. The remaining filters fulfill:

$$\begin{aligned} L'(n) &= (-1)^n H'(length - 1 - n) \\ H(n) &= H'(length - 1 - n) \\ L(n) &= L'(length - 1 - n) \end{aligned}$$

where

$length = \texttt{size}(H')$ &

$\texttt{is\_even}(length) = true$

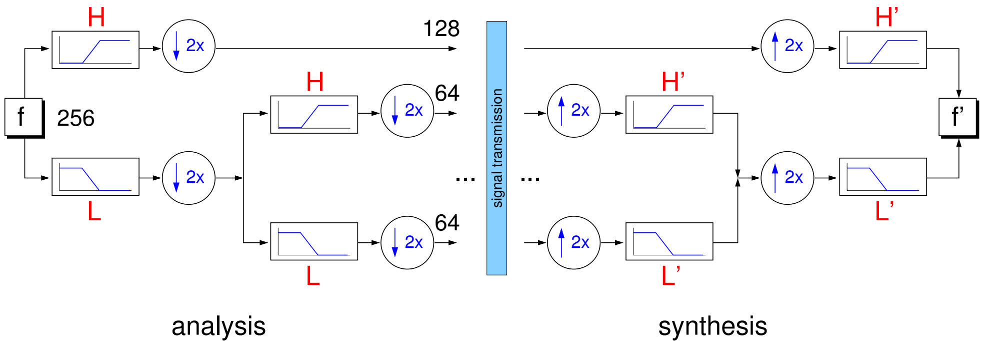


Notice: We will focus namely on the orthogonal filters.

Subband Coding

Recursive Signal Analysis

Once the input signal is decomposed into two parts (A and D), its approximation (A) can be further decomposed. In the reverse order, the same is valid for reconstruction.



Notice: Let us assume we have already employed (bi)orthogonal filters.

Subband Coding

The most common orthogonal filters

…and their scaling and wavelet functions

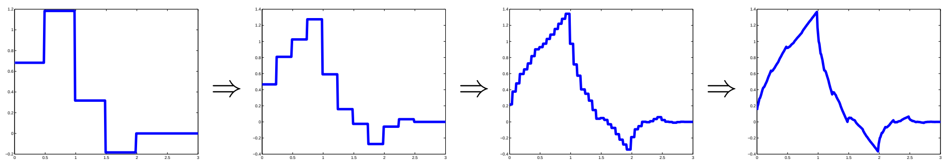
Notice: Useful web-pages: `http://wavelets.pybytes.com/`

From Filter Banks to Wavelets

Cascade algorithm for φ function (numerical solution)

Algorithm

1: $L' \leftarrow$ fetch low-pass synthesis filter from the selected filter bank
2: $h_\varphi = \texttt{fliplr}(L')$
3: $\varphi \leftarrow$ Dirac delta impulse
4: **while** (φ is converging) **do**
5: $\quad \varphi \leftarrow \texttt{conv}(\varphi, h_\varphi)$
6: $\quad \varphi \leftarrow \texttt{upsample}(\varphi, 2\times)$
7: **end while**
8: OUTPUT $\leftarrow \varphi$

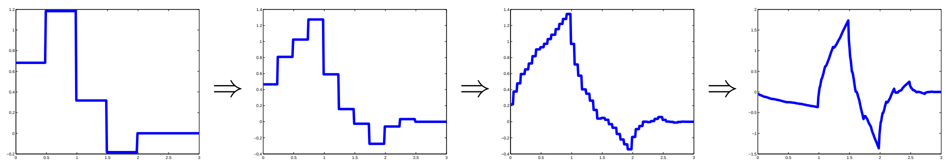


From Filter Banks to Wavelets

Cascade algorithm for ψ function (numerical solution)

Algorithm

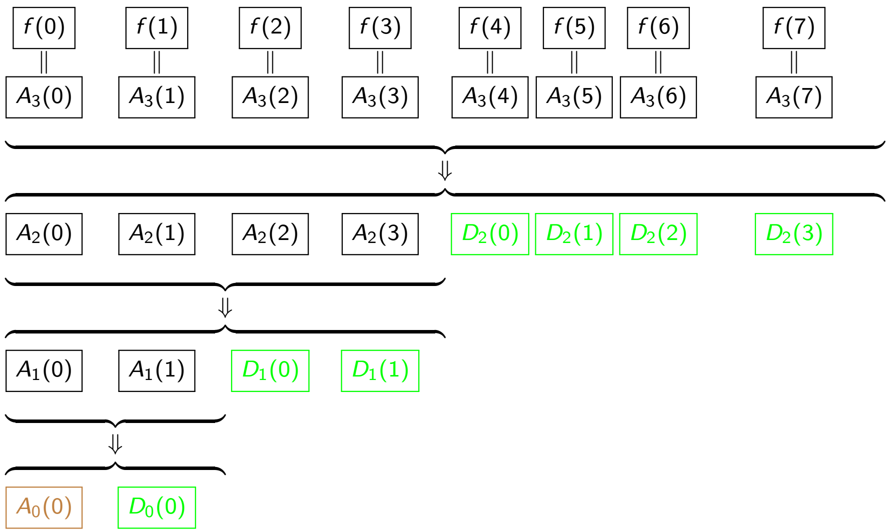
1: $\varphi \leftarrow$ call Cascade algorithm to get φ function
2: $H' \leftarrow$ fetch high-pass synthesis filter from the selected filter bank
3: $h_\psi = \texttt{fliplr}(H')$
4: $\psi \leftarrow \texttt{conv}(\varphi, h_\psi)$
5: $\psi \leftarrow \texttt{upsample}(\psi, 2\times)$
6: OUTPUT $\leftarrow \psi$



1D Fast Discrete Wavelet Transform

Basic scheme

Let $|f| = M = 8 = 2^3 = 2^J$ and $j_0 = 0$



1D Fast Discrete Wavelet Transform

Definition

$$D_j(k) = \sum_{r=0}^{|A_{j+1}|-1} H'(2k+1-r)A_{j+1}(r)$$

$$A_j(k) = \sum_{r=0}^{|A_{j+1}|-1} L'(2k+1-r)A_{j+1}(r)$$

$$A_J(k) = f(k)$$

Each step in FWT corresponds to convolution with high-pass and low-pass analysis filter followed by down-sampling ($\downarrow 2\times$).

1D-DWT $\equiv$ Subband coding

Fast Wavelet Transform

An example

Given $f(k) = [1, 4, -3, 0]$ and Haar scaling and wavelet coefficients

$$L'(k) = \begin{cases} 1/\sqrt{2} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad /\text{and}/ \quad H'(k) = \begin{cases} -1/\sqrt{2} & k = 0 \\ 1/\sqrt{2} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

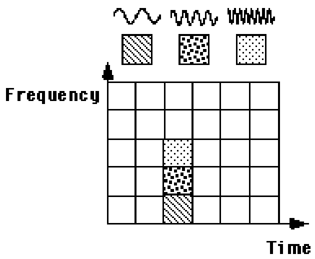
we can evaluate the following:

$$\begin{aligned}
\text{level 2:} \quad A_2(k) &= f(k) = [1, 4, -3, 0] \\
\text{level 1:} \quad A_1(k) &= \sum_{r=0}^{3} L'(2k+1-r)A_2(r) = [5/\sqrt{2}, -3/\sqrt{2}] \\
D_1(k) &= \sum_{r=0}^{3} H'(2k+1-r)A_2(r) = [-3/\sqrt{2}, -3/\sqrt{2}] \\
\text{level 0:} \quad A_0(k) &= \sum_{r=0}^{1} L'(2k+1-r)A_1(r) = [1] \\
D_0(k) &= \sum_{r=0}^{1} H'(2k+1-r)A_1(r) = [4]
\end{aligned}$$

Comparison of FWT and FFT

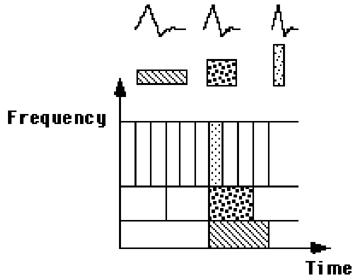
Fast Fourier Transform

- complexity: $O(n \log n)$
- existence: at any time
- time *versus* frequency domain



Fast Wavelet Transform

- complexity $O(cn)$
  c ... support of L' filter (typically small)
- existence: depends upon the availability of scaling function and the orthogonality of the scaling function
- time & frequency changes simultaneously



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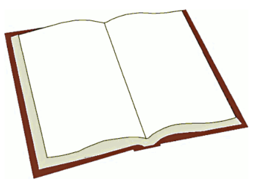

You should know the answers ...

- Explain the difference between Fourier basis functions and scaling and wavelet functions.
- Given a signal of fixed length and given a particular scaling a wavelet function we can perform discrete wavelet transform. The result is however not unique. Which parameter controls the behaviour of DWT? Demonstrate on some sample data.
- Explain the meaning of A and D coefficients.
- Derive the complexity for DWT and separately for FWT.
- What would happen if the quadrature mirror filters are not *perfect reconstruction filters*.
- Describe the *Cascade algorithm*.
- Design an algorithm for computing 2D-FWT.