

IA168 Algorithmic Game Theory

Tomáš Brázdil

Organization of This Course

Sources:

- ▶ Lectures (slides, notes)
 - ▶ based on several sources
 - ▶ slides are prepared for lectures, some stuff on greenboard
(\Rightarrow attend the lectures)

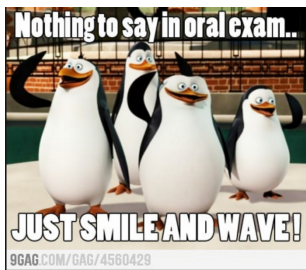
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- ▶ Books:
 - ▶ Nisan/Roughgarden/Tardos/Vazirani, **Algorithmic Game Theory**, Cambridge University, 2007.
Available online for free:
http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf
 - ▶ Tadelis, **Game Theory: An Introduction**, Princeton University Press, 2013

(I use various resources, so please, attend the lectures)

- ▶ Oral exam
- ▶ Homework



- ▶ 3 homework assignments

Notable features of the course

- ▶ No computer games course!
- ▶ **Very demanding!**
- ▶ Mathematical!

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An example of an instruction email (from another course with the same system):

It is typically not sufficient to devote a single afternoon to the preparation for the exam.

You have to know everything (which means every single thing) starting with the slide 42 and ending with the slide 245 with notable exceptions of slides: 121 - 123, 137 - 140, 165, 167.

Proofs presented on the whiteboard are also mandatory.

Most importantly,

The previous slide is not
a joke!

What is Algorithmic Game Theory?

First, what is the game theory?

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What does the "algorithmic" mean?

- ▶ It means that we are "concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."

Let's have a look at some examples

Prisoner's Dilemma

Prisoners' dilemma


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		confess		remain silent	
prisoner A	confess	 5 years 5 years	 0 year 20 years		
	remain silent	 20 years 0 year	 1 year 1 year		

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The problem: What would the suspects do?

Prisoner's Dilemma – Solution(?)

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C	-5, -5	0, -20
S	-20, 0	-1, -1

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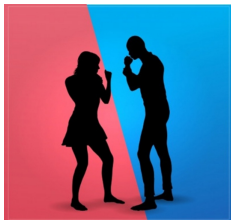
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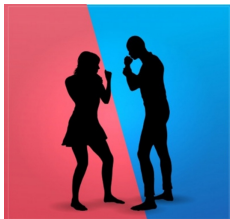
Are there always "dominant" strategies?

Nash equilibria – Battle of Sexes



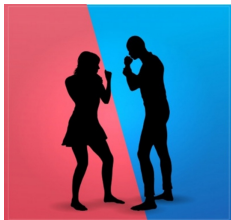
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If they cannot communicate, where should they go?

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Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

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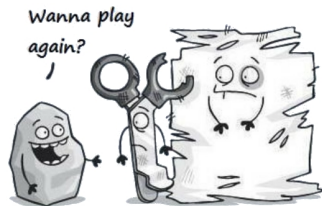
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(O, O) is an example of a *Nash equilibrium* (as is (F, F))

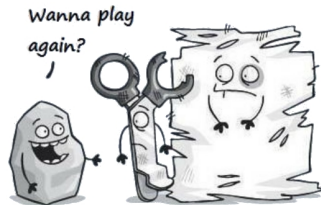
Mixed Equilibria – Rock-Paper-Scissors

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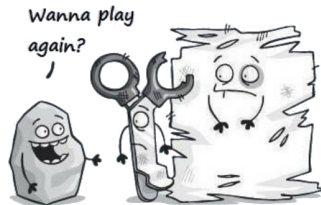
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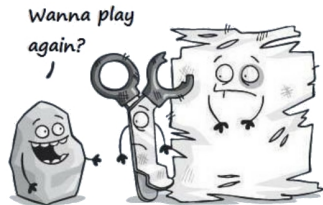
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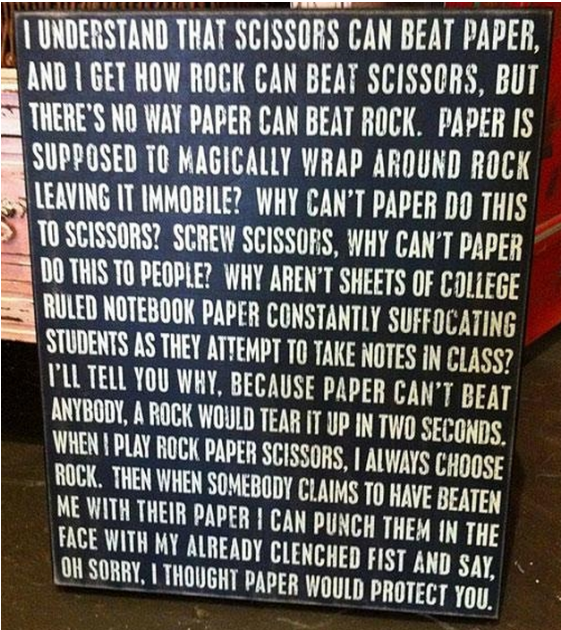
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Use *mixed strategies*: Each player plays each pure strategy with probability $1/3$. The expected payoff of each player is 0 (even if one of the players changes his strategy, he still gets 0!).

Philosophical Issues in Games



I UNDERSTAND THAT SCISSORS CAN BEAT PAPER, AND I GET HOW ROCK CAN BEAT SCISSORS, BUT THERE'S NO WAY PAPER CAN BEAT ROCK. PAPER IS SUPPOSED TO MAGICALLY WRAP AROUND ROCK LEAVING IT IMMOBILE? WHY CAN'T PAPER DO THIS TO SCISSORS? SCREW SCISSORS, WHY CAN'T PAPER DO THIS TO PEOPLE? WHY AREN'T SHEETS OF COLLEGE RULED NOTEBOOK PAPER CONSTANTLY SUFFOCATING STUDENTS AS THEY ATTEMPT TO TAKE NOTES IN CLASS? I'LL TELL YOU WHY, BECAUSE PAPER CAN'T BEAT ANYBODY, A ROCK WOULD TEAR IT UP IN TWO SECONDS. WHEN I PLAY ROCK PAPER SCISSORS, I ALWAYS CHOOSE ROCK. THEN WHEN SOMEBODY CLAIMS TO HAVE BEATEN ME WITH THEIR PAPER I CAN PUNCH THEM IN THE FACE WITH MY ALREADY CLENCHED FIST AND SAY, OH SORRY, I THOUGHT PAPER WOULD PROTECT YOU.

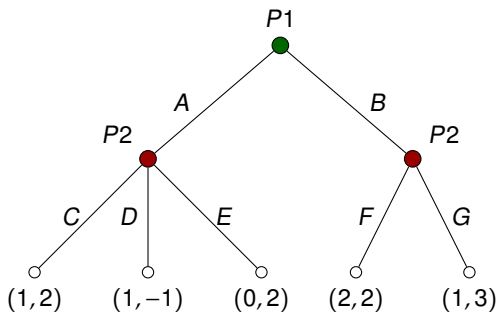
Dynamic Games

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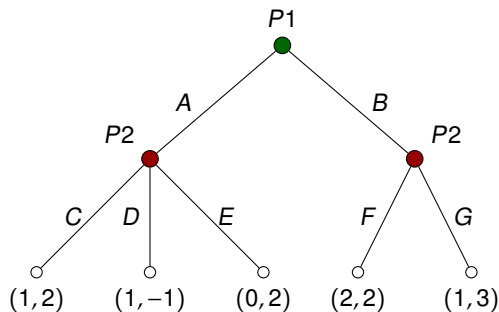
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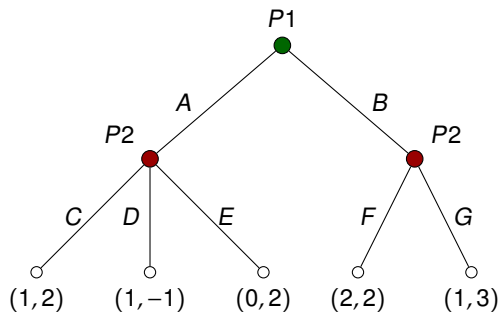


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What is their relationship to the strategic form games?

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Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

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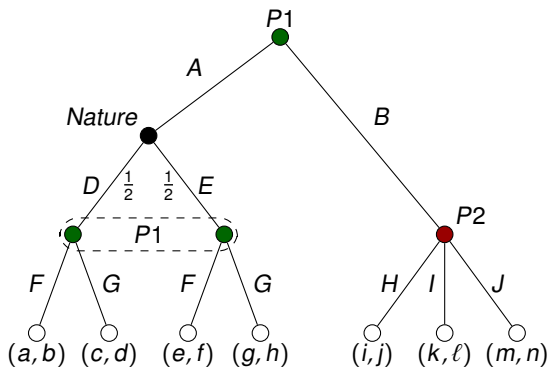
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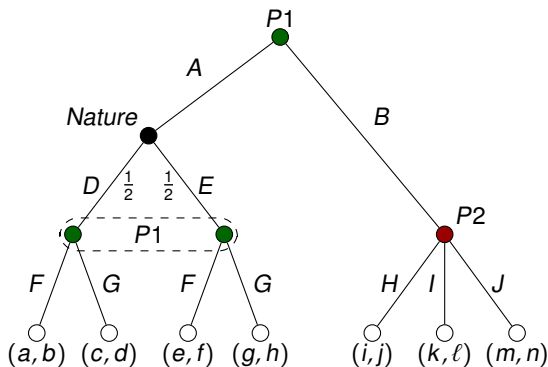
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Again, how to solve such games?

Games of Incomplete Information

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$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

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How to deal with such a game? Assume the “worst” private value?
What if we have a partial knowledge about the private values?

Inefficiency of Equilibria

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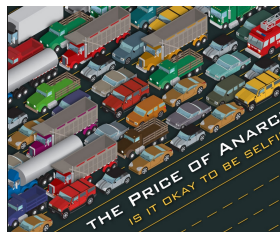
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Price of Anarchy is the maximum ratio between values of equilibria and the value of an optimal solution.

Inefficiency of Equilibria – Selfish Routing

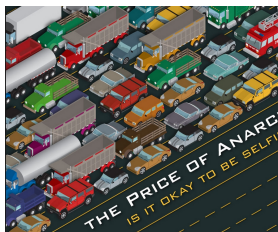
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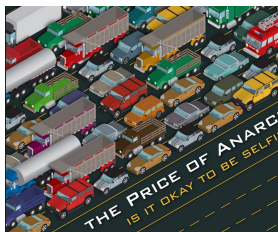
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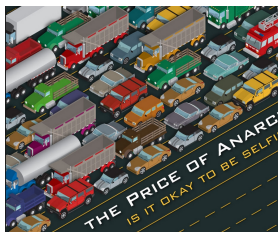


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Price of Anarchy measure the ratio between average travel time in these two cases.



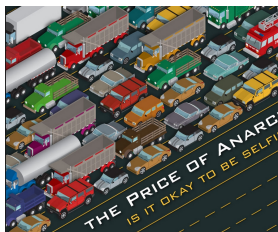
Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:

- ▶ “Centralized”: A central authority tells each agent where to go.
- ▶ “Decentralized”: Each agent selfishly minimizes his travel time.

Price of Anarchy measure the ratio between average travel time in these two cases.

Problem: Bound the price of anarchy over all routing games?



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- ▶ Games in Logic: modal and temporal logics, Ehrenfeucht-Fraisse games, etc.

Games, the Internet and E-commerce: An extremely active research area at the intersection of CS and Economics

Basic idea: “The internet is a HUGE experiment in interaction between agents (both human and automated)”

How do we set up the rules of this game to harness “socially optimal” results?

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- ▶ Remaining time will be devoted to selected topics from extensive form games, games on graphs etc.

Static Games of Complete Information

Strategic-Form Games

Solution concepts

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Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.

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A fact E is a *common knowledge* among players $\{1, \dots, n\}$ if for every sequence $i_1, \dots, i_k \in \{1, \dots, n\}$ we have that i_1 knows that i_2 knows that ... i_{k-1} knows that i_k knows E .

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The goal of each player is to maximize his payoff (and this fact is a common knowledge).

Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

Definition 2

A game in *strategic-form* (or normal-form) is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, in which:

- ▶ $N = \{1, 2, \dots, n\}$ is a finite set of *players*.
- ▶ S_i is a set of (*pure*) *strategies* of player i , for every $i \in N$.

A *strategy profile* is a vector of strategies of all players $(s_1, \dots, s_n) \in S_1 \times \dots \times S_n$.

We denote the set of all strategy profiles by $S = S_1 \times \dots \times S_n$.

- ▶ $u_i : S \rightarrow \mathbb{R}$ is a function associating each strategy profile $s = (s_1, \dots, s_n) \in S$ with the *payoff* $u_i(s)$ to player i , for every player $i \in N$.

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Definition 3

A *zero-sum* game G is one in which for all $s = (s_1, \dots, s_n) \in S$ we have $u_1(s) + u_2(s) + \dots + u_n(s) = 0$.

Example: Prisoner's Dilemma

- ▶ $N = \{1, 2\}$
- ▶ $S_1 = S_2 = \{S, C\}$
- ▶ u_1, u_2 are defined as follows:
 - ▶ $u_1(C, C) = -5, u_1(C, S) = 0, u_1(S, C) = -20,$
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(Is it zero sum?)

We usually write payoffs in the following form:

	C	S
C	-5, -5	0, -20
S	-20, 0	-1, -1

or as two matrices:

	C	S
C	-5	0
S	-20	-1

	C	S
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Strategic-form game model $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- ▶ $N = \{1, 2\}$
- ▶ $S_i = [0, \infty)$
- ▶ $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1$
 $u_2(q_1, q_2) = q_2(\kappa - q_1 - q_2) - q_2 c_2$

Solution Concepts

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(I follow the approach of Steven Tadelis here, it is not completely standard)

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Example 4

Nash equilibrium is a solution concept. That is, we “solve” games by finding Nash equilibria and declare them to be reasonable outcomes.

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Here 4. implies non-cooperative game theory: Each player is in control of his actions, and he will stick to an action only if he finds it to be in his best interest.

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For now, let us concentrate on

pure strategies only!

I.e., no mixed strategies are allowed. We will generalize to mixed setting later.

- ▶ Let $N = \{1, \dots, n\}$ be a finite set and for each $i \in N$ let X_i be a set. Let $X := \prod_{i \in N} X_i = \{(x_1, \dots, x_n) \mid x_j \in X_j, j \in N\}$.
 - ▶ For $i \in N$ we define $X_{-i} := \prod_{j \neq i} X_j$, i.e.,

$$X_{-i} = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mid x_j \in X_j, \forall j \neq i\}$$

- ▶ An element of X_{-i} will be denoted by

$$x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

We slightly abuse notation and write (x_i, x_{-i}) to denote $(x_1, \dots, x_i, \dots, x_n) \in X$.

Strict Dominance in Pure Strategies

Definition 5

Let $s_i, s'_i \in S_i$ be strategies of player i . Then s'_i is *strictly dominated* by s_i (write $s_i \succ s'_i$) if for any possible profile of the other players' strategies, $s_{-i} \in S_{-i}$, we have

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Claim 1

An intelligent and rational player will never play a strictly dominated strategy.

Clearly, intelligence implies that the player should recognize dominated strategies, rationality implies that the player will avoid playing them.

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$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i .

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Any rational player will play the strictly dominant strategy (if it exists).

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Corollary 8

If the strictly dominant strategy equilibrium exists, it is unique and rational players will play it.

Examples

In the Prisoner's dilemma:

	<i>C</i>	<i>S</i>
<i>C</i>	-5, -5	0, -20
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In the Battle of Sexes:

	<i>O</i>	<i>F</i>
<i>O</i>	2, 1	0, 0
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no strictly dominant strategies exist.

Indiana Jones and the Last Crusade

(Taken from Dixit & Nalebuff's "The Art of Strategy" and a lecture of Robert Marks)

Indiana Jones, his father, and the Nazis have all converged at the site of the Holy Grail. The two Joneses refuse to help the Nazis reach the last step. So the Nazis shoot Indiana's dad. Only the healing power of the Holy Grail can save the senior Dr. Jones from his mortal wound. Suitably motivated, Indiana leads the way to the Holy Grail. But there is one final challenge. He must choose between literally scores of chalices, only one of which is the cup of Christ. While the right cup brings eternal life, the wrong choice is fatal. The Nazi leader impatiently chooses a beautiful gold chalice, drinks the holy water, and dies from the sudden death that follows from the wrong choice. Indiana picks a wooden chalice, the cup of a carpenter. Exclaiming "There's only one way to find out" he dips the chalice into the font and drinks what he hopes is the cup of life. Upon discovering that he has chosen wisely, Indiana brings the cup to his father and the water heals the mortal wound.

Indy Goofed

- ▶ Although this scene adds excitement, it is somewhat embarrassing that such a distinguished professor as Dr. Indiana Jones would overlook his dominant strategy.
- ▶ He should have given the water to his father without testing it first.
 - ▶ If Indiana has chosen the right cup, his father is still saved.
 - ▶ If Indiana has chosen the wrong cup, then his father dies but Indiana is spared.
- ▶ Testing the cup before giving it to his father doesn't help, since if Indiana has made the wrong choice, there is no second chance – Indiana dies from the water and his father dies from the wound.