Limity formálních systémů, důkazů a výpočtů

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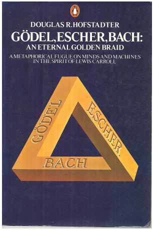
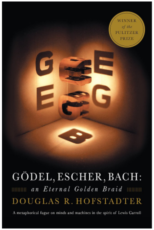
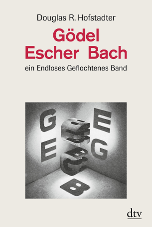
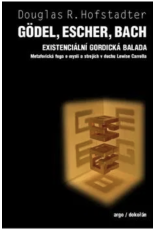
IV134
FI MU
Podzim 2024

- language: Czech/English
- voluntary course
- lecture on Monday, 10–12 or 12–14.
- Gödel, Escher, Bach: an Eternal Golden Braid
  by Douglas R. Hofstadter









- story of “strange loops” as limits of mathematics and computer science and foundations of intelligence

Bach

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- improvised 6-part fugue
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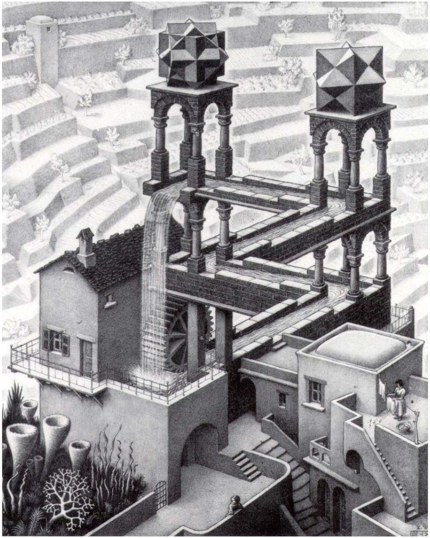
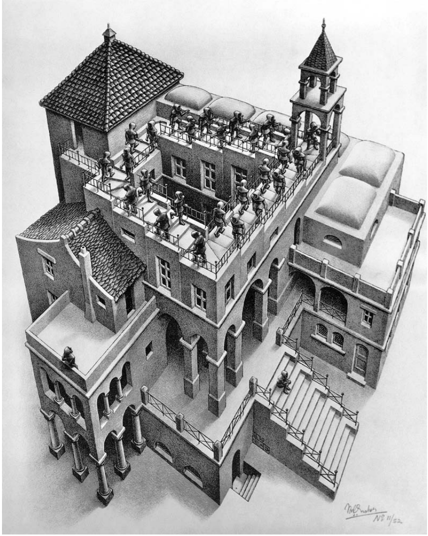
- canons
  - copies differing in time, pitch, speed, direction (upside down, crab)
  - isomorphic
  - multiple meanings of each note
  - canon endlessly rising in 6 steps – “strange loop”

Escher



“Waterfall”
6-step endlessly falling loop

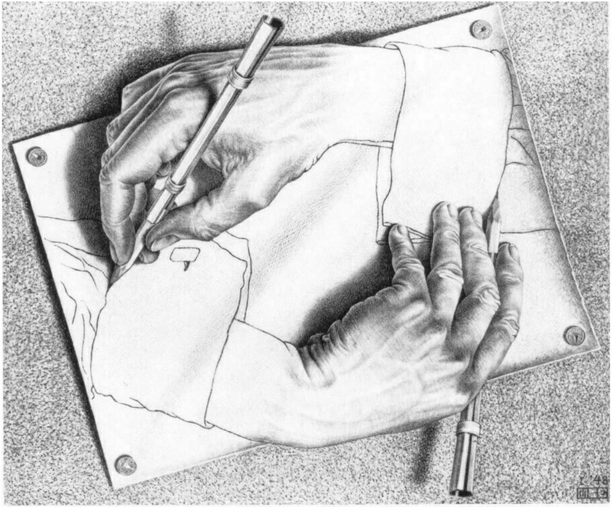
Escher



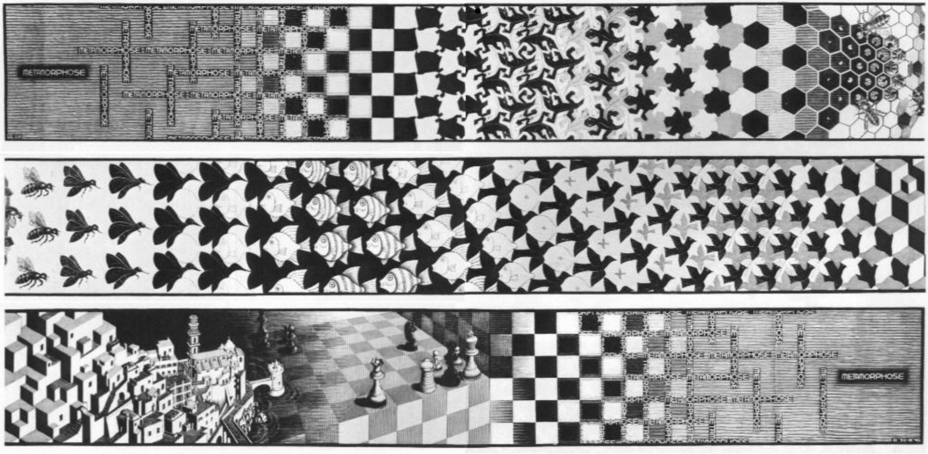
"Ascending and Descending"
illusion by Roger Penrose

Penrose triangle
Faculty of Informatics, Brno

Escher



“Drawing hands”
his first strange loop



“Metamorphosis”
copies of one theme

Gödel

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  215473077557

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  215473077557 is in binary
  0011001000101011001100100011110100110101 read as ASCII
  2+2=5
- homework:

34723379178930453204433293597543819411782291432109326918654063662

Mathematical logic

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- real world?
- proof?
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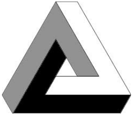
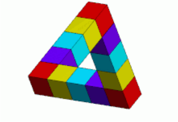
- Grelling's paradox
  - self-descriptive adjectives ("pentasyllabic") vs non-self-descriptive
  - what about "non-self-descriptive"?
- self-reference
  drawing hands
  The following sentence is false. The preceding sentence is true.

Way out?

- prohibition (Principia mathematica)
- types, metalanguage
- “In this lecture, I criticize the theory of types”
  cannot discuss the type theory

Computers

- Babbage
  The course through which I arrived at it was the most entangled and perplexed which probably ever occupied the human mind.

  Ada Lovelace (daughter of Lord Byron)
  Mechanized intelligence
  "Eating its own tail" (altering own program)
- axiomatic reasoning, mechanical computation, psycholgy of intelligence
- Alan Turing $\sim$ Gödel's counterpart in computation theory
  Halting problem is undecidable.
  Can intelligent behaviour be programmed? Rules for inventing new rules...
  Strange loops in the core of intelligence
- materialism, de la Metrie: L'homme machine

Formal system

Example (over alphabet `M,I,U`)

- initial string ("axiom"):
  - `MI`
- rules ("inference/production rules") to enlarge your collection (of "theorems")
  requirement of formality: not outside the rules
  - last letter `I` $\Rightarrow$ put `U` at the end
  - `M`x $\Rightarrow$ `M`xx where x can be any string
  - replace `III` by `U`
  - drop `UU`

Homework: Can you produce/derive/prove `MU` ?

- Which rule to use? That's the art.

Theorems of MIU system

Axiom: MI
Rules:

1. $x\mathtt{I} \Rightarrow x\mathtt{IU}$
2. $\mathtt{M}x \Rightarrow \mathtt{M}xx$
3. $x\mathtt{III}y \Rightarrow x\mathtt{U}y$
4. $x\mathtt{UU}y \Rightarrow xy$