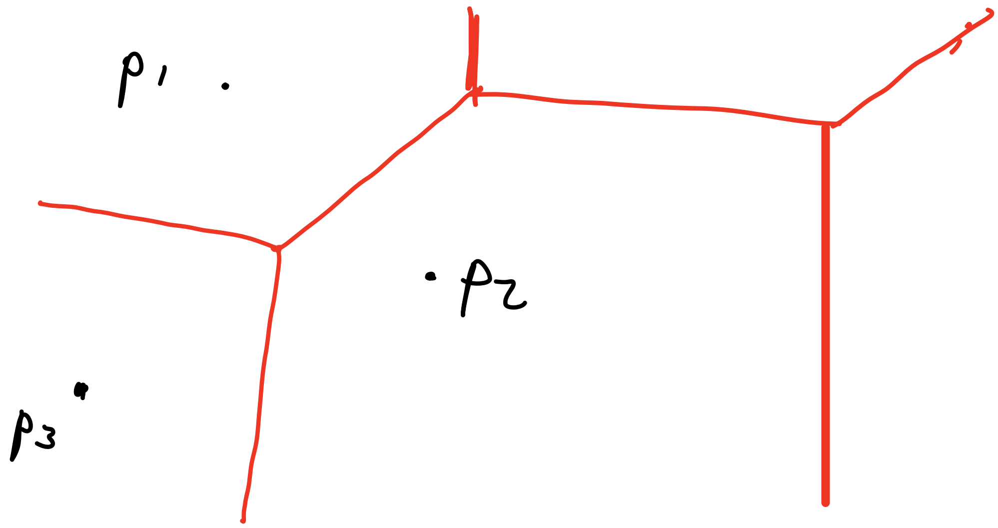
# Lecture 11 - Voronoi diagrams



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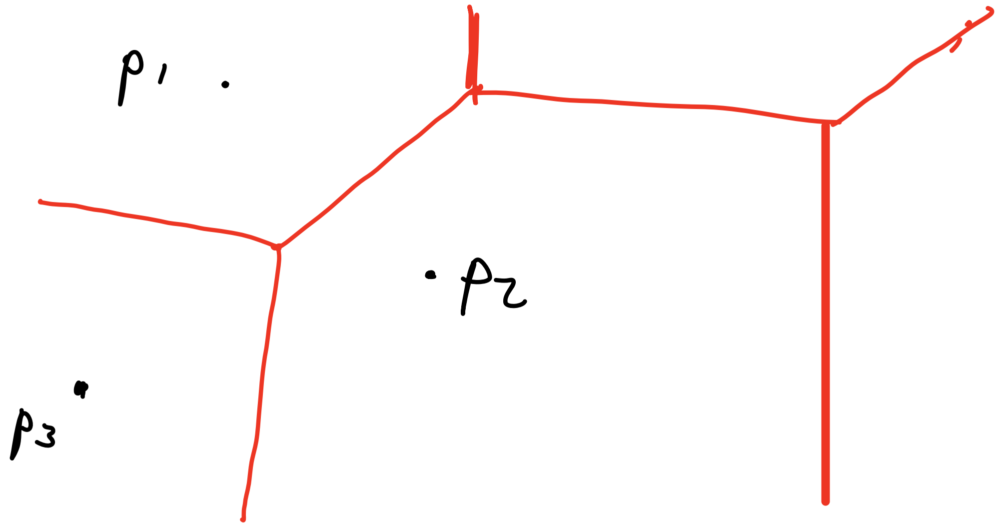


## Post-office problem

- City with post-offices $P = \{p_1, \ldots, p_n\}$.

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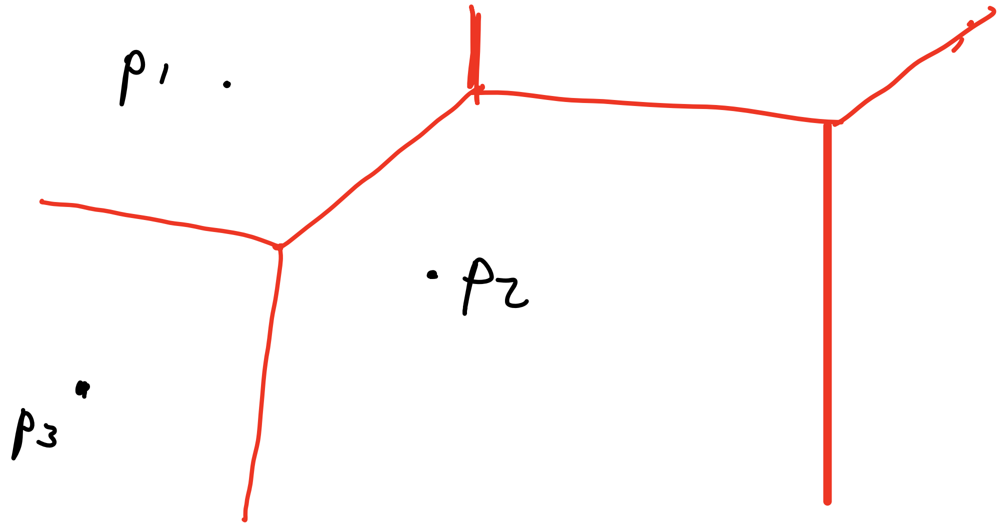


## Post-office problem

- City with post-offices $P = \{p_1, \ldots, p_n\}$.
- Divide city into regions $V(p_i)$ around each post-office $p_i$ such that any point in $V(p_i)$ is at least as close to $p_i$ as it is to any other $p_j$.

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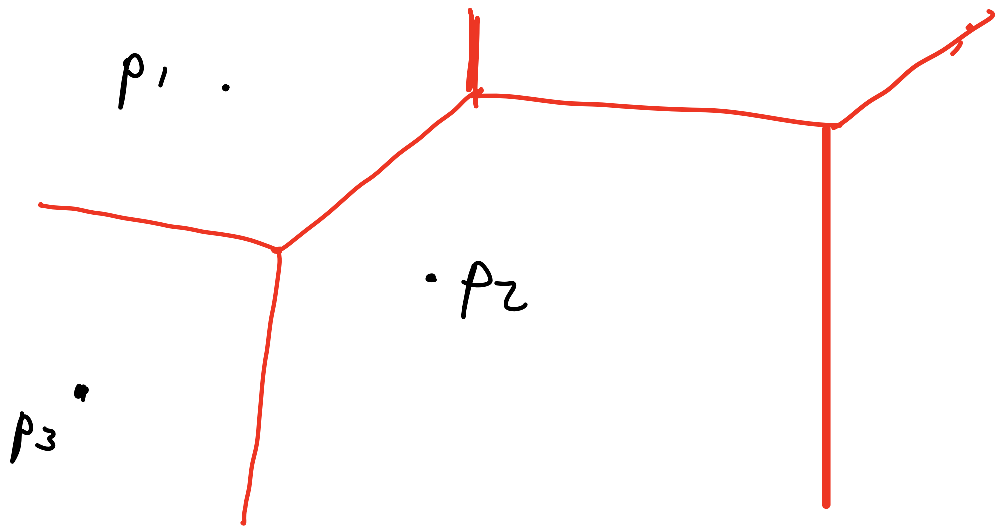


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## Note

Examples in nature:

- giraffe's skin (areas around cells secreting melatonin)
- show picture

Given $P = \{p_1, \ldots, p_n\}$, the Voronoi diagram $V(P)$ is subdivision of plane $\mathbb{R}^2$ into $n$ regions

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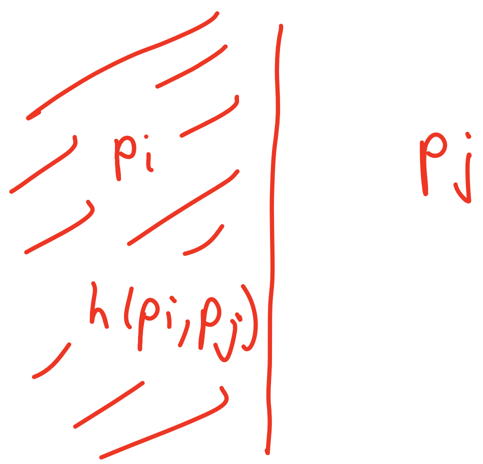
half-plane

- Let $h(p_i, p_j) = \{q : d(q, p_i) \leq d(q, p_j)\}$



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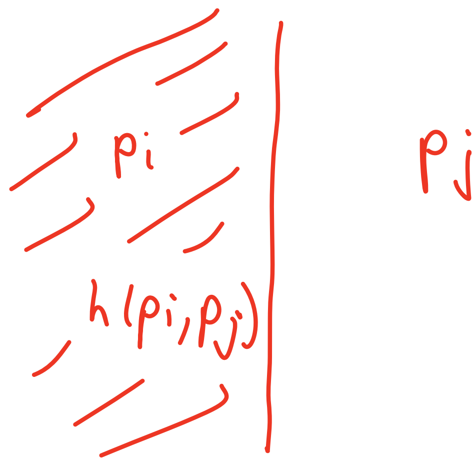


- Then $V(p_i) = \bigcap_{j \neq i} h(p_i, p_j)$

& as an intersection of half-planes is a convex polygon.

# Slow algorithm

Formula $$V(p_i) = \bigcap_{j \neq i} h(p_i, p_j)$$

- By Lecture 5, intersection of $n$ half-planes takes time $O(n \log n)$.

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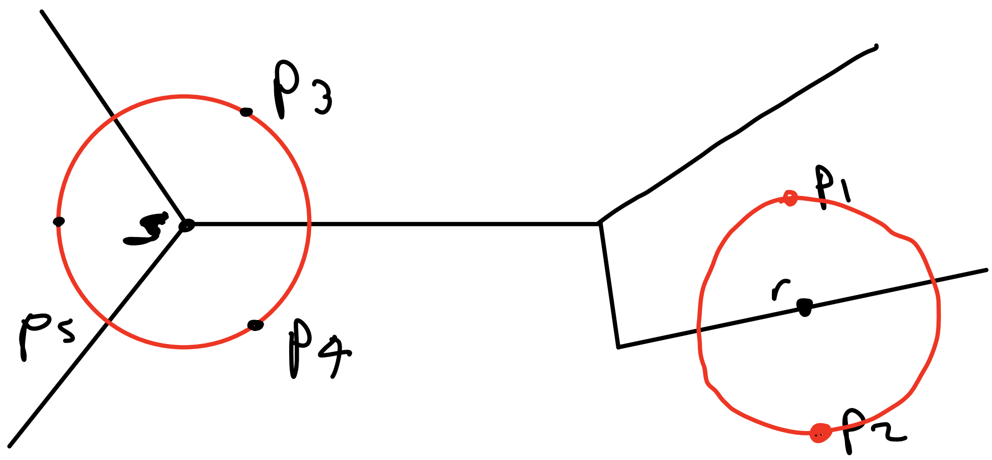
Today: Faster algorithm - $O(n \log n)$ ( Fortune's algorithm )

- A Sweepline algorithm

# Edges & vertices of Voronoi diagram

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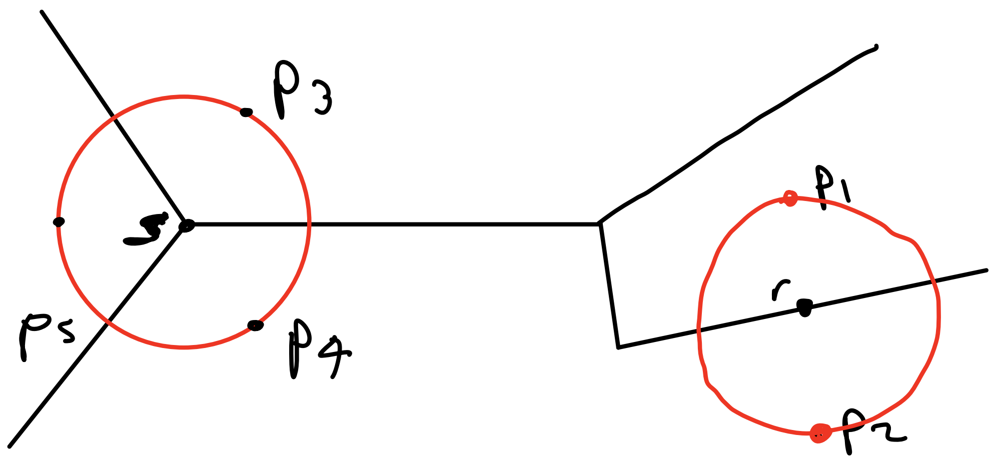


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(1) $r \in \mathbb{R}^2$ lie on an edge of V-diagram
$\Longleftrightarrow$ $r$ is equidistant from its nearest two points of $P$

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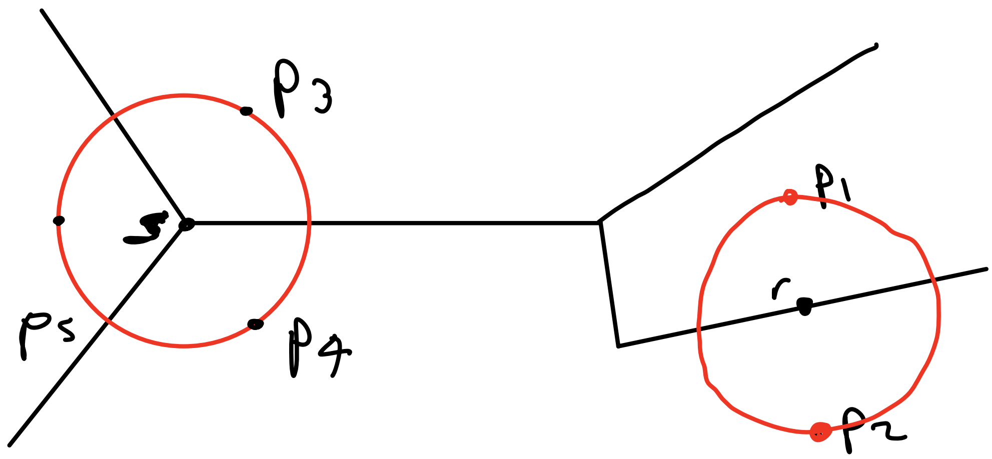


· It is not hard to see that:

① $r \in \mathbb{R}^2$ lie on an edge of V-diagram $\iff$ $r$ is equidistant from its nearest two points of $P$

② $r$ is a vertex of V-diagram $\iff$ $r$ is equidistant from its nearest three points.

See Fig 9.3 in E-Learning.

**Theorem)** Any U-diagram for a set of $n \geq 3$ points (not on a line) has at most $2n-5$ vertices & at most $3n-6$ edges.

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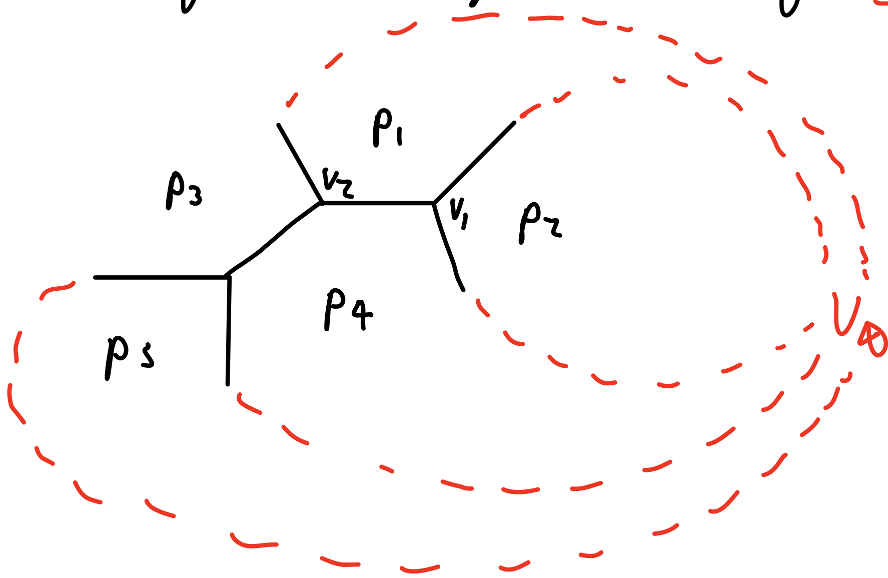
Proof) $m$ = no of vertices, $h$ = no of edges



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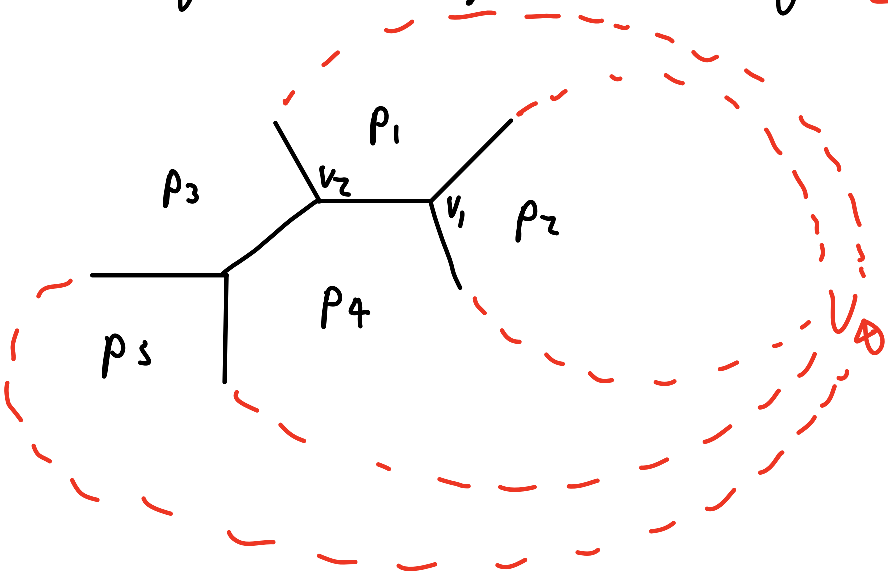


- Add vertex $v_\infty$ "at infinity" as endpoint for all half-lines.

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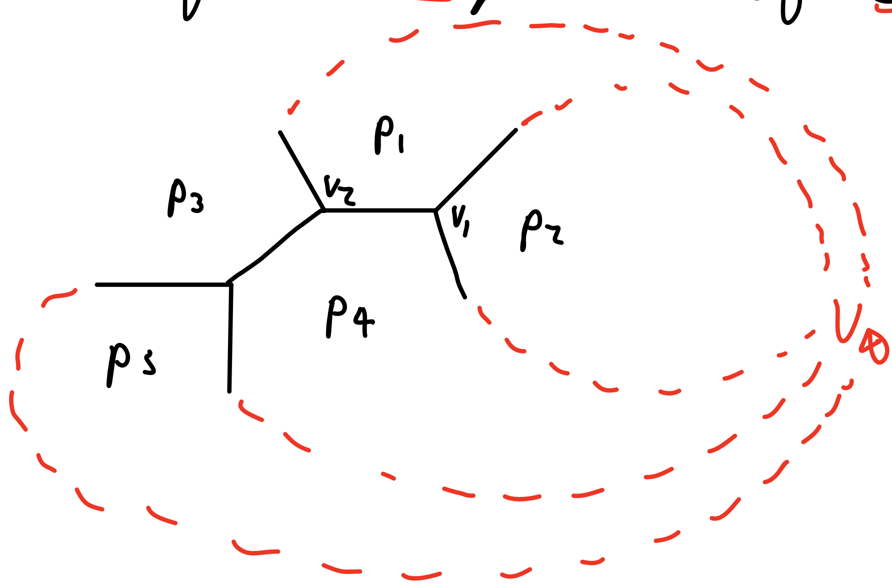


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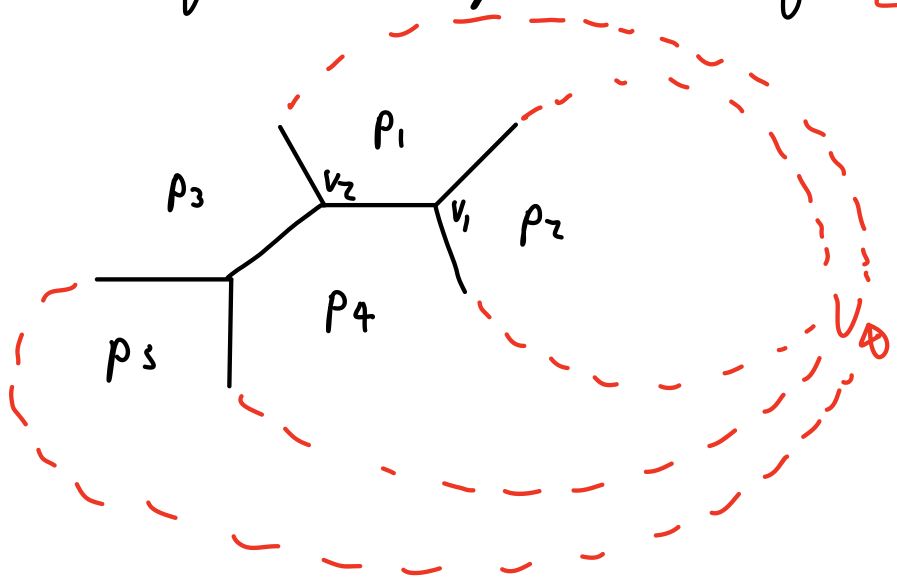


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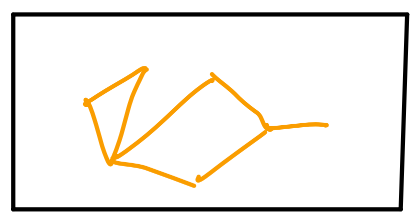
Aside: Euler Formula

$V - E + F = 2$

vertices edges faces

$7 - 8 + 3 = 2$



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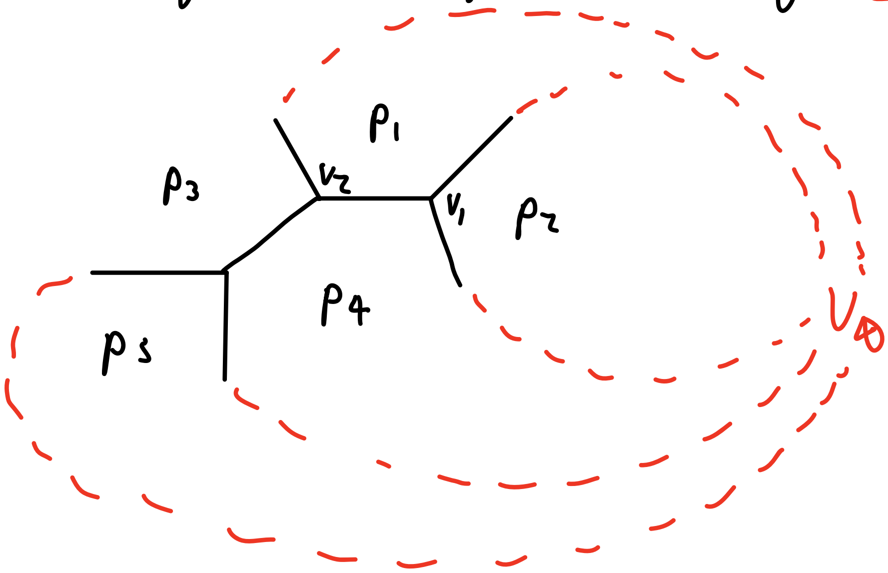


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Euler's formula

$(m+1) - h + n = 2$.

Aside: Euler formula

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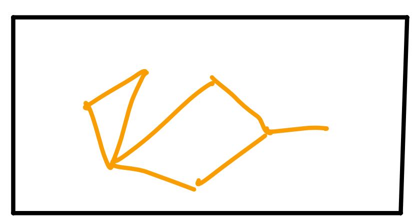

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$$\Rightarrow m \leq 2n-5.$$

- Sim., subbing $m = h-n+1$ into ✱ gives $h \leq 3n-6$.

$\square$

# Sweep-line algorithm - Fortune's algorithm

(See animation "Voronoi tesselation" on YouTube by K. Schaal )

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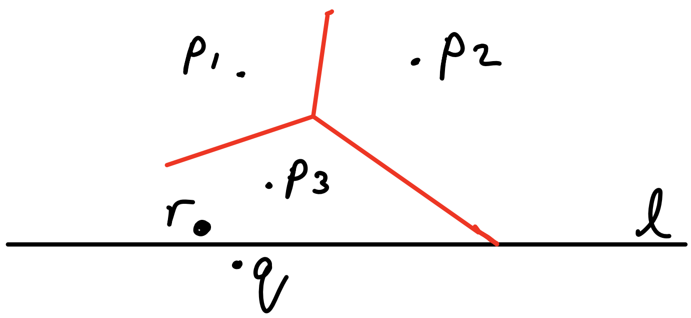
- Would like to use sweep-line approach to compute V-diag above sweep-line $\ell$.



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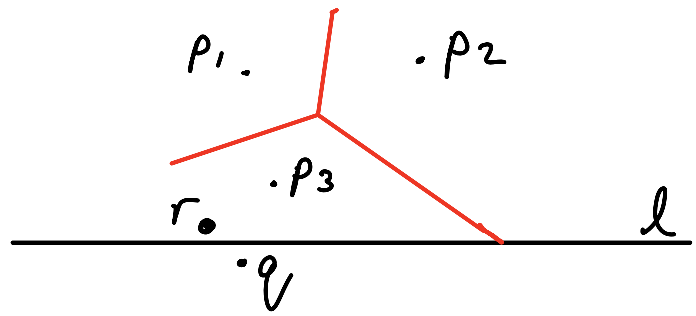


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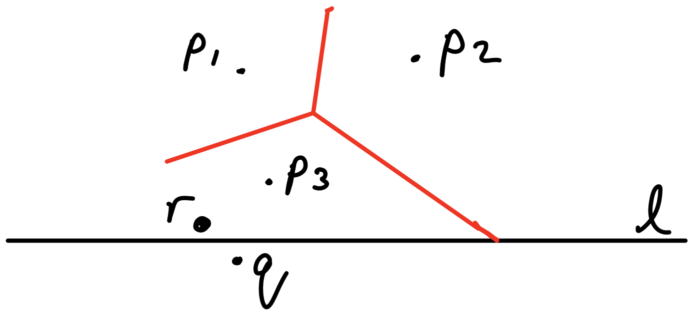


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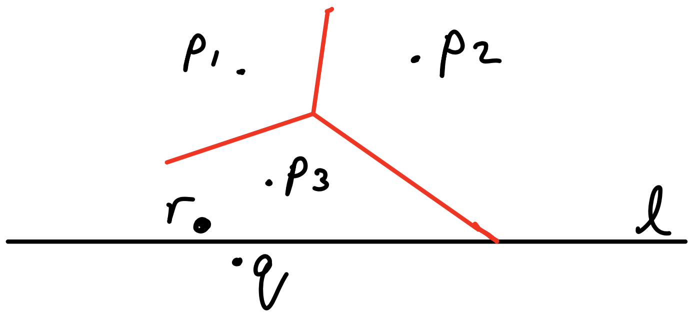


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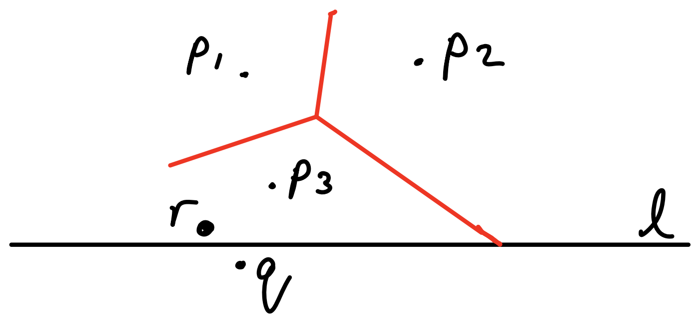

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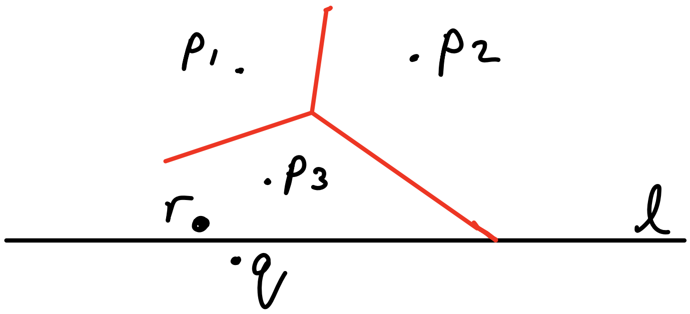


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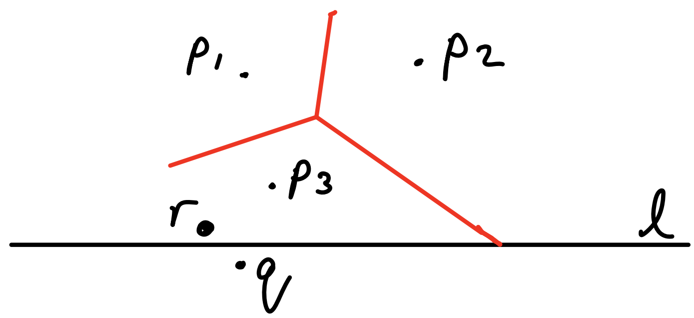


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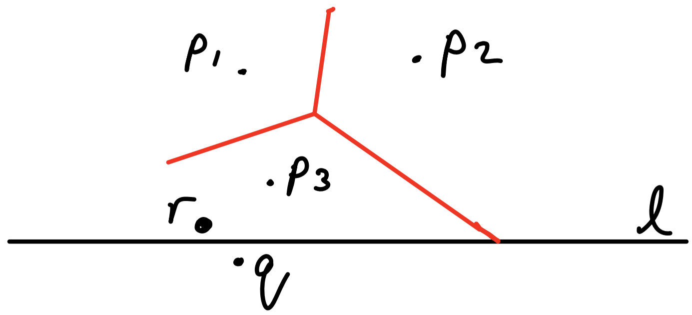


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- For $p$ above $\ell$,

  let $\alpha^+(p,\ell) = \{ x : d(x,p) \leq d(x,\ell) \}$

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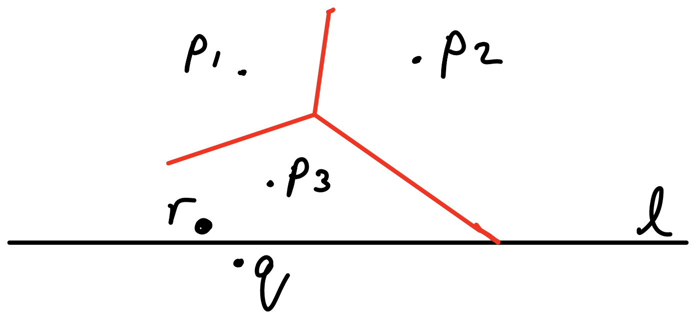


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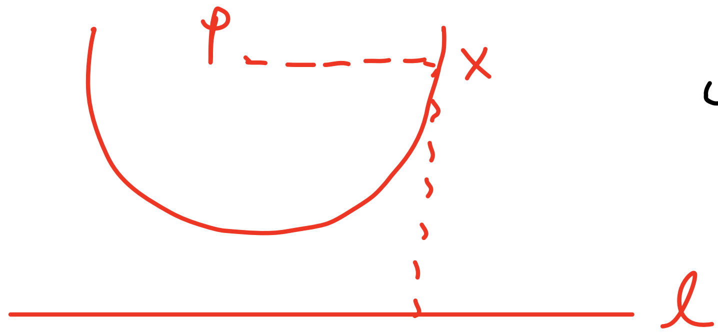
- By the above reasoning, we can correctly compute V-diagram in the region $\bigcup_{p \text{ above } \ell} \alpha^+(p,\ell)$

  as any point in this region is closer to some $p$ above $\ell$ than to any point below it.

Each $\alpha(p,\ell) = \{x : d(x,p) = d(x,\ell)\}$ is a parabola




w/ focus $p$ & directrix $\ell$

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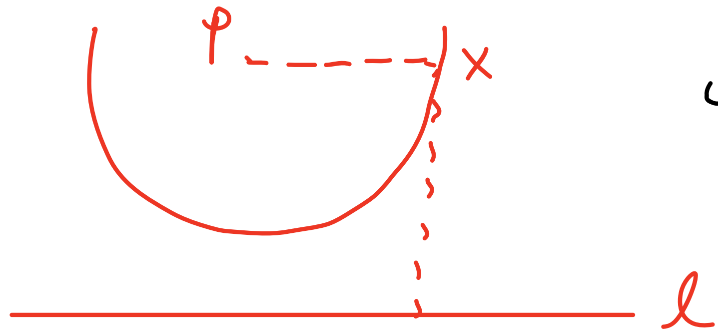


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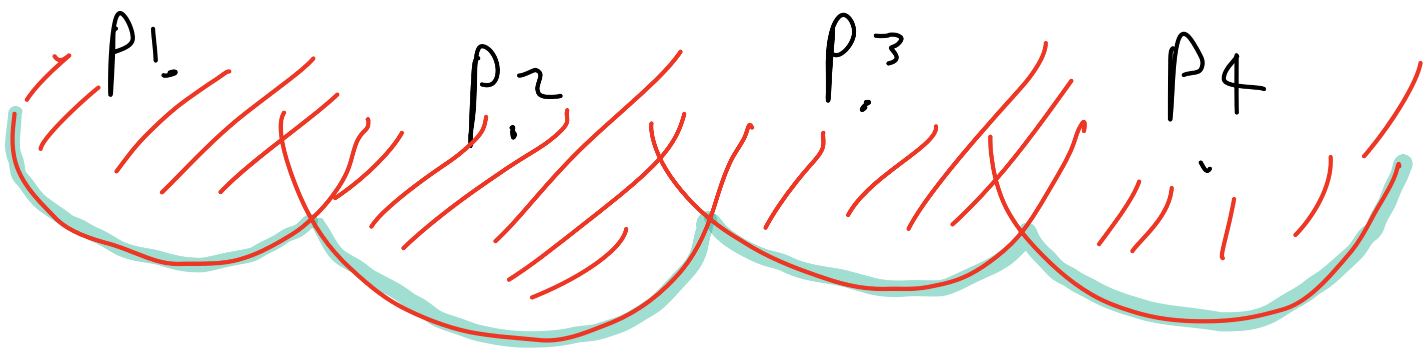
The boundary of $\bigcup_{p \text{ above } \ell} \alpha^+(p,\ell)$ consists
of arcs of paraboloi




boundary of this region
is called the beach-line

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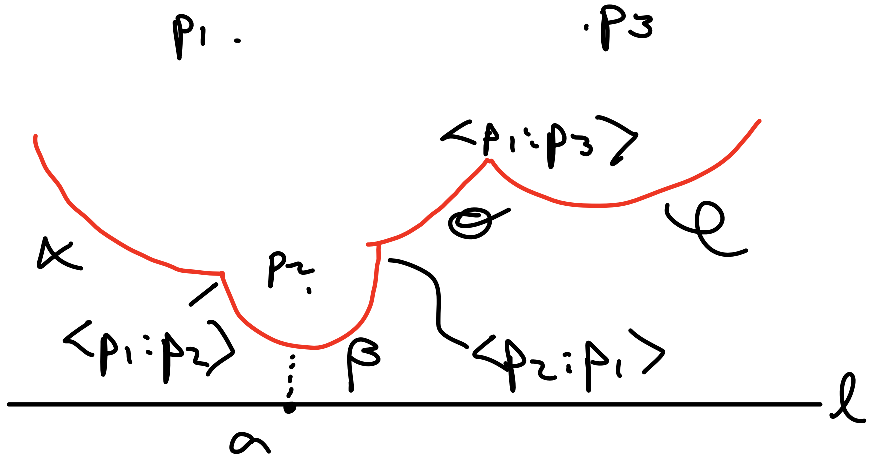
<u>Picture of beach-line @ $\ell$</u>:



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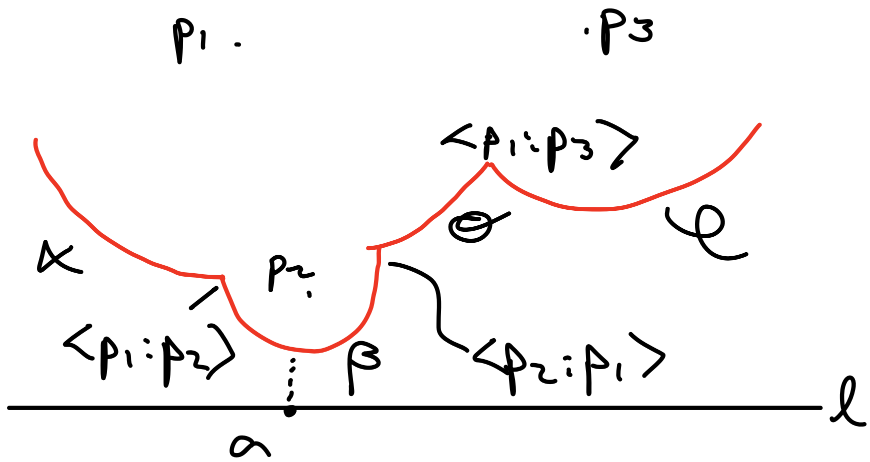
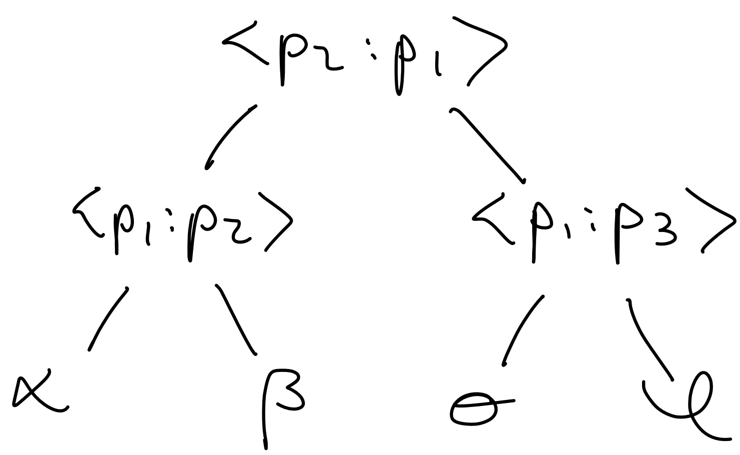



Associated tree :

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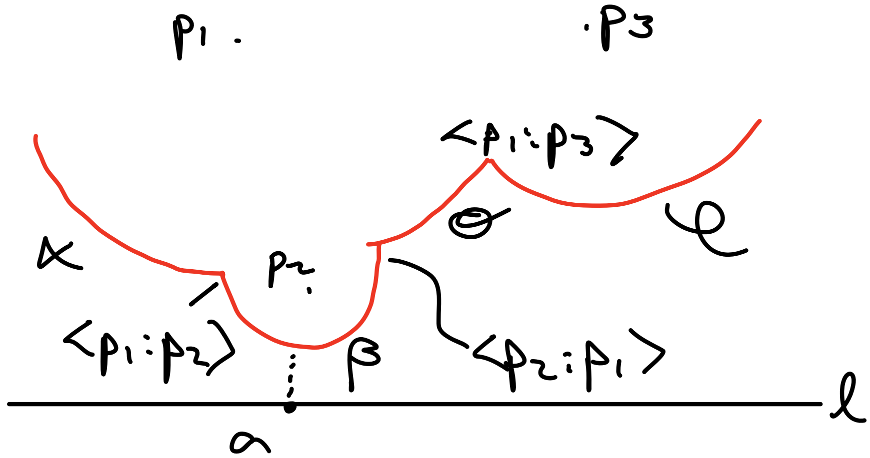


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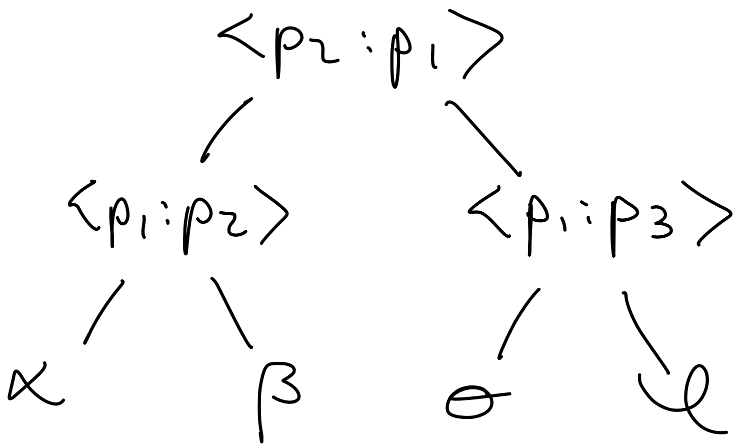


- leaves = arcs of beach-line
- Internal nodes of tree $\langle p_i : p_j \rangle$ represent "breakpoints" on beach-line, at which paraboloi around $p_i$ & $p_j$ meet with arc of $p_i$ to left & arc of $p_j$ to the right.

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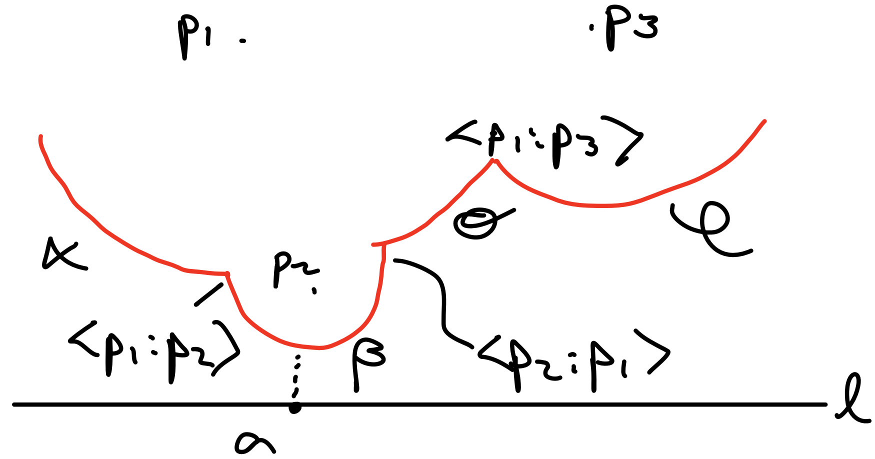
Picture of beach-line @ $\ell$ :

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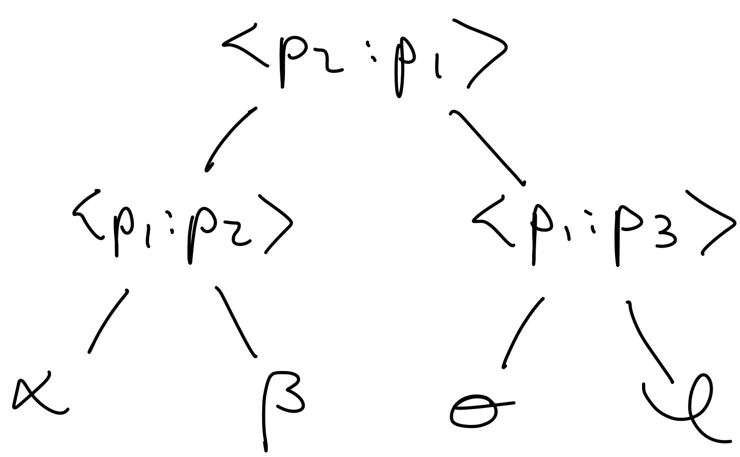


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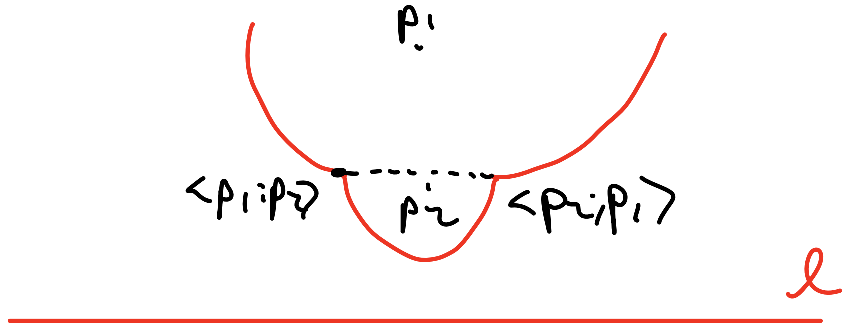
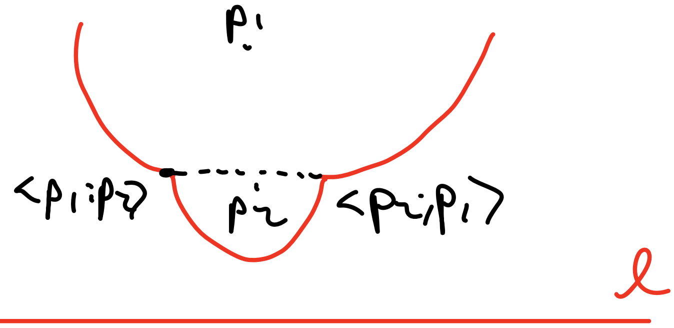
- Given a point $a$ on $\ell$, can search for arc of beach-line above $a$.

# Creating edges of Voronoi diagram 1



# Creating edges of Voronoi diagram I

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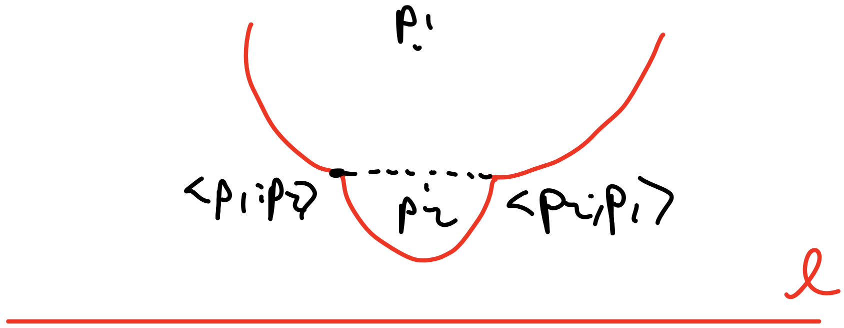


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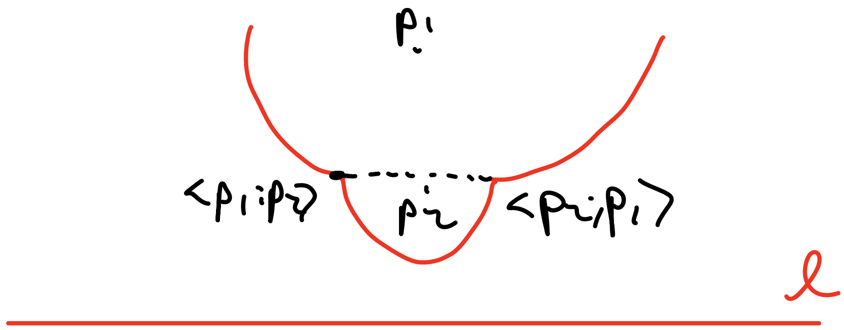
- This means that $r$ lies on an edge of the V-diagram.

# Creating edges of Voronoi diagram 1

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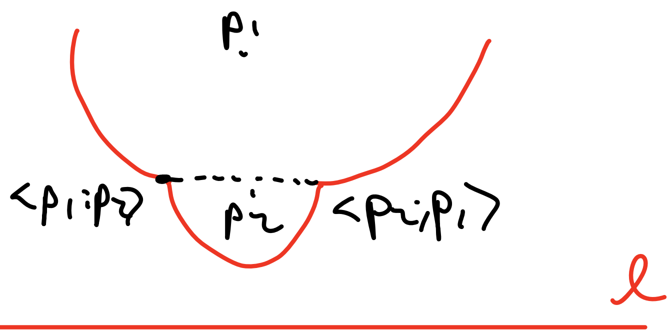
If $\langle p_2 : p_1 \rangle$ is on beach-line, then it has same distance from $p_1$ & $p_2$.

# Creating edges of Voronoi diagram I



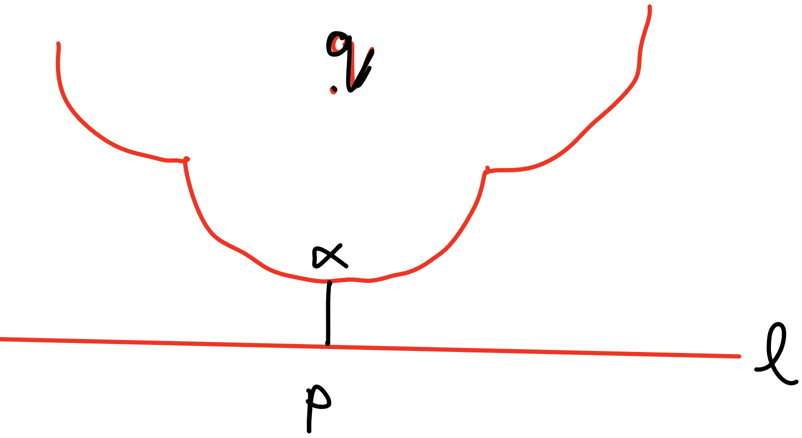

- At breakpoint $r = \langle p_1 : p_2 \rangle$, we have $d(r, p_1) = d(r, \ell) = d(r, p_2)$.
- This means that $r$ lies on an edge of the V-diagram.

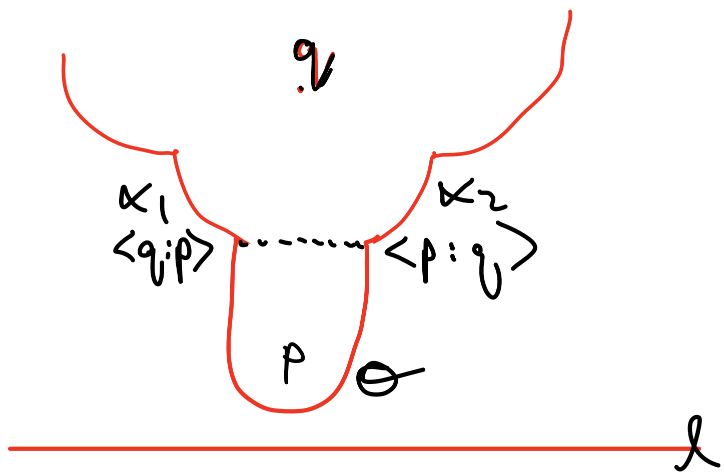
- If $\langle p_2 : p_1 \rangle$ is on beach-line, then it has same distance from $p_1$ & $p_2$.

- Therefore the edge from $\langle p_1 : p_2 \rangle$ to $\langle p_2 : p_1 \rangle$ will lie on $V(P)$

# Creating edges of Voronoi diagram 2

- A new arc appears just when the sweep-line passes a point of P.



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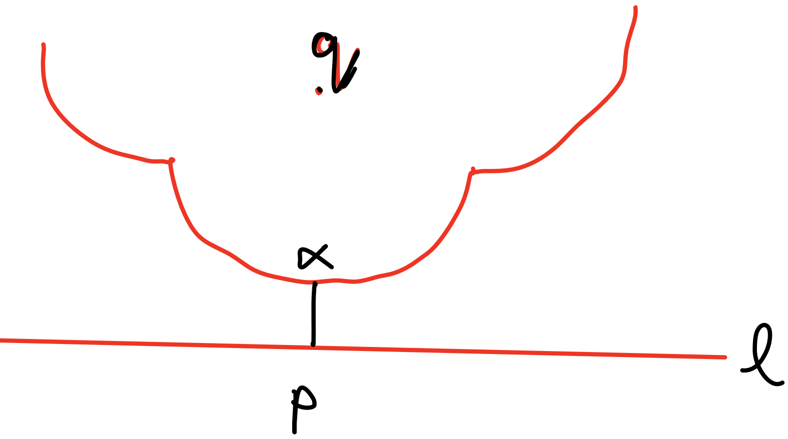


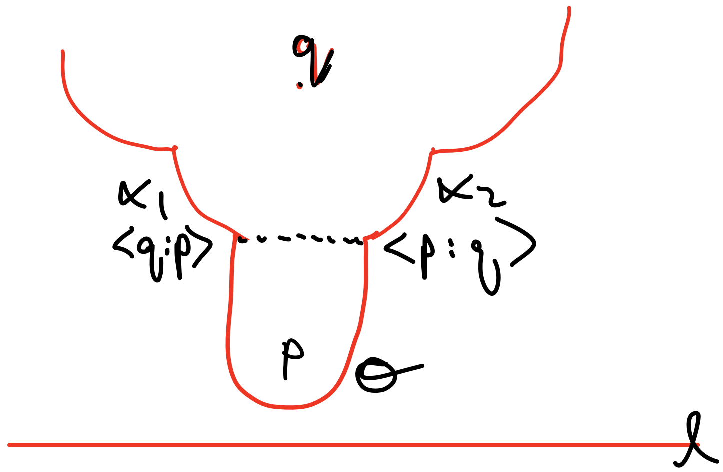


- In this case, we add edge between the new breakpoints.

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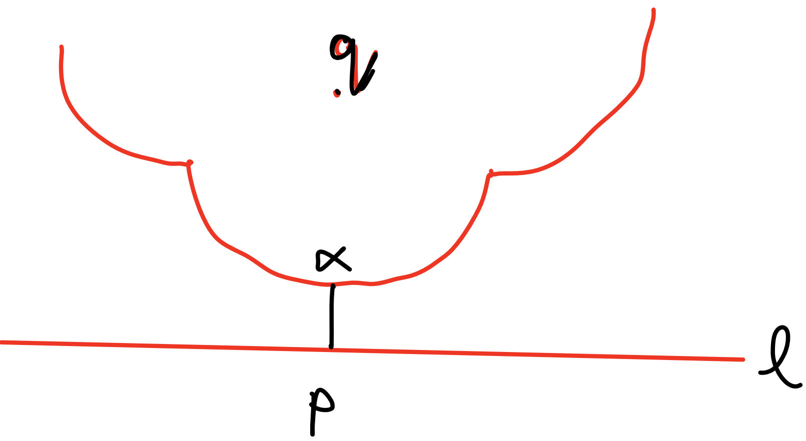
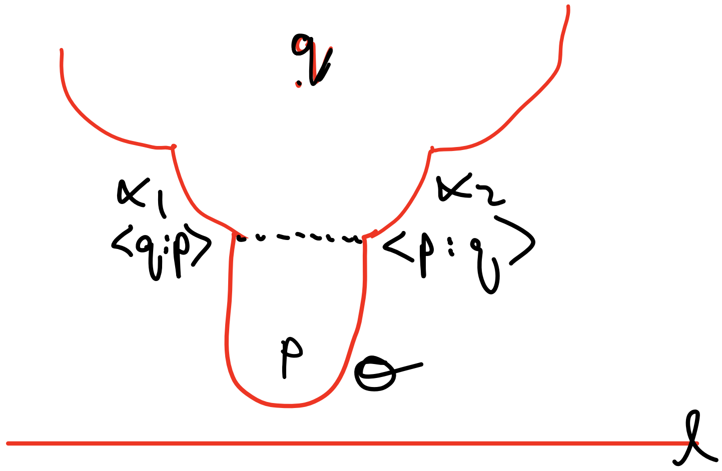


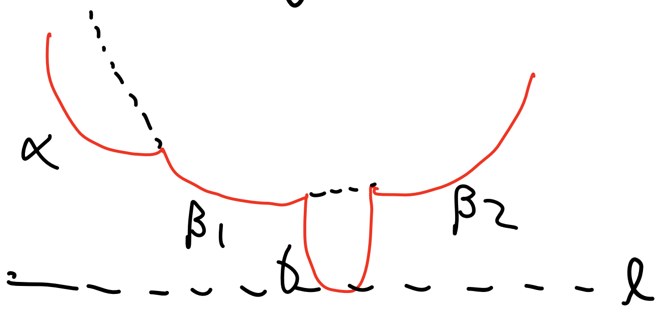




- In this case, we add edge between the new breakpoints.

- We will see that vertices of V(P) are created when an arc disappears from the beach-line.

# When does an arc disappear?

- Intuitively, when squeezed out by two adjacent arcs.



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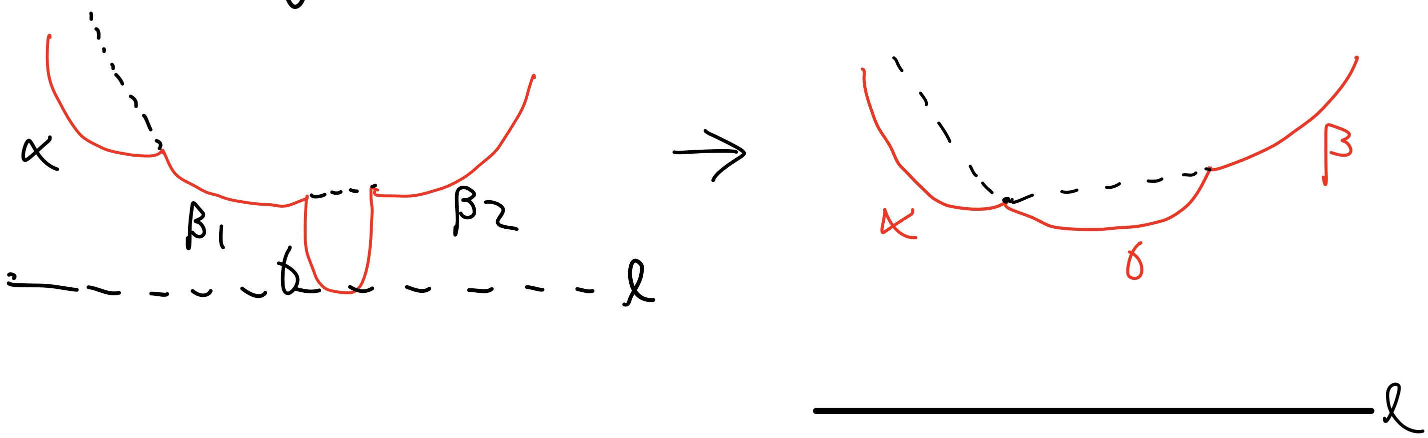

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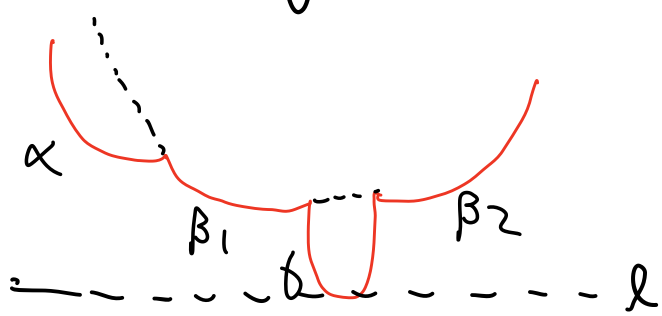


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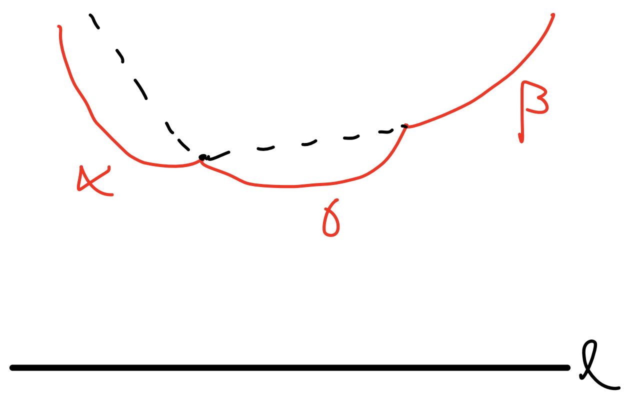


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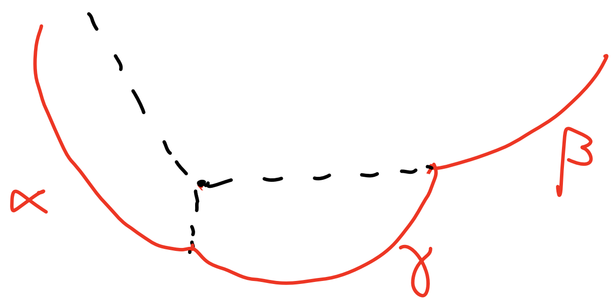


See Fig 9.8.

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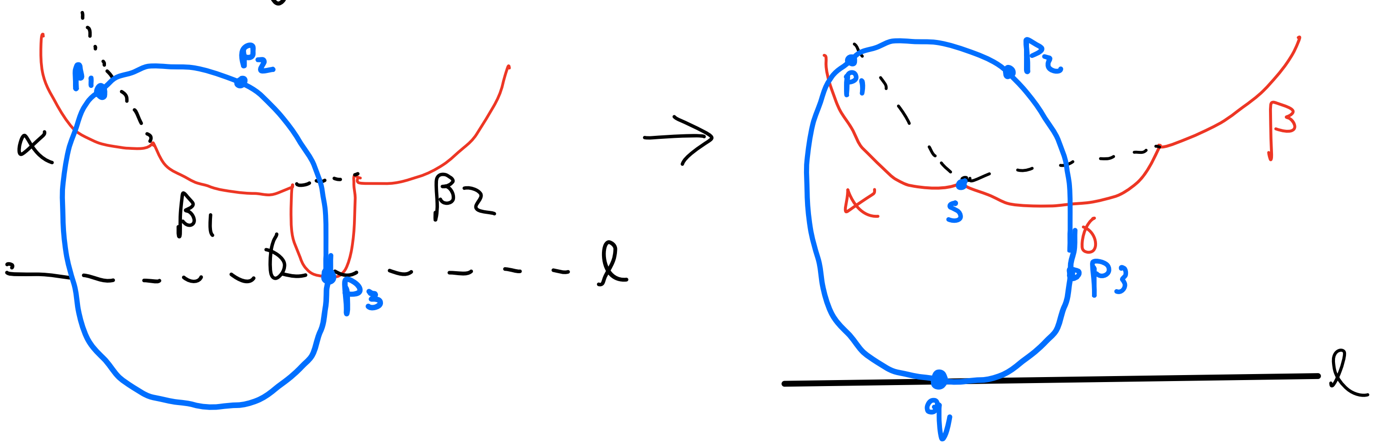


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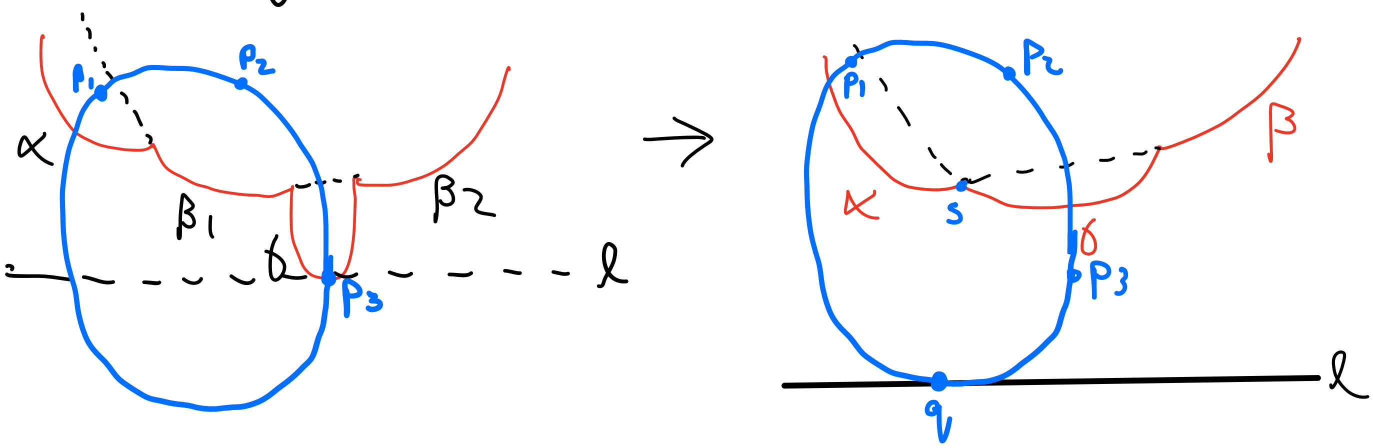


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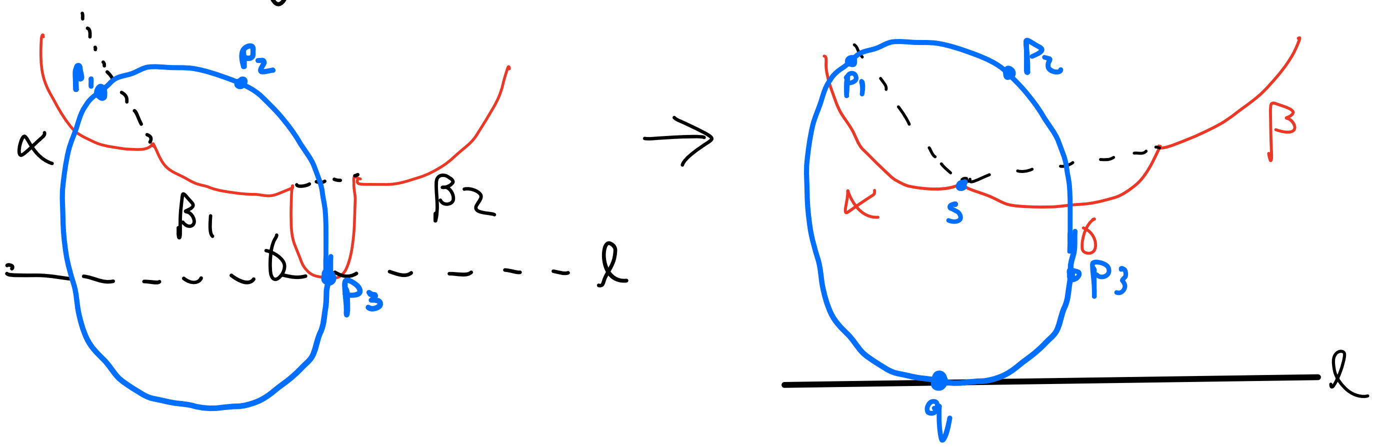


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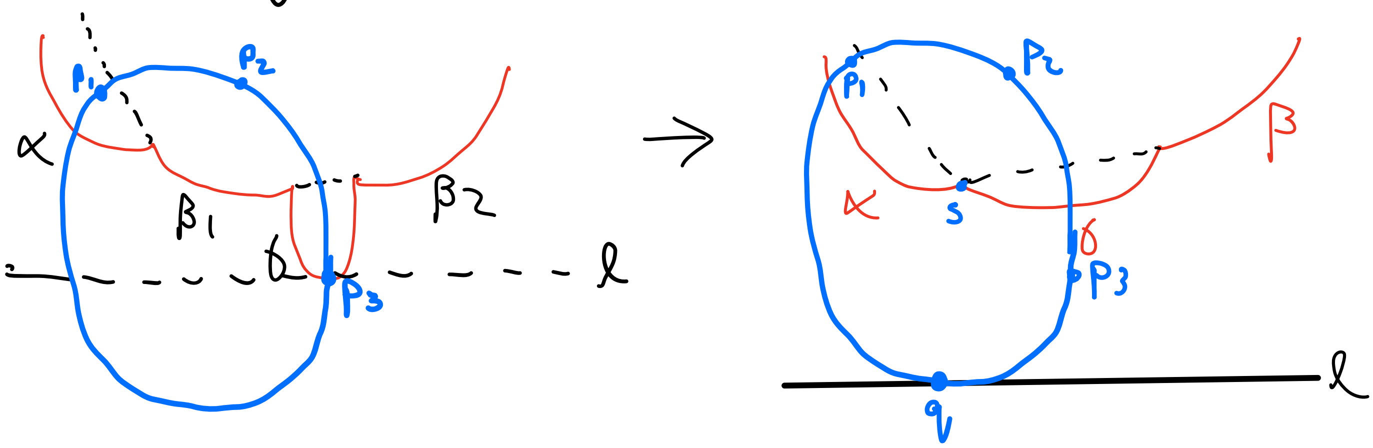


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- Because when $\ell$ passes $q$,
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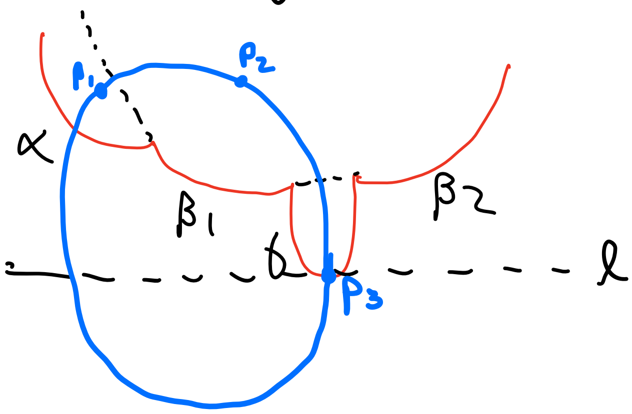


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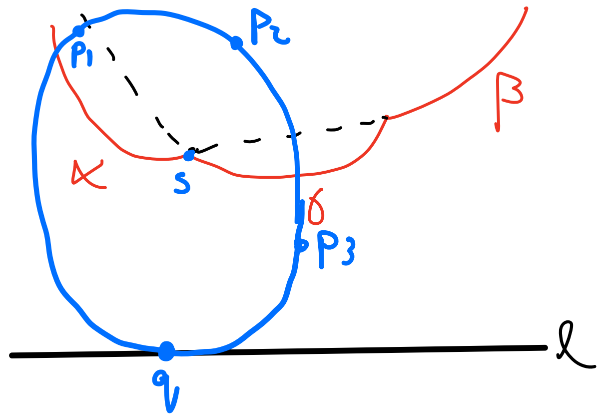


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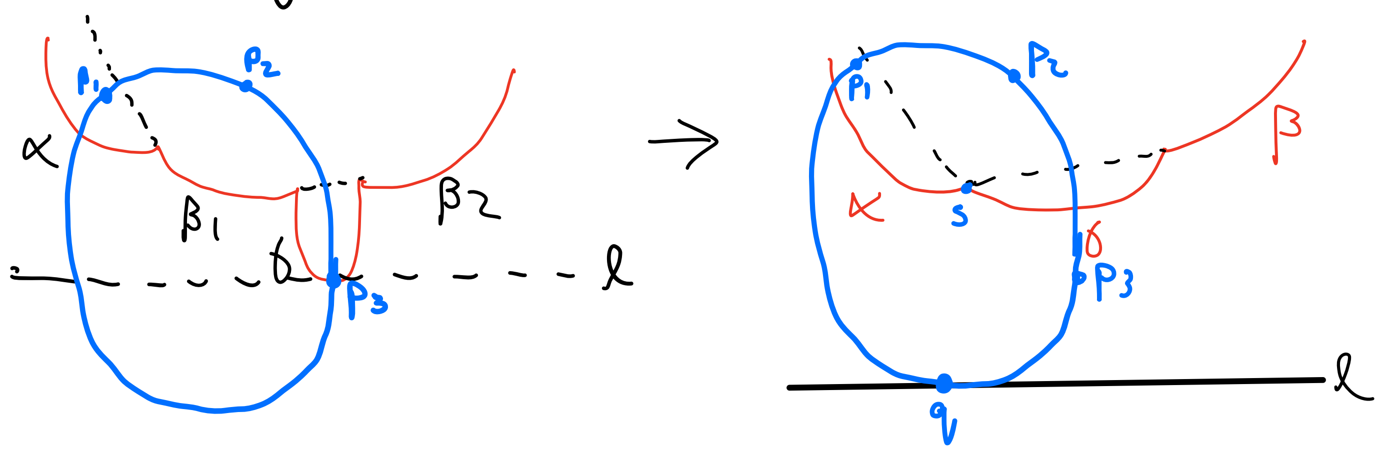


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- A new vertex is created at $s$, & a new half-edge.

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- event queue Q (actually a binbalT)
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- bbtree T for beach line
- DCEL for Voronoi diagram.

- Moreover each leaf of T (arc of beach line) has a pointer to the event queue, its circle event (if it exists, else nil.).

- Each internal node (break point on beach line) has pointer to half-edge in DCEL being traced out by break point.

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- At beginning, add all points of P to Q - called <u>site events</u>.

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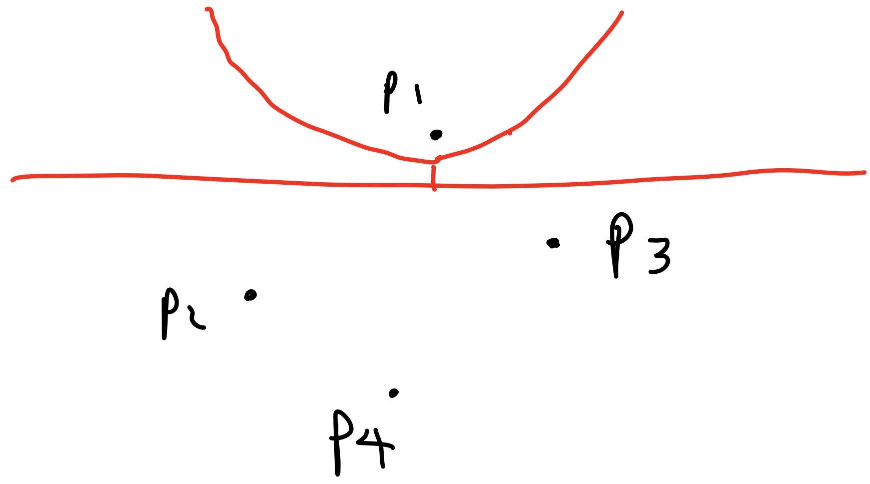
Complexity : $O(n \log n)$

- Lecture focused on geometric ideas involved in constructing V-diagram - see E-Learning for further detail on implementation, also thesis linked to at the end.

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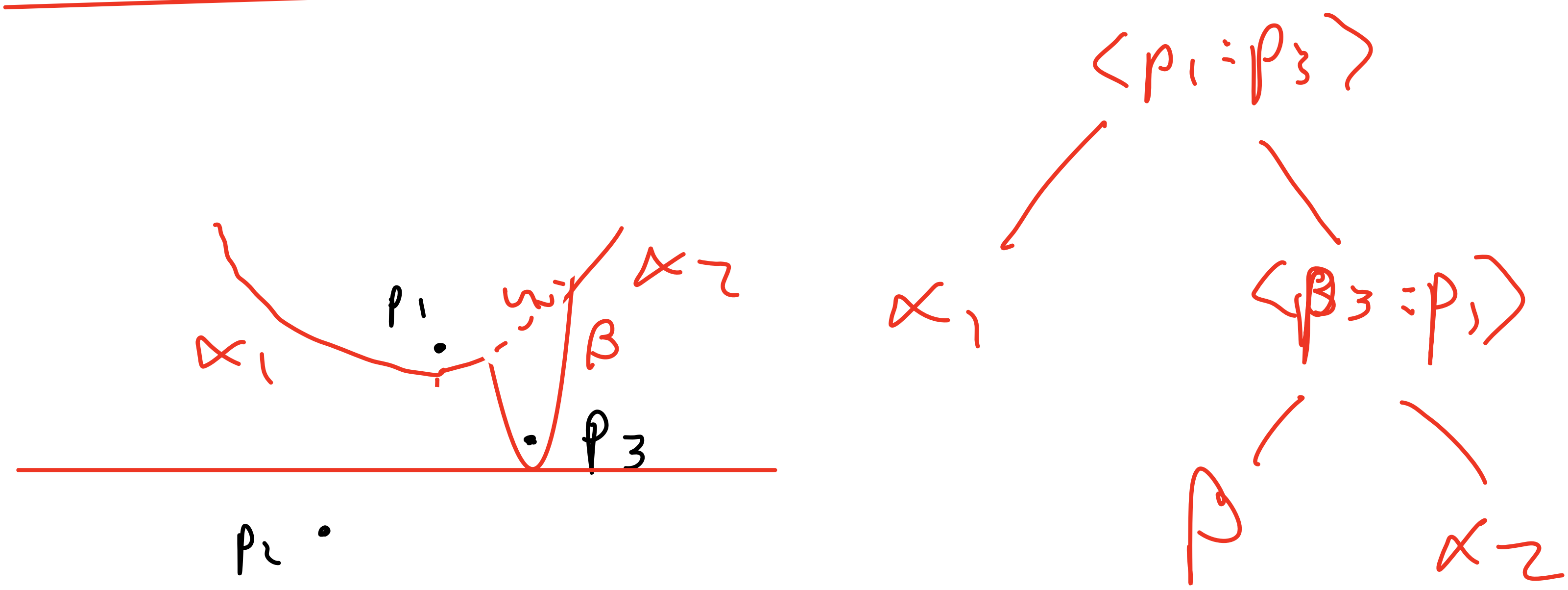


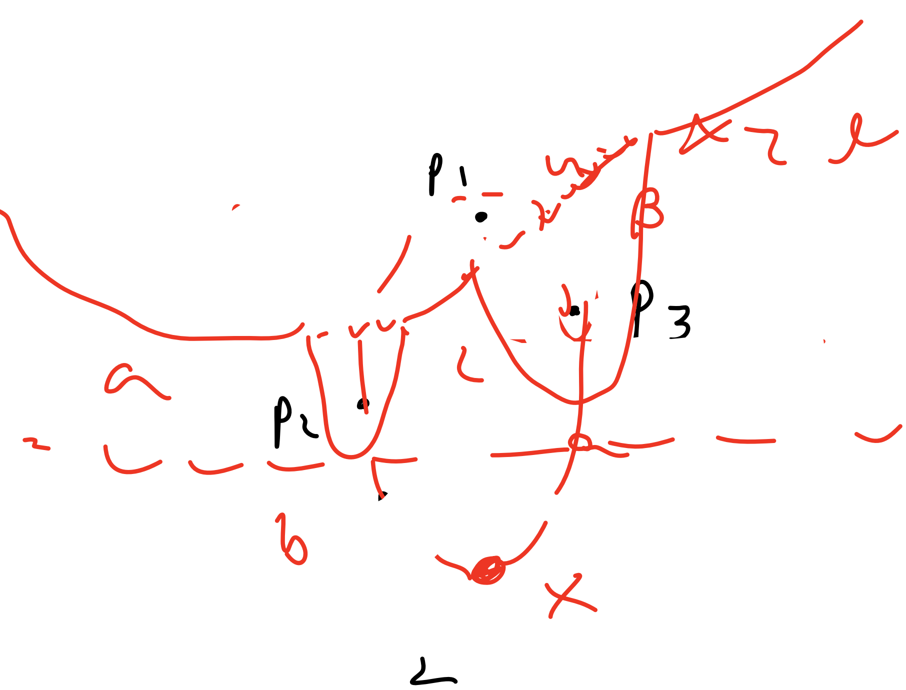


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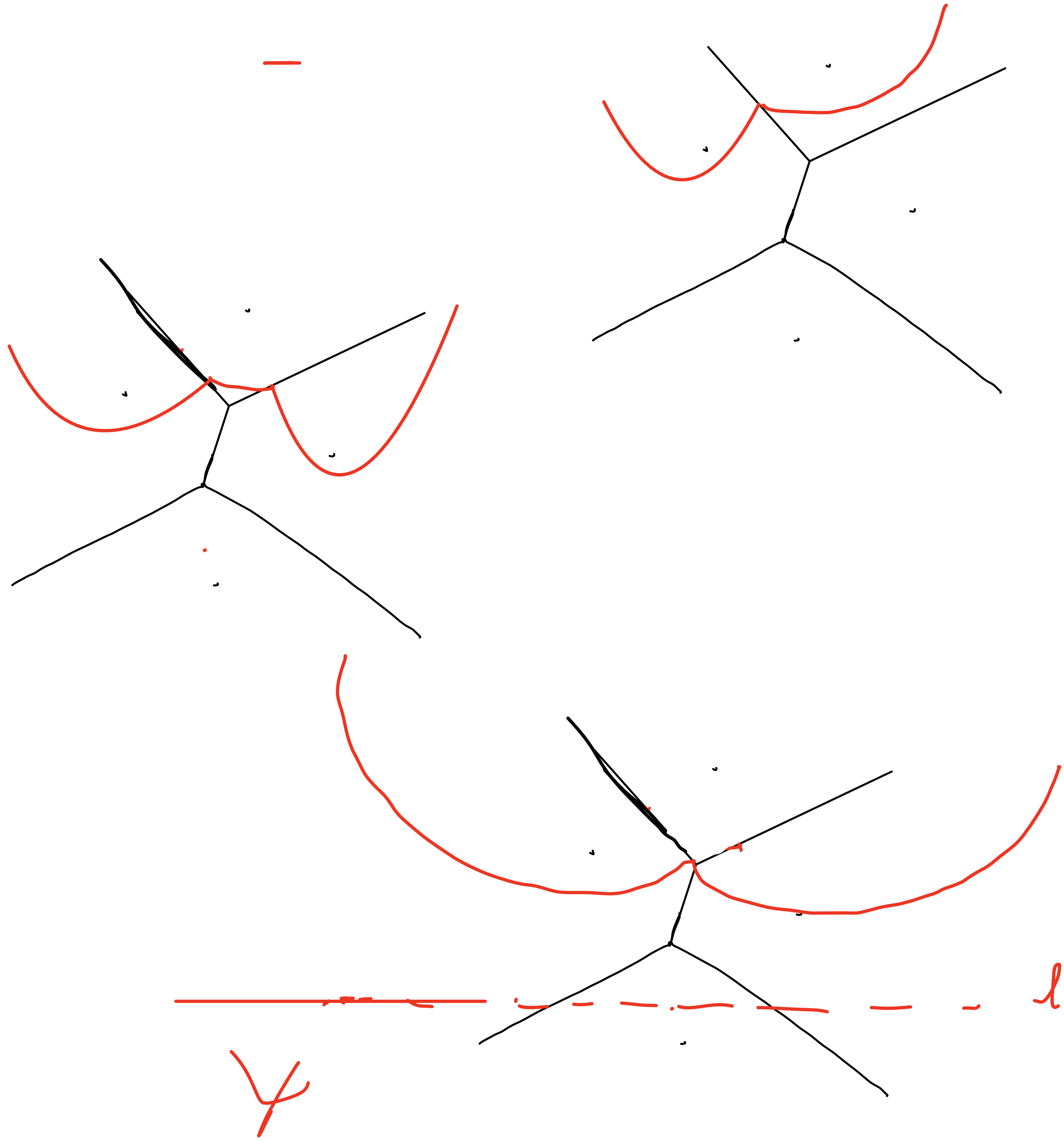
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