Integrace goniometrických funkcí.

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Obsah

Najděte $\displaystyle\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x.$

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Funkce je vzhledem k funkci $\cos x$ rac. lomená a v násobení se $\sin x$.

Najděte $\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x$.

$$\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x =$$

$\cos x = t$

Zavedeme substituci $\cos x = t$.

Najděte $\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x$.

$$\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x = \boxed{\begin{array}{c} \cos x = t \\ -\sin x\,\mathrm{d}x = \mathrm{d}t \\ \end{array}}$$

Diferencujeme.

Najděte $\displaystyle\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x$.

$$\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x = \boxed{\begin{aligned} \cos x &= t \\ -\sin x\,\mathrm{d}x &= \mathrm{d}t \\ \sin x\,\mathrm{d}x &= -\,\mathrm{d}t \end{aligned}}$$

Vyjádříme $\sin x\,\mathrm{d}x$.

Najděte $\displaystyle\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x.$

$$\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x = \boxed{\begin{aligned} \cos x &= t \\ -\sin x\,\mathrm{d}x &= \mathrm{d}t \\ \sin x\,\mathrm{d}x &= -\,\mathrm{d}t \end{aligned}}$$

$$= -\int \frac{\mathrm{d}t}{t^2 + 1}$$

Dosadíme.

Najděte $\displaystyle\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x.$

$$\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x = \boxed{\begin{aligned} \cos x &= t \\ -\sin x\,\mathrm{d}x &= \mathrm{d}t \\ \sin x\,\mathrm{d}x &= -\,\mathrm{d}t \end{aligned}}$$

$$= -\int \frac{\mathrm{d}t}{t^2 + 1} = -\operatorname{arctg} t + c$$

Integrujeme.

Najděte $\displaystyle\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x$.

$$\int \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x = \boxed{\begin{array}{c} \cos x = t \\ -\sin x\,\mathrm{d}x = \mathrm{d}t \\ \sin x\,\mathrm{d}x = -\,\mathrm{d}t \end{array}}$$

$$= -\int \frac{\mathrm{d}t}{t^2+1} = -\operatorname{arctg} t + c = -\operatorname{arctg}(\cos x) + c$$

Navrátíme se k původní proměnné.

Najděte $\int \sin^2 x \cos^3 x \,\mathrm{d}x$.

$$\int \sin^2 x \cos^3 x \,\mathrm{d}x =$$

Najděte $\int \sin^2 x \cos^3 x \, dx$.

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$$

Funkce, které jsou vzhledem ke $\cos x$ v liché mocnině, je vhodné rozepsat vytknutím $\cos x$

Najděte $\int \sin^2 x \cos^3 x \, dx$.

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

a přepisem pomocí vzorce $\cos^2 x = 1 - \sin^2 x$ (analogicky pro $\sin x$).

Najděte $\int \sin^2 x \cos^3 x \,\mathrm{d}x$.

$$\int \sin^2 x \cos^3 x \,\mathrm{d}x = \int \sin^2 x \cos^2 x \cos x \,\mathrm{d}x$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \,\mathrm{d}x =$$

$$\boxed{\sin x = t}$$

Zavedeme substituci $\sin x = t$.

Najděte $\int \sin^2 x \cos^3 x \,\mathrm{d}x$.

$$\int \sin^2 x \cos^3 x \,\mathrm{d}x = \int \sin^2 x \cos^2 x \cos x \,\mathrm{d}x$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \,\mathrm{d}x = \boxed{\begin{array}{c} \sin x = t \\ \cos x \,\mathrm{d}x = \mathrm{d}t \end{array}}$$

Diferencujeme.

Najděte $\int \sin^2 x \cos^3 x \, dx$.

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx = \boxed{\begin{array}{c} \sin x = t \\ \cos x \, dx = dt \end{array}}$$

$$= \int t^2 (1 - t^2) \, dt$$

Dosadíme.

Najděte $\int \sin^2 x \cos^3 x \, \mathrm{d}x$.

$$\int \sin^2 x \cos^3 x \, \mathrm{d}x = \int \sin^2 x \cos^2 x \cos x \, \mathrm{d}x$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, \mathrm{d}x = \boxed{\begin{aligned} \sin x &= t \\ \cos x \, \mathrm{d}x &= \mathrm{d}t \end{aligned}}$$

$$= \int t^2 (1 - t^2) \, \mathrm{d}t = \int t^2 - t^4 \, \mathrm{d}t$$

Roznásobíme.

Najděte $\int \sin^2 x \cos^3 x \, \mathrm{d}x$.

$$\int \sin^2 x \cos^3 x \, \mathrm{d}x = \int \sin^2 x \cos^2 x \cos x \, \mathrm{d}x$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, \mathrm{d}x = \boxed{\begin{array}{c} \sin x = t \\ \cos x \, \mathrm{d}x = \mathrm{d}t \end{array}}$$

$$= \int t^2 (1 - t^2) \, \mathrm{d}t = \int t^2 - t^4 \, \mathrm{d}t = \frac{t^3}{3} - \frac{t^5}{5} + c$$

Integrujeme.

Najděte $\int \sin^2 x \cos^3 x \, dx$.

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx = \boxed{\begin{array}{c} \sin x = t \\ \cos x \, dx = dt \end{array}}$$

$$= \int t^2 (1 - t^2) \, dt = \int t^2 - t^4 \, dt = \frac{t^3}{3} - \frac{t^5}{5} + c$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

Navrátíme se k původní proměnné.

Najděte $\displaystyle\int \frac{1}{\sin x}\,\mathrm{d}x.$

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$$\int \frac{1}{\sin x}\,\mathrm{d}x$$

Integrand je vzhledem k funkci $\sin x$ v liché mocnině, proto budeme volit substituci $t = \cos x$. Musíme tedy dostat do čitatele $\sin x$.

Najděte $\int \frac{1}{\sin x}\,\mathrm{d}x$.

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x$$

Rozšíříme zlomek.

Najděte $\displaystyle\int \frac{1}{\sin x}\,\mathrm{d}x$.

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

Jmenovatel přepíšeme pomocí vzorce $\sin^2 x = 1 - \cos^2 x$.

Najděte $\int \frac{1}{\sin x}\,\mathrm{d}x$.

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

$$= \boxed{\cos x = t}$$

Zavedeme substituci $\cos x = t$.

Najděte $\displaystyle\int \frac{1}{\sin x}\,\mathrm{d}x$.

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} \cos x = t \\ -\sin x\,\mathrm{d}x = \mathrm{d}t \\ \end{array}}$$

Diferencujeme.

Najděte $\displaystyle\int \frac{1}{\sin x}\,\mathrm{d}x$.

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

$$= \boxed{\begin{aligned} \cos x &= t \\ -\sin x\,\mathrm{d}x &= \mathrm{d}t \\ \color{blue}{\sin x\,\mathrm{d}x} &\color{blue}{= -\,\mathrm{d}t} \end{aligned}}$$

Vyjádříme $\sin x\,\mathrm{d}x$.

$$\boxed{\text{Najděte } \int \frac{1}{\sin x}\,\mathrm{d}x.}$$

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

$$= \boxed{\begin{aligned} \cos x &= t \\ -\sin x\,\mathrm{d}x &= \mathrm{d}t \\ \sin x\,\mathrm{d}x &= -\mathrm{d}t \end{aligned}} = -\int \frac{\mathrm{d}t}{1-t^2}$$

Dosadíme.

Najděte $\displaystyle\int \frac{1}{\sin x}\,\mathrm{d}x$.

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

$$= \boxed{\begin{aligned} \cos x &= t \\ -\sin x\,\mathrm{d}x &= \mathrm{d}t \\ \sin x\,\mathrm{d}x &= -\,\mathrm{d}t \end{aligned}} = -\int \frac{\mathrm{d}t}{1-t^2} = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c$$

Integrujeme.

$$\boxed{\text{Najděte } \int \frac{1}{\sin x}\,\mathrm{d}x.}$$

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} \cos x = t \\ -\sin x\,\mathrm{d}x = \mathrm{d}t \\ \sin x\,\mathrm{d}x = -\,\mathrm{d}t \end{array}} = -\int \frac{\mathrm{d}t}{1-t^2} = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c$$

$$= -\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + c =$$

Navrátíme se k původní proměnné.

$$\boxed{\text{Najděte } \int \frac{1}{\sin x}\,\mathrm{d}x.}$$

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\,\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\,\mathrm{d}x$$

$$= \boxed{\begin{array}{c} \cos x = t \\ -\sin x\,\mathrm{d}x = \mathrm{d}t \\ \sin x\,\mathrm{d}x = -\,\mathrm{d}t \end{array}} = -\int \frac{\mathrm{d}t}{1-t^2} = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c$$

$$= -\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + c = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + c$$

Lze upravit.

KONEC