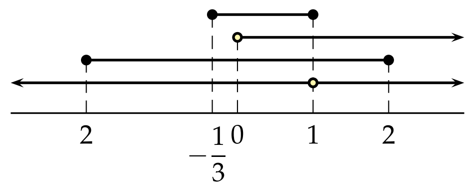
A.

1.
Najděte definiční obor funkce

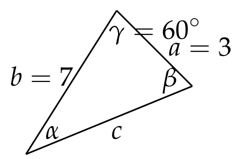
$$f(x) = \frac{\arcsin \frac{3x-1}{2}}{\ln x} + \sqrt{4-x^2}$$

Řešení:
$\ln x \neq 0 \;\Rightarrow\; x \neq 1$
$4 - x^2 \geq 0 \;\Rightarrow\; 4 \geq x^2 \;\Rightarrow\; 2 \geq |x|$
$x > 0$
$-1 \leq \dfrac{3x-1}{2} \leq 1 \;\Rightarrow\; -2 \leq 3x-1 \leq 2 \;\Rightarrow\; -1 \leq 3x \leq 3 \;\Rightarrow\; -\dfrac{1}{3} \leq x \leq 1$

$D(f) = (0,1)$



2.
V trojúhelníku $\triangle ABC$ je dáno $a = 3$, $b = 7$, $\gamma = 60^\circ$. Vypočtěte c a α s přesností na minuty.
Řešení:



$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$c^2 = 9 + 49 - 2\cdot 3\cdot 7\cdot \frac{1}{2} = 37$$
$$c = \sqrt{37}$$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$$
$$\sin\alpha = \frac{3}{\sqrt{37}}\cdot\frac{\sqrt{3}}{2} \doteq 0.427$$
$$\alpha \doteq 25.285^\circ = 25^\circ 17'$$

3.
Řešte rovnici v $\mathbb{C}$:

$$\frac{z+1}{z+i} = 3+i$$

Řešení:

$$z+1 = (3+i)(z+i)$$
$$z+1 = 3z+3i+iz-1$$
$$-2z - iz = 3i - 2$$
$$z(2+i) = 2-3i$$

$$z = \frac{2-3i}{2+i}\cdot\frac{2-i}{2-i}$$
$$z = \frac{4-2i-6i-3}{5} = \frac{1}{5} - \frac{8}{5}i$$

4.
Vydělte polynomy:

$$(x^4 + 3x^3 - 2x + 1) : (x^2 + 3x - 1)$$

Řešení:

$$\begin{array}{l}
(x^4 \;\; +3x^3 \; -2x \; +1) : (x^2+3x-1) = x^2 + 1 + \dfrac{-5x+2}{x^2+3x-1} \\
\underline{-(x^4 + 3x^3 - x^2} \\
\qquad\quad x^2 \quad -2x \; +1 \\
\qquad \underline{-(x^2 + 3x - 1)} \\
\qquad\qquad\quad -5x \; +2
\end{array}$$

5.
Najděte rozklad polynomu na kořenové činitele (kořeny jsou celá čísla):

$$P(x) = x^4 + 10x^3 + 24x^2 - 10x - 25$$

Řešení:

	1	10	24	−10	−25
1	1	11	35	25	0
−1	1	10	25	0	

$$P(x) = (x-1)(x+1)(x^2+10x+25) = (x-1)(x+1)(x+5)^2$$
$$x_{1,2} = \frac{-10 \pm \sqrt{100-100}}{2} = -5$$

6.
Rozložte na parciální zlomky:

$$\frac{x^2-3x+2}{(x+1)(x^2+2)}$$

Řešení:

$$\begin{aligned}\frac{x^2-3x+2}{(x+1)(x^2+2)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+2}\\ x^2-3x+2 &= A(x^2+2) + (Bx+C)(x+1)\\ x=-1:\ 6 &= 3A \Rightarrow A = 2\\ x^2:\ 1 &= A+B = 2+B \Rightarrow B = -1\\ x^0:\ 2 &= 2A+C = 4+C \Rightarrow C = -2\end{aligned}$$

7.
Spočtěte X, je-li $A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $B = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ a

$$AX + B = 2X + 3BA$$

Řešení:

$$\begin{aligned}AX - 2X &= 3BA - B\\ (A-2I)X &= 3BA - B\\ X &= (A-2I)^{-1}(3BA-B)\end{aligned}$$

$$BA = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$
$$3BA - B = \begin{pmatrix}0 & 3\\ 3 & 3\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 2\\ 2 & 3\end{pmatrix}$$
$$(A-2I) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} - \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$$

$$\left(\begin{array}{cc|cc}-1 & 1 & 1 & 0\\ 0 & -1 & 0 & 1\end{array}\right) \sim \left(\begin{array}{cc|cc}-1 & 0 & 1 & 1\\ 0 & -1 & 0 & 1\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 0 & -1 & -1\\ 0 & 1 & 0 & -1\end{array}\right) \qquad X = \begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 2\\ 2 & 3\end{pmatrix} = \begin{pmatrix}-2 & -5\\ -2 & -3\end{pmatrix}$$

8.
Vypočtěte inverzní matici k matici

$$A = \begin{pmatrix}2 & -1 & 3\\ 0 & 1 & -4\\ 4 & -1 & 1\end{pmatrix}$$

Řešení:

$$\left(\begin{array}{ccc|ccc}2 & -1 & 3 & 1 & 0 & 0\\ 0 & 1 & -4 & 0 & 1 & 0\\ 4 & -1 & 1 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}2 & -1 & 3 & 1 & 0 & 0\\ 0 & 1 & -4 & 0 & 1 & 0\\ 0 & 1 & -5 & -2 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}2 & 0 & -1 & 1 & 1 & 0\\ 0 & 1 & -4 & 0 & 1 & 0\\ 0 & 0 & -1 & -2 & -1 & 1\end{array}\right) \sim$$
$$\left(\begin{array}{ccc|ccc}2 & 0 & 0 & 3 & 2 & -1\\ 0 & 1 & 0 & 8 & 5 & -4\\ 0 & 0 & -1 & -2 & -1 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{2} & 1 & -\frac{1}{2}\\ 0 & 1 & 0 & 8 & 5 & -4\\ 0 & 0 & 1 & 2 & 1 & -1\end{array}\right) \qquad A^{-1} = \begin{pmatrix}\frac{3}{2} & 1 & -\frac{1}{2}\\ 8 & 5 & -4\\ 2 & 1 & -1\end{pmatrix}$$

9.
Vypočtěte determinant

$$\begin{vmatrix}2 & -2 & -5 & 1\\ 2 & 0 & 1 & -2\\ 6 & 0 & -1 & 0\\ 8 & 2 & 2 & 4\end{vmatrix}$$

Řešení:

$$\begin{vmatrix}2 & -2 & -5 & 1\\ 2 & 0 & 1 & -2\\ 6 & 0 & -1 & 0\\ 8 & 2 & 2 & 4\end{vmatrix} = 2\begin{vmatrix}2 & -2 & -5 & 1\\ 2 & 0 & 1 & -2\\ 6 & 0 & -1 & 0\\ 4 & 1 & 1 & 2\end{vmatrix} = 2 \cdot 2 \cdot \begin{vmatrix}1 & -2 & -5 & 1\\ 1 & 0 & 1 & -2\\ 3 & 0 & -1 & 0\\ 2 & 1 & 1 & 2\end{vmatrix} = 4\begin{vmatrix}5 & 0 & -3 & 5\\ 1 & 0 & 1 & -2\\ 3 & 0 & -1 & 0\\ 2 & 1 & 1 & 2\end{vmatrix} =$$

$$4\cdot 1\cdot(-1)^{4+2}\begin{vmatrix}5 & -3 & 5\\ 1 & 1 & -2\\ 3 & -1 & 0\end{vmatrix} = 4(-5+18+15-10) = 4\cdot 18 = 72$$

10.
Řešte soustavu rovnic

$$\begin{aligned}
x_1 + 3x_2 - 4x_3 + 9x_4 &= 15\\
-x_1 - x_2 + 3x_3 + 2x_4 &= 2\\
2x_1 \qquad - 5x_3 + 13x_4 &= 7\\
2x_1 + 4x_2 - 7x_3 + 5x_4 &= 11
\end{aligned}$$

Řešení:

$$\left(\begin{array}{cccc|c}1 & 3 & -4 & 9 & 15\\ -1 & -1 & 3 & 2 & 2\\ 2 & 0 & -5 & 13 & 7\\ 2 & 4 & -7 & 5 & 11\end{array}\right) \sim \left(\begin{array}{cccc|c}-1 & -1 & 3 & 2 & 2\\ 0 & 2 & -1 & 11 & 17\\ 0 & -2 & 1 & 17 & 11\\ 0 & 2 & -1 & 9 & 15\end{array}\right) \sim \left(\begin{array}{cccc|c}-1 & -1 & 3 & 2 & 2\\ 0 & 2 & -1 & 11 & 17\\ 0 & 0 & 0 & 28 & 28\\ 0 & 0 & 0 & -2 & -2\end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c}-1 & -1 & 3 & 2 & 2\\ 0 & 2 & -1 & 11 & 17\\ 0 & 0 & 0 & 1 & 1\end{array}\right) \qquad h(A) = h(A_r) = 3 \quad \Rightarrow \quad \text{úloha má řešení}$$

4 neznámé, 3 nezávislé rovnice ⇒ nekonečně mnoho řešení s jedním parametrem.

$$x_4 = 1$$

$$\begin{aligned}
2x_2 - x_3 + 11x_4 &= 17\\
2x_2 - x_3 + 11 &= 17\\
2x_2 - x_3 &= 6\\
x_2 &= t\\
x_3 &= 2t - 6
\end{aligned}
\qquad
\begin{aligned}
-x_1 - x_2 + 3x_3 + 2x_4 &= 2\\
-x_1 - t + 3(2t-6) + 2 &= 2\\
x_1 &= 5t - 18
\end{aligned}$$

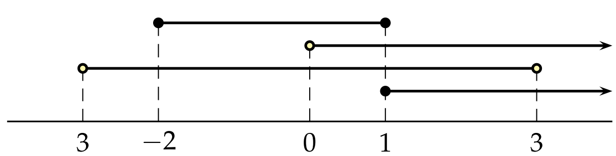
B.

1.
Najděte definiční obor funkce

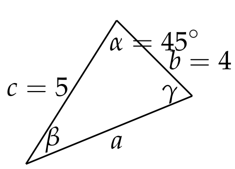
$$f(x) = \ln(9 - x^2) + \sqrt{\ln x} - \arccos\frac{2x+1}{3}$$

Řešení:
$\ln x \geq 0 \;\Rightarrow\; x \geq 1$
$9 - x^2 > 0 \;\Rightarrow\; 9 > x^2 \;\Rightarrow\; 3 > |x|$
$x > 0$
$-1 \leq \dfrac{2x+1}{3} \leq 1 \;\Rightarrow\; -3 \leq 2x+1 \leq 3 \;\Rightarrow\; -4 \leq 2x \leq 2 \;\Rightarrow\; -2 \leq x \leq 1$

$D(f) = \{1\}$



2.
V trojúhelníku $\triangle ABC$ je dáno $b = 4$, $c = 5$, $\alpha = 45^\circ$. Vypočtěte a a β s přesností na minuty.
Řešení:



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$a^2 = 16 + 25 - 2\cdot 4\cdot 5\cdot\frac{\sqrt{2}}{2} \doteq 12.7$$

$$a \doteq 3.566$$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a}$$

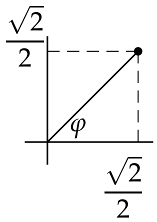
$$\sin\beta \doteq \frac{4}{3.566}\cdot\frac{\sqrt{2}}{2} \doteq 0.793$$

$$\beta \doteq 52.48^\circ = 52^\circ 29'$$

3.
Spočtěte z^{10}, je-li $z = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} i$
Řešení:

$$|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$\varphi = \operatorname{arctg} \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \operatorname{arctg} 1 = \frac{\pi}{4}$$



$$\begin{aligned} z^{10} &= 1^{10} \cdot \left(\cos 10\frac{\pi}{4} + i \sin 10\frac{\pi}{4}\right) = \\ &\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \end{aligned}$$

4.
Vydělte polynomy:

$$(2x^3 + x^2 - x + 3) : (x^2 - 2)$$

Řešení:

$$\begin{array}{rrrl} (2x^3 & +x^2 & -x & +3) : (x^2 - 2) = 2x + 1 + \dfrac{3x+5}{x^2-2} \\ -(2x^3 & & -4x &) \\ \hline & x^2 & +3x & +3 \\ & -(x^2 & & -2) \\ \hline & & 3x & +5 \end{array}$$

5.
Najděte rozklad polynomu na kořenové činitele (kořeny jsou celá čísla):

$$P(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$$

Řešení:

	1	3	−3	−11	−6
1	1	4	1	−10	−16
−1	1	2	−5	−6	0
−1	1	1	−6	0	

$$P(x) = (x+1)^2(x^2 + x - 6) = (x+1)^2(x+3)(x-2)$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} = -3,\ 2$$

6.
Rozložte na parciální zlomky:

$$\frac{x^2 - x + 3}{x^2(x-1)}$$

Řešení:

$$\begin{aligned} \frac{x^2 - x + 3}{x^2(x-1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \\ x^2 - 3x + 2 &= A(x^2 + 2) + (Bx + C)(x+1) \\ x = 0:\ 3 &= -B \Rightarrow B = -3 \\ x = 1:\ 3 &= C \\ x^2:\ 1 &= A + C = A + 3 \Rightarrow A = -2 \end{aligned}$$

7.
Spočtěte X, je-li $A = \begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ a

$$XA - B = 3X - 2BA$$

Řešení:

$$\begin{aligned} XA - 3X &= B - 2BA \\ X(A - 3I) &= B - 2BA \\ X &= (B - 2BA)(A - 3I)^{-1} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -4 & 1 \end{pmatrix} \\ B - 2BA &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 8 \\ -8 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -7 \\ 7 & -2 \end{pmatrix} \\ (A - 3I) &= \begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\left(\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 0 & 1 & 0 & 1\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\end{array}\right) \qquad X = \begin{pmatrix}0 & -7\\ 7 & -2\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & -7\\ 7 & 5\end{pmatrix}$$

8.

Vypočtěte inverzní matici k matici

$$A = \begin{pmatrix}3 & -2 & 4\\ -1 & 2 & -1\\ -3 & 4 & -4\end{pmatrix}$$

Řešení:

$$\left(\begin{array}{ccc|ccc}3 & -2 & 4 & 1 & 0 & 0\\ -1 & 2 & -1 & 0 & 1 & 0\\ -3 & 4 & -4 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}-1 & 2 & -1 & 0 & 1 & 0\\ 0 & 4 & 1 & 1 & 3 & 0\\ 0 & -2 & -1 & 0 & -3 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}-1 & 0 & -2 & 0 & -2 & 1\\ 0 & -2 & -1 & 0 & -3 & 1\\ 0 & 0 & -1 & 1 & -3 & 2\end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc}-1 & 0 & 0 & -2 & 4 & -3\\ 0 & -2 & 0 & -1 & 0 & -1\\ 0 & 0 & -1 & 1 & -3 & 2\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 2 & -4 & 3\\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & -1 & 3 & -2\end{array}\right) \quad A^{-1} = \begin{pmatrix}2 & -4 & 3\\ \frac{1}{2} & 0 & \frac{1}{2}\\ -1 & 3 & -2\end{pmatrix}$$

9.

Vypočtěte determinant

$$\begin{vmatrix}1 & -5 & 2 & 2\\ -2 & 1 & 0 & -4\\ 0 & -1 & 2 & 4\\ 4 & 2 & 0 & 8\end{vmatrix}$$

Řešení:

$$\begin{vmatrix}1 & -5 & 2 & 2\\ -2 & 1 & 0 & -4\\ 0 & -1 & 2 & 4\\ 4 & 2 & 0 & 8\end{vmatrix} = 2\begin{vmatrix}1 & -5 & 2 & 2\\ -2 & 1 & 0 & -4\\ 0 & -1 & 2 & 4\\ 2 & 1 & 0 & 4\end{vmatrix} = 2\cdot 2\cdot 2\cdot\begin{vmatrix}1 & -5 & 1 & 1\\ -2 & 1 & 0 & -2\\ 0 & -1 & 1 & 2\\ 2 & 1 & 0 & 2\end{vmatrix} = 8\begin{vmatrix}1 & -5 & 1 & 1\\ -2 & 1 & 0 & -2\\ -1 & 4 & 0 & 1\\ 2 & 1 & 0 & 2\end{vmatrix} =$$

$$8\cdot 1\cdot(-1)^{1+3}\begin{vmatrix}-2 & 1 & -2\\ -1 & 4 & 1\\ 2 & 1 & 2\end{vmatrix} = 8(-16+2+2+16+2+2) = 64$$

10.

Řešte soustavu rovnic

$$\begin{aligned}2x_1 + 4x_2 + 3x_3 + 4x_4 &= 1\\ 2x_1 + x_2 + 2x_3 - 2x_4 &= 8\\ 4x_1 - x_2 + 3x_3 + 11x_4 &= 2\\ 8x_1 + 10x_2 + 10x_3 + x_4 &= 21\end{aligned}$$

Řešení:

$$\left(\begin{array}{cccc|c}2 & 4 & 3 & 4 & 1\\ 2 & 1 & 2 & -2 & 8\\ 4 & -1 & 3 & 11 & 2\\ 8 & 10 & 10 & 1 & 21\end{array}\right) \sim \left(\begin{array}{cccc|c}2 & 4 & 3 & 4 & 1\\ 0 & -3 & -1 & -6 & 7\\ 0 & -9 & -3 & 3 & 0\\ 0 & -6 & -2 & -15 & 17\end{array}\right) \sim \left(\begin{array}{cccc|c}2 & 4 & 3 & 4 & 1\\ 0 & -3 & -1 & -6 & 7\\ 0 & 0 & 0 & 21 & -21\\ 0 & 0 & 0 & -3 & 3\end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c}2 & 4 & 3 & 4 & 1\\ 0 & -3 & -1 & -6 & 7\\ 0 & 0 & 0 & 1 & -1\end{array}\right) \qquad h(A) = h(A_r) = 3 \quad \Rightarrow \quad \text{úloha má řešení,}$$

4 neznámé, 3 nezávislé rovnice ⇒ nekonečně mnoho řešení s jedním parametrem.

$$x_4 = -1$$

$$\begin{aligned}-3x_2 - x_3 - 6x_4 &= 7\\ -3x_2 - x_3 + 6 &= 7\\ -3x_2 - x_3 &= 1\\ x_2 &= t\\ x_3 &= -3t - 1\end{aligned} \qquad \begin{aligned}2x_1 + 4x_2 + 3x_3 + 4x_4 &= 1\\ 2x_1 + 4t + 3(-3t-1) - 4 &= 1\\ x_1 &= \frac{5t+8}{2}\end{aligned}$$