8.2 Some Applications of Eigenvalue Problems

In this section we discuss a few typical examples from the range of applications of matrix eigenvalue problems, which is incredibly large. Chapter 4 shows matrix eigenvalue problems related to ODEs governing mechanical systems and electrical networks. To keep our present discussion independent of Chap. 4, we include a typical application of that kind as our last example.

EXAMPLE 1 Stretching of an Elastic Membrane

An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ (Fig. 158) is stretched so that a point P: (x_1, x_2) goes over into the point Q: (y_1, y_2) given by

$$(1) \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \text{in components,} \qquad \begin{aligned} y_1 &= 5x_1 + 3x_2 \\ y_2 &= 3x_1 + 5x_2. \end{aligned}$$

Find the **principal directions,** that is, the directions of the position vector $\mathbf{x}$ of P for which the direction of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Solution. We are looking for vectors $\mathbf{x}$ such that $\mathbf{y} = \lambda\mathbf{x}$. Since $\mathbf{y} = \mathbf{Ax}$, this gives $\mathbf{Ax} = \lambda\mathbf{x}$, the equation of an eigenvalue problem. In components, $\mathbf{Ax} = \lambda\mathbf{x}$ is

$$(2) \qquad \begin{aligned} 5x_1 + 3x_2 &= \lambda x_1 \\ 3x_1 + 5x_2 &= \lambda x_2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} (5-\lambda)x_1 + \quad 3x_2 \quad &= 0 \\ 3x_1 \quad + (5-\lambda)x_2 &= 0. \end{aligned}$$

The characteristic equation is

$$(3) \qquad \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0.$$

Its solutions are $\lambda_1 = 8$ and $\lambda_2 = 2$. These are the eigenvalues of our problem. For $\lambda = \lambda_1 = 8$, our system (2) becomes

$$\begin{aligned} -3x_1 + 3x_2 &= 0, \\ 3x_1 - 3x_2 &= 0. \end{aligned} \qquad \left| \quad \begin{aligned} &\text{Solution } x_2 = x_1, \quad x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = x_2 = 1. \end{aligned} \right.$$

For $\lambda_2 = 2$, our system (2) becomes

$$\begin{aligned} 3x_1 + 3x_2 &= 0, \\ 3x_1 + 3x_2 &= 0. \end{aligned} \qquad \left| \quad \begin{aligned} &\text{Solution } x_2 = -x_1, \quad x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = 1, x_2 = -1. \end{aligned} \right.$$

We thus obtain as eigenvectors of $\mathbf{A}$, for instance, $[1 \;\; 1]^T$ corresponding to λ_1 and $[1 \;\; -1]^T$ corresponding to λ_2 (or a nonzero scalar multiple of these). These vectors make 45° and 135° angles with the positive x_1-direction. They give the principal directions, the answer to our problem. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively; see Fig. 158.

Accordingly, if we choose the principal directions as directions of a new Cartesian u_1u_2-coordinate system, say, with the positive u_1-semi-axis in the first quadrant and the positive u_2-semi-axis in the second quadrant of the x_1x_2-system, and if we set $u_1 = r\cos\phi$, $u_2 = r\sin\phi$, then a boundary point of the unstretched circular membrane has coordinates $\cos\phi$, $\sin\phi$. Hence, after the stretch we have

$$z_1 = 8\cos\phi, \qquad z_2 = 2\sin\phi.$$

Since $\cos^2\phi + \sin^2\phi = 1$, this shows that the deformed boundary is an ellipse (Fig. 158)

$$(4) \qquad \frac{z_1^2}{8^2} + \frac{z_2^2}{2^2} = 1. \qquad \blacksquare$$

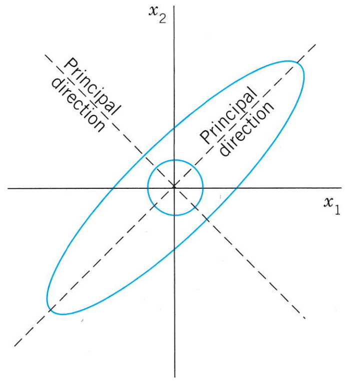

Fig. 158. Undeformed and deformed membrane in Example 1

EXAMPLE 2 Eigenvalue Problems Arising from Markov Processes

Markov processes as considered in Example 13 of Sec. 7.2 lead to eigenvalue problems if we ask for the limit state of the process in which the state vector **x** is reproduced under the multiplication by the stochastic matrix **A** governing the process, that is, $\mathbf{Ax} = \mathbf{x}$. Hence **A** should have the eigenvalue 1, and **x** should be a corresponding eigenvector. This is of practical interest because it shows the long-term tendency of the development modeled by the process.

In that example,

$$\mathbf{A} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix}. \qquad \text{For the transpose,} \qquad \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.9 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $\mathbf{A}^\mathsf{T}$ has the eigenvalue 1, and the same is true for **A** by Theorem 3 in Sec. 8.1. An eigenvector **x** of **A** for $\lambda = 1$ is obtained from

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.1 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix}, \qquad \text{row-reduced to} \qquad \begin{bmatrix} -3/10 & 1/10 & 0 \\ 0 & -1/30 & 1/5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking $x_3 = 1$, we get $x_2 = 6$ from $-x_2/30 + x_3/5 = 0$ and then $x_1 = 2$ from $-3x_1/10 + x_2/10 = 0$. This gives $\mathbf{x} = [2 \quad 6 \quad 1]^\mathsf{T}$. It means that in the long run, the ratio Commercial : Industrial : Residential will approach 2 : 6 : 1, provided that the probabilities given by **A** remain (about) the same. (We switched to ordinary fractions to avoid rounding errors.) ■

EXAMPLE 3 Eigenvalue Problems Arising from Population Models. Leslie Model

The Leslie model describes age-specified population growth, as follows. Let the oldest age attained by the females in some animal population be 9 years. Divide the population into three age classes of 3 years each. Let the *"Leslie matrix"* be

$$\mathbf{L} = [l_{jk}] = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \tag{5}$$

where l_{1k} is the average number of daughters born to a single female during the time she is in age class k, and $l_{j,j-1}$ ($j = 2, 3$) is the fraction of females in age class $j - 1$ that will survive and pass into class j. (a) What is the number of females in each class after 3, 6, 9 years if each class initially consists of 400 females? (b) For what initial distribution will the number of females in each class change by the same proportion? What is this rate of change?