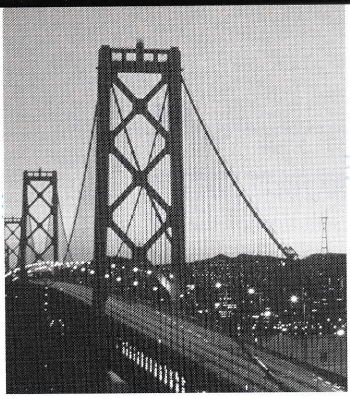

CHAPTER 5

Series Solutions of ODEs. Special Functions

In Chaps. 2 and 3 we have seen that linear ODEs with *constant* coefficients can be solved by functions known from calculus. However, if a linear ODE has *variable* coefficients (functions of x), it must usually be solved by other methods, as we shall see in this chapter.

Legendre polynomials, Bessel functions, and eigenfunction expansions are the three main topics in this chapter. These are of greatest importance to the applied mathematician.

Legendre's ODE and **Legendre polynomials** (Sec. 5.3) are likely to occur in problems showing *spherical symmetry*. They are obtained by the **power series method** (Secs. 5.1, 5.2), which gives solutions of ODEs in power series.

Bessel's ODE and **Bessel functions** (Secs. 5.5, 5.6) are likely to occur in problems showing *cylindrical symmetry*. They are obtained by the **Frobenius method** (Sec. 5.4), an extension of the power series method which gives solutions of ODEs in power series, possibly multiplied by a logarithmic term or by a fractional power.

Eigenfunction expansions (Sec. 5.8) are infinite series obtained by the **Sturm–Liouville theory** (Sec. 5.7). The terms of these series may be Legendre polynomials or other functions, and their coefficients are obtained by the **orthogonality** of those functions. These expansions include **Fourier series** in terms of cosine and sine, which are so important that we shall devote a whole chapter (Chap. 11) to them.

Special functions (also called **higher functions**) is a name for more advanced functions not considered in calculus. If a function occurs in many applications, it gets a name, and its properties and values are investigated in all details, resulting in hundreds of formulas which together with the underlying theory often fill whole books. This is what has happened to the gamma, Legendre, Bessel, and several other functions (take a look into Refs. [GR1], [GR10], [A11] in App. 1).

Your CAS knows most of the special functions and corresponding formulas that you will ever need in your later work in industry, and this chapter will give you a feel for the basics of their theory and their application in modeling.

COMMENT. You can study this chapter directly after Chap. 2 because it needs no material from Chaps. 3 or 4.

Prerequisite: Chap. 2.
Sections that may be omitted in a shorter course: 5.2, 5.6–5.8.
References and Answers to Problems: App. 1 Part A, and App. 2.

5.1 Power Series Method

The **power series method** is the standard method for solving linear ODEs with ***variable*** coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions, as we shall see. In this section we begin by explaining the idea of the power series method.

Power Series

From calculus we recall that a **power series** (in powers of $x - x_0$) is an infinite series of the form

(1) $$\sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots.$$

Here, x is a variable. $a_0, a_1, a_2, \cdots$ are constants, called the **coefficients** of the series. x_0 is a constant, called the **center** of the series. In particular, if $x_0 = 0$, we obtain a **power series in powers of x**

(2) $$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

We shall assume that all variables and constants are real.

Familiar examples of power series are the Maclaurin series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \textbf{ geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots.$$

We note that the term "power series" usually refers to a series of the form (1) [or (2)] but ***does not include*** series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the Legendre and Bessel equations for integer values of the parameter.

Idea of the Power Series Method

The idea of the power series method for solving ODEs is simple and natural. We describe the practical procedure and illustrate it for two ODEs whose solution we know, so that we can see what is going on. The mathematical justification of the method follows in the next section.

For a given ODE

$$y'' + p(x)y' + q(x)y = 0$$

we first represent $p(x)$ and $q(x)$ by power series in powers of x (or of $x - x_0$ if solutions in powers of $x - x_0$ are wanted). Often $p(x)$ and $q(x)$ are polynomials, and then nothing needs to be done in this first step. Next we assume a solution in the form of a power series with unknown coefficients,

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{3}$$

and insert this series and the series obtained by termwise differentiation,

$$\text{(4)}\quad \begin{aligned} &\text{(a)}\quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\ &\text{(b)}\quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots \end{aligned}$$

into the ODE. Then we collect like powers of x and equate the sum of the coefficients of each occurring power of x to zero, starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of (3) successively.

Let us show this for two simple ODEs that can also be solved by elementary methods, so that we would not need power series.

EXAMPLE 1 Solve the following ODE by power series. To grasp the idea, do this by hand; do not use your CAS (for which you could program the whole process).

$$y' = 2xy.$$

Solution. We insert (3) and (4a) into the given ODE, obtaining

$$a_1 + 2a_2x + 3a_3x^2 + \cdots = 2x(a_0 + a_1x + a_2x^2 + \cdots).$$

We must perform the multiplication by $2x$ on the right and can write the resulting equation conveniently as

$$\begin{aligned} &a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots \\ = \quad &\qquad 2a_0x + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 + 2a_4x^5 + \cdots. \end{aligned}$$

For this equation to hold, the two coefficients of every power of x on both sides must be equal, that is,

$$a_1 = 0, \qquad 2a_2 = 2a_0, \qquad 3a_3 = 2a_1, \qquad 4a_4 = 2a_2, \qquad 5a_5 = 2a_3, \qquad 6a_6 = 2a_4, \cdots.$$

Hence $a_3 = 0$, $a_5 = 0$, $\cdots$ and for the coefficients with even subscripts,

$$a_2 = a_0, \qquad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \qquad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \cdots;$$

a_0 remains arbitrary. With these coefficients the series (3) gives the following solution, which you should confirm by the method of separating variables.

$$y = a_0 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) = a_0 e^{x^2}.$$

More rapidly, (3) and (4) give for the ODE $y' = 2xy$

$$1 \cdot a_1 x^0 + \sum_{m=2}^{\infty} m a_m x^{m-1} = 2x \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} 2a_m x^{m+1}.$$

Now, to get the same general power on both sides, we make a **"shift of index"** on the left by setting $m = s + 2$, thus $m - 1 = s + 1$. Then a_m becomes a_{s+2} and x^{m-1} becomes x^{s+1}. Also the summation, which started with $m = 2$, now starts with $s = 0$ because $s = m - 2$. On the right we simply make a change of notation $m = s$, hence $a_m = a_s$ and $x^{m+1} = x^{s+1}$; also the summation now starts with $s = 0$. This altogether gives

$$a_1 + \sum_{s=0}^{\infty} (s + 2)a_{s+2} x^{s+1} = \sum_{s=0}^{\infty} 2a_s x^{s+1}.$$

Every occurring power of x must have the same coefficient on both sides; hence

$$a_1 = 0 \qquad \text{and} \qquad (s + 2)a_{s+2} = 2a_s \qquad \text{or} \qquad a_{s+2} = \frac{2}{s + 2} a_s.$$

For $s = 0, 1, 2, \cdots$ we thus have $a_2 = (2/2)a_0$, $a_3 = (2/3)a_1 = 0$, $a_4 = (2/4)a_2, \cdots$ as before. ■

EXAMPLE 2 Solve

$$y'' + y = 0.$$

Solution. By inserting (3) and (4b) into the ODE we have

$$\sum_{m=2}^{\infty} m(m - 1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0.$$

To obtain the same general power on both series, we set $m = s + 2$ in the first series and $m = s$ in the second, and then we take the latter to the right side. This gives

$$\sum_{s=0}^{\infty} (s + 2)(s + 1)a_{s+2} x^s = -\sum_{s=0}^{\infty} a_s x^s.$$

Each power x^s must have the same coefficient on both sides. Hence $(s + 2)(s + 1)a_{s+2} = -a_s$. This gives the **recursion formula**

$$a_{s+2} = -\frac{a_s}{(s + 2)(s + 1)} \qquad (s = 0, 1, \cdots).$$

We thus obtain successively

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{a_0}{2!}, \qquad a_3 = -\frac{a_1}{3 \cdot 2} = -\frac{a_1}{3!}$$

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \qquad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}.$$

and so on. a_0 and a_1 remain arbitrary. With these coefficients the series (3) becomes

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots.$$

Reordering terms (which is permissible for a power series), we can write this in the form

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right)$$

and we recognize the familiar general solution

$$y = a_0 \cos x + a_1 \sin x.$$

■

Do we need the power series method for these or similar ODEs? Of course not; we used them just for explaining the idea of the method. What happens if we apply the method to an ODE not of the kind considered so far, even to an innocent-looking one such as $y'' + xy = 0$ ("Airy's equation")? We most likely end up with new special functions given by power series. And if such an ODE and its solutions are of practical (or theoretical) interest, we name and investigate them in terms of formulas and graphs and by numeric methods.

We shall discuss Legendre's, Bessel's, and the hypergeometric equations and their solutions, to mention just the most prominent of these ODEs. To do this with a good understanding, also in the light of your CAS, we first explain the power series method (and later an extension, the Frobenius method) in more detail.

PROBLEM SET 5.1

1–10 POWER SERIES METHOD: TECHNIQUE, FEATURES

Apply the power series method. Do this by hand, not by a CAS, so that you get a feel for the method, e.g., why a series may terminate, or has even powers only, or has no constant or linear terms, etc. Show the details of your work.

1. $y' - y = 0$
2. $y' + xy = 0$
3. $y'' + 4y = 0$
4. $y'' - y = 0$
5. $(2 + x)y' = y$
6. $y' + 3(1 + x^2)y = 0$
7. $y' = y + x$
8. $(x^5 + 4x^3)y' = (5x^4 + 12x^2)y$
9. $y'' - y' = 0$
10. $y'' - xy' + y = 0$

11–16 CAS PROBLEMS. INITIAL VALUE PROBLEMS

Solve the initial value problems by a power series. Graph the partial sum s of the powers up to and including x^5. Find the value of s (5 digits) at x_1.

11. $y' + 4y = 1, \quad y(0) = 1.25, \quad x_1 = 0.2$
12. $y' = 1 + y^2, \quad y(0) = 0, \quad x_1 = \frac{1}{4}\pi$
13. $y' = y - y^2, \quad y(0) = \frac{1}{2}, \quad x_1 = 1$
14. $(x - 2)y' = xy, \quad y(0) = 4, \quad x_1 = 2$
15. $y'' + 3xy' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1, \quad x_1 = 0.5$
16. $(1 - x^2)y'' - 2xy' + 30y = 0, \quad y(0) = 0, \quad y'(0) = 1.875, \quad x_1 = 0.5$
17. **WRITING PROJECT. Power Series.** Write a review (2–3 pages) on power series as they are discussed in calculus, using your own formulation and examples—do not just copy passages from calculus texts.
18. **LITERATURE PROJECT. Maclaurin Series.** Collect Maclaurin series of the functions known from calculus and arrange them systematically in a list that you can use for your work.

5.2 Theory of the Power Series Method

In the last section we saw that the power series method gives solutions of ODEs in the form of power series. In this section we justify the method mathematically as follows. We first review relevant facts on power series from calculus. Then we list the operations on power series needed in the method (differentiation, addition, multiplication, etc.). Near the end we state the basic existence theorem for power series solutions of ODEs.

Basic Concepts

Recall from calculus that a power series is an infinite series of the form

$$\text{(1)} \qquad \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

As before, we assume the variable x, the **center** x_0, and the **coefficients** $a_0, a_1, \cdots$ to be real. The ***n*th partial sum** of (1) is

$$\text{(2)} \qquad s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

where $n = 0, 1, \cdots$. Clearly, if we omit the terms of s_n from (1), the remaining expression is

$$\text{(3)} \qquad R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots .$$

This expression is called the **remainder** *of* (1) *after the term* $a_n(x - x_0)^n$.

For example, in the case of the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

we have

$$s_0 = 1, \qquad R_0 = x + x^2 + x^3 + \cdots,$$

$$s_1 = 1 + x, \qquad R_1 = x^2 + x^3 + x^4 + \cdots,$$

$$s_2 = 1 + x + x^2, \qquad R_2 = x^3 + x^4 + x^5 + \cdots, \qquad \text{etc.}$$

In this way we have now associated with (1) the sequence of the partial sums $s_0(x), s_1(x), s_2(x), \cdots$. If for some $x = x_1$ this sequence converges, say,

$$\lim_{n\to\infty} s_n(x_1) = s(x_1),$$

then the series (1) is called **convergent** *at* $x = x_1$, the number $s(x_1)$ is called the **value** or *sum* of (1) at x_1, and we write

$$s(x_1) = \sum_{m=0}^{\infty} a_m(x_1 - x_0)^m.$$

Then we have for every n,

$$\text{(4)} \qquad s(x_1) = s_n(x_1) + R_n(x_1).$$

If that sequence diverges at $x = x_1$, the series (1) is called **divergent** at $x = x_1$.

In the case of convergence, for any positive ϵ there is an N (depending on ϵ) such that, by (4),

$$\text{(5)} \qquad |R_n(x_1)| = |s(x_1) - s_n(x_1)| < \epsilon \qquad \text{for all } n > N.$$

Geometrically, this means that all $s_n(x_1)$ with $n > N$ lie between $s(x_1) - \epsilon$ and $s(x_1) + \epsilon$ (Fig. 102). Practically, this means that in the case of convergence we can approximate the sum $s(x_1)$ of (1) at x_1 by $s_n(x_1)$ as accurately as we please, by taking n large enough.

Convergence Interval. Radius of Convergence

With respect to the convergence of the power series (1) there are three cases, the useless Case 1, the usual Case 2, and the best Case 3, as follows.

Case 1. The series (1) always converges at $x = x_0$, because for $x = x_0$ all its terms are zero, perhaps except for the first one, a_0. In exceptional cases $x = x_0$ may be the only x for which (1) converges. Such a series is of no practical interest.

Case 2. If there are further values of x for which the series converges, these values form an interval, called the **convergence interval.** If this interval is finite, it has the midpoint x_0, so that it is of the form

$$|x - x_0| < R \qquad \text{(Fig. 103)} \tag{6}$$

and the series (1) converges for all x such that $|x - x_0| < R$ and diverges for all x such that $|x - x_0| > R$. (No general statement about convergence or divergence can be made for $x - x_0 = R$ or $-R$.) The number R is called the **radius of convergence** of (1). (R is called "radius" because for a *complex* power series it is the radius of a disk of convergence.) R can be obtained from either of the formulas

$$\text{(a)} \quad R = 1\Big/\lim_{m\to\infty} \sqrt[m]{|a_m|} \qquad \text{(b)} \quad R = 1\Big/\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| \tag{7}$$

provided these limits exist and are not zero. [If these limits are infinite, then (1) converges only at the center x_0.]

Case 3. The convergence interval may sometimes be infinite, that is, (1) converges for all x. For instance, if the limit in (7a) or (7b) is zero, this case occurs. One then writes $R = \infty$, for convenience. (Proofs of all these facts can be found in Sec. 15.2.)

For each x for which (1) converges, it has a certain value $s(x)$. We say that (1) **represents** the function $s(x)$ in the convergence interval and write

$$s(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m \qquad (|x - x_0| < R).$$

Let us illustrate these three possible cases with typical examples.

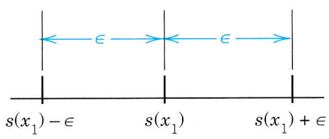

Fig. 102. Inequality (5)

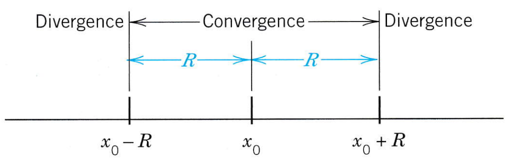

Fig. 103. Convergence interval (6) of a power series with center x_0

EXAMPLE 1 The Useless Case 1 of Convergence Only at the Center

In the case of the series

$$\sum_{m=0}^{\infty} m!x^m = 1 + x + 2x^2 + 6x^3 + \cdots$$

we have $a_m = m!$, and in (7b),

$$\frac{a_{m+1}}{a_m} = \frac{(m+1)!}{m!} = m + 1 \to \infty \qquad \text{as } m \to \infty.$$

Thus this series converges only at the center $x = 0$. Such a series is useless. ■

EXAMPLE 2 The Usual Case 2 of Convergence in a Finite Interval. Geometric Series

For the **geometric series** we have

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1).$$

In fact, $a_m = 1$ for all m, and from (7) we obtain $R = 1$, that is, the geometric series converges and represents $1/(1 - x)$ when $|x| < 1$. ■

EXAMPLE 3 The Best Case 3 of Convergence for All x

In the case of the series

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$$

we have $a_m = 1/m!$. Hence in (7b),

$$\frac{a_{m+1}}{a_m} = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0 \qquad \text{as } m \to \infty,$$

so that the series converges for all x. ■

EXAMPLE 4 Hint for Some of the Problems

Find the radius of convergence of the series

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m} = 1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + - \cdots.$$

Solution. This is a series in powers of $t = x^3$ with coefficients $a_m = (-1)^m/8^m$, so that in (7b),

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{8^m}{8^{m+1}} = \frac{1}{8}.$$

Thus $R = 8$. Hence the series converges for $|t| = |x^3| < 8$, that is, $|x| < 2$. ■

Operations on Power Series

In the power series method we differentiate, add, and multiply power series. These three operations are permissible, in the sense explained in what follows. We also list a condition about the vanishing of all coefficients of a power series, which is a basic tool of the power series method. (Proofs can be found in Sec. 15.3.)

Termwise Differentiation

A power series may be differentiated term by term. More precisely: if

$$y(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m$$

converges for $|x - x_0| < R$, where $R > 0$, then the series obtained by differentiating term by term also converges for those x and represents the derivative y' of y for those x, that is,

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1} \qquad (|x - x_0| < R).$$

Similarly,

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m (x - x_0)^{m-2} \qquad (|x - x_0| < R), \text{ etc.}$$

Termwise Addition

Two power series may be added term by term. More precisely: if the series

$$(8) \qquad \sum_{m=0}^{\infty} a_m(x - x_0)^m \qquad \text{and} \qquad \sum_{m=0}^{\infty} b_m(x - x_0)^m$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, then the series

$$\sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

converges and represents $f(x) + g(x)$ for each x that lies in the interior of the convergence interval of each of the two given series.

Termwise Multiplication

Two power series may be multiplied term by term. More precisely: Suppose that the series (8) have positive radii of convergence and let $f(x)$ and $g(x)$ be their sums. Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of $x - x_0$, that is,

$$\sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0)(x - x_0)^m$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x - x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x - x_0)^2 + \cdots$$

converges and represents $f(x)g(x)$ for each x in the interior of the convergence interval of each of the two given series.

Vanishing of All Coefficients

If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.

Existence of Power Series Solutions of ODEs. Real Analytic Functions

The properties of power series just discussed form the foundation of the power series method. The remaining question is whether an ODE has power series solutions at all. An answer is simple: If the coefficients p and q and the function r on the right side of

$$y'' + p(x)y' + q(x)y = r(x) \tag{9}$$

have power series representations, then (9) has power series solutions. The same is true if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = \tilde{r}(x) \tag{10}$$

have power series representations and $\tilde{h}(x_0) \neq 0$ (x_0 the center of the series). Almost all ODEs in practice have polynomials as coefficients (thus terminating power series), so that (when $r(x) \equiv 0$ or is a power series, too) those conditions are satisfied, except perhaps the condition $\tilde{h}(x_0) \neq 0$. If $\tilde{h}(x_0) \neq 0$, division of (10) by $\tilde{h}(x)$ gives (9) with $p = \tilde{p}/\tilde{h}$, $q = \tilde{q}/\tilde{h}$, $r = \tilde{r}/\tilde{h}$. This motivates our notation in (10).

To formulate all this in a precise and simple way, we use the following concept (which is of general interest).

DEFINITION

Real Analytic Function

A real function $f(x)$ is called **analytic** *at a point* $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Using this concept, we can state the following basic theorem.

THEOREM 1

Existence of Power Series Solutions

If p, q, and r in (9) *are analytic at $x = x_0$, then every solution of* (9) *is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$. Hence the same is true if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in* (10) *are analytic at $x = x_0$ and $\tilde{h}(x_0) \neq 0$.*

The proof of this theorem requires advanced methods of complex analysis and can be found in Ref. [A11] listed in App. 1.

We mention that the radius of convergence R in Theorem 1 is at least equal to the distance from the point $x = x_0$ to the point (or points) closest to x_0 at which one of the functions p, q, r, as functions of a *complex variable,* is not analytic. (Note that that point may not lie on the x-axis but somewhere in the complex plane.)

PROBLEM SET 5.2

1–12 RADIUS OF CONVERGENCE

Determine the radius of convergence. (Show the details.)

1. $\sum_{m=0}^{\infty} \frac{x^m}{c^m} \quad (c \neq 0)$

2. $\sum_{m=0}^{\infty} \frac{(-1)^m}{3^m(m+1)^2}(x+1)^{2m}$

3. $\sum_{m=1}^{\infty} \frac{(m+1)m}{2^m}(x-3)^{2m}$

4. $\sum_{m=0}^{\infty} (-1)^m x^{4m}$

5. $\sum_{m=0}^{\infty} \frac{(2m)!}{(2m+2)(2m+4)} x^m$

6. $\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} x^{2m+10}$

7. $\sum_{m=2}^{\infty} \frac{(-1)^m}{4^m}(x-1)^{2m}$

8. $\sum_{m=1}^{\infty} \frac{(4m)!}{(m!)^4} x^m$

9. $\sum_{m=4}^{\infty} \frac{(m+3)^2}{(m-3)^4} x^m$

10. $\sum_{m=1}^{\infty} \frac{(2m)!}{m^2} x^m$

11. $\sum_{m=1}^{\infty} \frac{1}{\pi^m}(x - \frac{1}{2}\pi)^m$

12. $\sum_{m=1}^{\infty} \frac{(m+1)m}{(2m+1)!} x^{2m+1}$

13–15 SHIFTING SUMMATION INDICES (CF. SEC. 5.1)

This is often convenient or necessary in the power series method. Shift the index so that the power under the summation sign is x^s. Check by writing the first few terms explicitly. Also determine the radius of convergence R.

13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n} x^{n+2}$

14. $\sum_{m=3}^{\infty} \frac{(-1)^{m+1}}{4^m} x^{m-3}$

15. $\sum_{p=1}^{\infty} \frac{p^2}{(p+1)!} x^{p+4}$

16–23 POWER SERIES SOLUTIONS

Find a power series solution in powers of x. (Show the details of your work.)

16. $y'' + xy = 0$
17. $y'' - y' + x^2y = 0$
18. $y'' - y' + xy = 0$
19. $y'' + 4xy' = 0$
20. $y'' + 2xy' + y = 0$
21. $y'' + (1 + x^2)y = 0$
22. $y'' - 4xy' + (4x^2 - 2)y = 0$
23. $(2x^2 - 3x + 1)y'' + 2xy' - 2y = 0$

24. **TEAM PROJECT. Properties from Power Series.** In the next sections we shall define new functions (Legendre functions, etc.) by power series, deriving properties of the functions directly from the series. To understand this idea, do the same for functions familiar from calculus, using Maclaurin series.

(a) Show that $\cosh x + \sinh x = e^x$. Show that $\cosh x > 0$ for all x. Show that $e^x \geqq e^{-x}$ for all $x \geqq 0$.

(b) Derive the differentiation formulas for e^x, $\cos x$, $\sin x$, $1/(1-x)$ and other functions of your choice. Show that $(\cos x)'' = -\cos x$, $(\cosh x)'' = \cosh x$. Consider integration similarly.

(c) What can you conclude if a series contains only odd powers? Only even powers? No constant term? If all its coefficients are positive? Give examples.

(d) What properties of $\cos x$ and $\sin x$ are *not* obvious from the Maclaurin series? What properties of other functions?

25. **CAS EXPERIMENT. Information from Graphs of Partial Sums.** In connection with power series in numerics we use partial sums. To get a feel for the accuracy for various x, experiment with $\sin x$ and graphs of partial sums of the Maclaurin series of an increasing number of terms, describing qualitatively the "breakaway points" of these graphs from the graph of $\sin x$. Consider other examples of your own choice.

5.3 Legendre's Equation. Legendre Polynomials $P_n(x)$

In order to first gain skill, we have applied the power series method to ODEs that can also be solved by other methods. We now turn to the first "big" equation of physics, for which we do need the power series method. This is **Legendre's equation**[1]

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \tag{1}$$

where n is a given constant. Legendre's equation arises in numerous problems, particularly in boundary value problems for spheres (take a quick look at Example 1 in Sec. 12.10). The **parameter** n in (1) is a given real number. Any solution of (1) is called a **Legendre function.** The study of these and other "higher" functions not occurring in calculus is called the **theory of special functions.** Further special functions will occur in the next sections.

Dividing (1) by the coefficient $1 - x^2$ of y'', we see that the coefficients $-2x/(1 - x^2)$ and $n(n + 1)/(1 - x^2)$ of the new equation are analytic at $x = 0$. Hence by Theorem 1, in Sec. 5.2, Legendre's equation has power series solutions of the form

$$y = \sum_{m=0}^{\infty} a_m x^m. \tag{2}$$

Substituting (2) and its derivatives into (1), and denoting the constant $n(n + 1)$ simply by k, we obtain

$$(1 - x^2)\sum_{m=2}^{\infty} m(m - 1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing the first expression as two separate series we have the equation

$$\sum_{m=2}^{\infty} m(m - 1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m - 1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.$$

To obtain the same general power x^s in all four series, we set $m - 2 = s$ (thus $m = s + 2$) in the first series and simply write s instead of m in the other three series. This gives

$$\sum_{s=0}^{\infty} (s + 2)(s + 1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s - 1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.$$

---

[1]ADRIEN-MARIE LEGENDRE (1752–1833), French mathematician, who became a professor in Paris in 1775 and made important contributions to special functions, elliptic integrals, number theory, and the calculus of variations, His book *Éléments de géométrie* (1794) became very famous and had 12 editions in less than 30 years.

Formulas on Legendre functions may be found in Refs. [GR1] and [GR10].

(Note that in the first series the summation begins with $s = 0$.) Since this equation with right side 0 must be an identity in x if (2) is to be a solution of (1), the sum of the coefficients of each power of x on the left must be zero. Now x^0 occurs in the first and fourth series and gives [remember that $k = n(n+1)$]

$$2 \cdot 1 a_2 + n(n+1)a_0 = 0. \tag{3a}$$

x^1 occurs in the first, third, and fourth series and gives

$$3 \cdot 2 a_3 + [-2 + n(n+1)]a_1 = 0. \tag{3b}$$

The higher powers $x^2, x^3, \cdots$ occur in all four series and give

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0. \tag{3c}$$

The expression in the brackets $[\cdots]$ can be written $(n-s)(n+s+1)$, as you may readily verify. Solving (3a) for a_2 and (3b) for a_3 as well as (3c) for a_{s+2}, we obtain the general formula

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}\, a_s \qquad (s = 0, 1, \cdots). \tag{4}$$

This is called a **recurrence relation** or **recursion formula.** (Its derivation you may verify with your CAS.) It gives each coefficient in terms of the second one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively

$$a_2 = -\frac{n(n+1)}{2!}\, a_0 \qquad\qquad a_3 = -\frac{(n-1)(n+2)}{3!}\, a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4 \cdot 3}\, a_2 \qquad\qquad a_5 = -\frac{(n-3)(n+4)}{5 \cdot 4}\, a_3$$

$$= \frac{(n-2)n(n+1)(n+3)}{4!}\, a_0 \qquad\qquad = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\, a_1$$

and so on. By inserting these expressions for the coefficients into (2) we obtain

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{5}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots \tag{6}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots. \tag{7}$$

These series converge for $|x| < 1$ (see Prob. 4; or they may terminate, see below). Since (6) contains even powers of x only, while (7) contains odd powers of x only, the ratio y_1/y_2 is not a constant, so that y_1 and y_2 are not proportional and are thus linearly independent solutions. Hence (5) is a general solution of (1) on the interval $-1 < x < 1$.

Legendre Polynomials $P_n(x)$

In various applications, power series solutions of ODEs reduce to polynomials, that is, they terminate after finitely many terms. This is a great advantage and is quite common for special functions, leading to various important families of polynomials (see Refs. [GR1] or [GR10] in App. 1). For Legendre's equation this happens when the parameter n is a nonnegative integer because then the right side of (4) is zero for $s = n$, so that $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0, \cdots$. Hence if n is even, $y_1(x)$ reduces to a polynomial of degree n. If n is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called **Legendre polynomials** and are denoted by $P_n(x)$. The standard choice of a constant is done as follows. We choose the coefficient a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \qquad (n \text{ a positive integer}) \tag{8}$$

(and $a_n = 1$ if $n = 0$). Then we calculate the other coefficients from (4), solved for a_s in terms of a_{s+2}, that is,

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \leqq n-2). \tag{9}$$

The choice (8) makes $P_n(1) = 1$ for every n (see Fig. 104 on p. 180); this motivates (8). From (9) with $s = n - 2$ and (8) we obtain

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2}.$$

Using $(2n)! = 2n(2n-1)(2n-2)!$, $n! = n(n-1)!$, and $n! = n(n-1)(n-2)!$, we obtain

$$a_{n-2} = -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!\, n(n-1)(n-2)!}.$$

$n(n-1)2n(2n-1)$ cancels, so that we get

$$a_{n-2} = -\frac{(2n-2)!}{2^n(n-1)!\,(n-2)!}.$$

Similarly,

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!\,(n-2)!\,(n-4)!}$$

and so on, and in general, when $n - 2m \geqq 0$,

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!}. \tag{10}$$

The resulting solution of Legendre's differential equation (1) is called the **Legendre polynomial** *of degree n* and is denoted by $P_n(x)$.

From (10) we obtain

$$
\begin{aligned}
P_n(x) &= \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m} \\
&= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!\,(n-1)!\,(n-2)!} x^{n-2} + - \cdots
\end{aligned} \tag{11}
$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are (Fig. 104)

$$
\begin{aligned}
&P_0(x) = 1, && P_1(x) = x \\
&P_2(x) = \tfrac{1}{2}(3x^2 - 1), && P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \\
&P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), && P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)
\end{aligned} \tag{11$'$}
$$

and so on. You may now program (11) on your CAS and calculate $P_n(x)$ as needed.

The so-called **orthogonality** of the Legendre polynomials will be considered in Secs. 5.7 and 5.8.

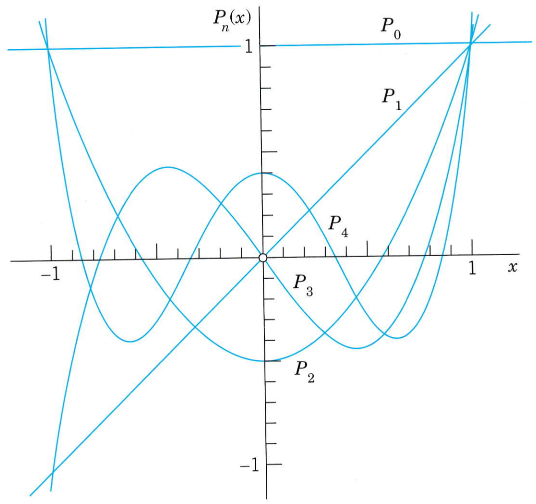

Fig. 104. Legendre polynomials

PROBLEM SET 5.3

1. Verify that the polynomials in (11′) satisfy Legendre's equation.
2. Derive (11′) from (11).
3. Obtain P_6 and P_7 from (11).
4. **(Convergence)** Show that for any n for which (6) or (7) does not reduce to a polynomial, the series has radius of convergence 1.
5. **(Legendre function $Q_0(x)$ for $n = 0$)** Show that (6) with $n = 0$ gives $y_1(x) = P_0(x) = 1$ and (7) gives

$$
\begin{aligned}
y_2(x) &= x + \frac{2}{3!}x^3 + \frac{(-3)(-1)\cdot 2\cdot 4}{5!}x^5 + \cdots \\
&= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \frac{1}{2}\ln\frac{1+x}{1-x}.
\end{aligned}
$$

Verify this by solving (1) with $n = 0$, setting $z = y'$ and separating variables.

6. (Legendre function $-Q_1(x)$ for $n = 1$) Show that (7) with $n = 1$ gives $y_2(x) = P_1(x) = x$ and (6) gives $y_1(x) = -Q_1(x)$ (the minus sign in the notation being conventional),

$$y_1(x) = 1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} - \cdots$$
$$= 1 - x\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$
$$= 1 - \frac{1}{2} x \ln \frac{1+x}{1-x}.$$

7. (ODE) Find a solution of $(a^2 - x^2)y'' - 2xy' + n(n+1)y = 0$, $a \neq 0$, by reduction to the Legendre equation.

8. [Rodrigues's formula (12)][2] Applying the binomial theorem to $(x^2 - 1)^n$, differentiating it n times term by term, and comparing the result with (11), show that

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2 - 1)^n\right]. \tag{12}$$

9. (Rodrigues's formula) Obtain (11′) from (12).

10–13 CAS PROBLEMS

10. Graph $P_2(x), \cdots, P_{10}(x)$ on common axes. For what x (approximately) and $n = 2, \cdots, 10$ is $|P_n(x)| < \frac{1}{2}$?

11. From what n on will your CAS no longer produce faithful graphs of $P_n(x)$? Why?

12. Graph $Q_0(x)$, $Q_1(x)$, and some further Legendre functions.

13. Substitute $a_s x^s + a_{s+1}x^{s+1} + a_{s+2}x^{s+2}$ into Legendre's equation and obtain the coefficient recursion (4).

14. TEAM PROJECT. Generating Functions. Generating functions play a significant role in modern applied mathematics (see [GR5]). The idea is simple. If we want to study a certain sequence $(f_n(x))$ and can find a function

$$G(u, x) = \sum_{n=0}^{\infty} f_n(x)u^n,$$

we may obtain properties of $(f_n(x))$ from those of G, which "generates" this sequence and is called a **generating function** of the sequence.

(a) Legendre polynomials. Show that

$$G(u, x) = \frac{1}{\sqrt{1 - 2xu + u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n \tag{13}$$

is a generating function of the Legendre polynomials. *Hint:* Start from the binomial expansion of $1/\sqrt{1-v}$, then set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

(b) Potential theory. Let A_1 and A_2 be two points in space (Fig. 105, $r_2 > 0$). Using (13), show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}} = \frac{1}{r_2}\sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{r_1}{r_2}\right)^m.$$

This formula has applications in potential theory. (Q/r is the electrostatic potential at A_2 due to a charge Q located at A_1. And the series expresses $1/r$ in terms of the distances of A_1 and A_2 from any origin O and the angle θ between the segments OA_1 and OA_2.)

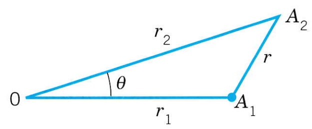

Fig. 105. Team Project 14

(c) Further applications of (13). Show that $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $P_{2n+1}(0) = 0$, and

$$P_{2n}(0) = (-1)^n \cdot 1 \cdot 3 \cdots (2n-1)/[2 \cdot 4 \cdots (2n)].$$

(d) Bonnet's recursion.[3] Differentiating (13) with respect to u, using (13) in the resulting formula, and comparing coefficients of u^n, obtain the *Bonnet recursion*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \tag{14}$$

where $n = 1, 2, \cdots$. This formula is useful for computations, the loss of significant digits being small (except near zeros). Try (14) out for a few computations of your own choice.

[2] OLINDE RODRIGUES (1794–1851), French mathematician and economist.
[3] OSSIAN BONNET (1819–1892), French mathematician, whose main work was in differential geometry.

15. **(Associated Legendre functions)** The associated Legendre functions $P_n^{\,k}(x)$ play a role in quantum physics. They are defined by

$$P_n^{\,k}(x) = (1 - x^2)^{k/2} \frac{d^k P_n}{dx^k} \tag{15}$$

and are solutions of the ODE

$$(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{k^2}{1 - x^2} \right] y = 0. \tag{16}$$

Find $P_1^{\,1}(x)$, $P_2^{\,1}(x)$, $P_2^{\,2}(x)$, and $P_4^{\,2}(x)$ and verify that they satisfy (16).

5.4 Frobenius Method

Several second-order ODEs of considerable practical importance—the famous Bessel equation among them—have coefficients that are not analytic (definition in Sec. 5.2), but are "not too bad," so that these ODEs can still be solved by series (power series times a logarithm or times a fractional power of x, etc.). Indeed, the following theorem permits an extension of the power series method that is called the **Frobenius method.** The latter—as well as the power series method itself—has gained in significance due to the use of software in the actual calculations.

THEOREM 1

Frobenius Method

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \tag{1}$$

has at least one solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0) \tag{2}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE (1) also has a second solution (such that these two solutions are linearly independent) that may be similar to (2) (with a different r and different coefficients) or may contain a logarithmic term. (Details in Theorem 2 below.)[4]

For example, Bessel's equation (to be discussed in the next section)

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2} \right) y = 0 \qquad (\nu \text{ a parameter})$$

[4]GEORG FROBENIUS (1849–1917), German mathematician, also known for his work on matrices and in group theory.

In this theorem we may replace x by $x - x_0$ with any number x_0. The condition $a_0 \neq 0$ is no restriction; it simply means that we factor out the highest possible power of x.

The singular point of (1) at $x = 0$ is sometimes called a **regular singular point,** a term confusing to the student, which we shall not use.

is of the form (1) with $b(x) = 1$ and $c(x) = x^2 - \nu^2$ analytic at $x = 0$, so that the theorem applies. This ODE could not be handled in full generality by the power series method.

Similarly, the so-called hypergeometric differential equation (see Problem Set 5.4) also requires the Frobenius method.

The point is that in (2) we have a power series times a single power of x whose exponent r is not restricted to be a nonnegative integer. (The latter restriction would make the whole expression a power series, by definition; see Sec. 5.1.)

The proof of the theorem requires advanced methods of complex analysis and can be found in Ref. [A11] listed in App. 1.

Regular and Singular Points

The following commonly used terms are practical. A **regular point** of

$$y'' + p(x)y' + q(x)y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Then the power series method can be applied. If x_0 is not regular, it is called **singular.** Similarly, a **regular point** of the ODE

$$\tilde{h}(x)y'' + \tilde{p}(x)y'(x) + \tilde{q}(x)y = 0$$

is an x_0 at which $\tilde{h}, \tilde{p}, \tilde{q}$ are analytic and $\tilde{h}(x_0) \neq 0$ (so what we can divide by $\tilde{h}$ and get the previous standard form). If x_0 is not regular, it is called **singular.**

Indicial Equation, Indicating the Form of Solutions

We shall now explain the Frobenius method for solving (1). Multiplication of (1) by x^2 gives the more convenient form

$$x^2y'' + xb(x)y' + c(x)y = 0. \qquad (1')$$

We first expand $b(x)$ and $c(x)$ in power series,

$$b(x) = b_0 + b_1x + b_2x^2 + \cdots, \qquad c(x) = c_0 + c_1x + c_2x^2 + \cdots$$

or we do nothing if $b(x)$ and $c(x)$ are polynomials. Then we differentiate (2) term by term, finding

$$y'(x) = \sum_{m=0}^{\infty} (m + r)a_m x^{m+r-1} = x^{r-1}\,[ra_0 + (r + 1)a_1x + \cdots]$$

$$y''(x) = \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-2} \qquad (2^*)$$

$$= x^{r-2}[r(r - 1)a_0 + (r + 1)ra_1x + \cdots].$$

By inserting all these series into (1′) we readily obtain

$$x^r[r(r - 1)a_0 + \cdots] + (b_0 + b_1x + \cdots)x^r(ra_0 + \cdots) \qquad (3)$$

$$+ (c_0 + c_1x + \cdots)x^r(a_0 + a_1x + \cdots) = 0.$$

We now equate the sum of the coefficients of each power $x^r, x^{r+1}, x^{r+2}, \cdots$ to zero. This yields a system of equations involving the unknown coefficients a_m. The equation corresponding to the power x^r is

$$[r(r-1) + b_0 r + c_0]a_0 = 0.$$

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives

$$r(r-1) + b_0 r + c_0 = 0. \tag{4}$$

This important quadratic equation is called the **indicial equation** of the ODE (1). Its role is as follows.

The Frobenius method yields a basis of solutions. One of the two solutions will always be of the form (2), where r is a root of (4). The other solution will be of a form indicated by the indicial equation. There are three cases:

Case 1. Distinct roots not differing by an integer $1, 2, 3, \cdots$.

Case 2. A double root.

Case 3. Roots differing by an integer $1, 2, 3, \cdots$.

Cases 1 and 2 are not unexpected because of the Euler–Cauchy equation (Sec. 2.5), the simplest ODE of the form (1). Case 1 includes complex conjugate roots r_1 and $r_2 = \bar{r}_1$ because $r_1 - r_2 = r_1 - \bar{r}_1 = 2i \operatorname{Im} r_1$ is imaginary, so it cannot be a *real* integer. The form of a basis will be given in Theorem 2 (which is proved in App. 4), without a general theory of convergence, but convergence of the occurring series can be tested in each individual case as usual. Note that in Case 2 we ***must*** have a logarithm, whereas in Case 3 we *may* or *may not*.

THEOREM 2

Frobenius Method. Basis of Solutions. Three Cases

Suppose that the ODE (1) satisfies the assumptions in Theorem 1. Let r_1 and r_2 be the roots of the indicial equation (4). Then we have the following three cases.

Case 1. Distinct Roots Not Differing by an Integer. *A basis is*

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5}$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{6}$$

with coefficients obtained successively from (3) with $r = r_1$ and $r = r_2$, respectively.

Case 2. Double Root $r_1 = r_2 = r$. *A basis is*

$$y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \left[r = \tfrac{1}{2}(1 - b_0)\right] \tag{7}$$

(of the same general form as before) and

$$y_2(x) = y_1(x) \ln x + x^r(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0). \tag{8}$$

Case 3. Roots Differing by an Integer. *A basis is*

(9) $$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

(of the same general form as before) and

(10) $$y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots),$$

where the roots are so denoted that $r_1 - r_2 > 0$ *and* k *may turn out to be zero.*

Typical Applications

Technically, the Frobenius method is similar to the power series method, once the roots of the indicial equation have been determined. However, (5)–(10) merely indicate the general form of a basis, and a second solution can often be obtained more rapidly by reduction of order (Sec. 2.1).

EXAMPLE 1 **Euler–Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm**

For the Euler–Cauchy equation (Sec. 2.5)

$$x^2 y'' + b_0 x y' + c_0 y = 0 \qquad (b_0, c_0 \text{ constant})$$

substitution of $y = x^r$ gives the auxiliary equation

$$r(r-1) + b_0 r + c_0 = 0,$$

which is the indicial equation [and $y = x^r$ is a very special form of (2)!]. For different roots r_1, r_2 we get a basis $y_1 = x^{r_1}$, $y_2 = x^{r_2}$, and for a double root r we get a basis x^r, $x^r \ln x$. Accordingly, for this simple ODE, Case 3 plays no extra role. ■

EXAMPLE 2 **Illustration of Case 2 (Double Root)**

Solve the ODE

(11) $$x(x-1)y'' + (3x-1)y' + y = 0.$$

(This is a special hypergeometric equation, as we shall see in the problem set.)

Solution. Writing (11) in the standard form (1), we see that it satisfies the assumptions in Theorem 1. [What are $b(x)$ and $c(x)$ in (11)?] By inserting (2) and its derivatives (2*) into (11) we obtain

(12) $$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-1}$$
$$+ 3\sum_{m=0}^{\infty} (m+r)a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

The smallest power is x^{r-1}, occurring in the second and the fourth series; by equating the sum of its coefficients to zero we have

$$[-r(r-1) - r]a_0 = 0, \qquad \text{thus} \qquad r^2 = 0.$$

Hence this indicial equation has the double root $r = 0$.

First Solution. We insert this value $r = 0$ into (12) and equate the sum of the coefficients of the power x^s to zero, obtaining

$$s(s-1)a_s - (s+1)sa_{s+1} + 3sa_s - (s+1)a_{s+1} + a_s = 0$$

thus $a_{s+1} = a_s$. Hence $a_0 = a_1 = a_2 = \cdots$, and by choosing $a_0 = 1$ we obtain the solution

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad (|x| < 1).$$

Second Solution. We get a second independent solution y_2 by the method of reduction of order (Sec. 2.1), substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9), Sec. 2.1, which we shall use in this example, instead of starting reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x-1)/(x^2-x)$, the coefficient of y' in (11) ***in standard form.*** By partial fractions,

$$-\int p\,dx = -\int \frac{3x-1}{x(x-1)}\,dx = -\int \left(\frac{2}{x-1} + \frac{1}{x}\right) dx = -2\ln(x-1) - \ln x.$$

Hence (9), Sec. 2.1, becomes

$$u' = U = y_1^{-2} e^{-\int p\,dx} = \frac{(x-1)^2}{(x-1)^2 x} = \frac{1}{x}, \qquad u = \ln x, \qquad y_2 = uy_1 = \frac{\ln x}{1-x}.$$

y_1 and y_2 are shown in Fig. 106. These functions are linearly independent and thus form a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$). ■

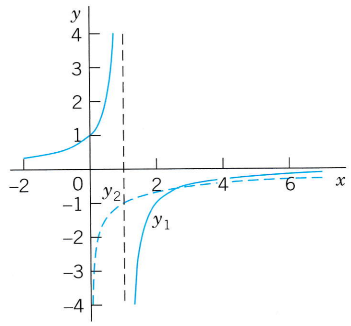

Fig. 106. Solutions in Example 2

EXAMPLE 3 Case 3, Second Solution with Logarithmic Term

Solve the ODE

$$(x^2 - x)y'' - xy' + y = 0. \tag{13}$$

Solution. Substituting (2) and (2*) into (13), we have

$$(x^2 - x)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} - x\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We now take x^2, x, and x inside the summations and collect all terms with power x^{m+r} and simplify algebraically,

$$\sum_{m=0}^{\infty}(m+r-1)^2 a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} = 0.$$

In the first series we set $m = s$ and in the second $m = s + 1$, thus $s = m - 1$. Then

$$\sum_{s=0}^{\infty}(s+r-1)^2 a_s x^{s+r} - \sum_{s=-1}^{\infty}(s+r+1)(s+r)a_{s+1}x^{s+r} = 0. \tag{14}$$

The lowest power is x^{r-1} (take $s = -1$ in the second series) and gives the indicial equation

$$r(r-1) = 0.$$

The roots are $r_1 = 1$ and $r_2 = 0$. They differ by an integer. This is Case 3.

First Solution. From (14) with $r = r_1 = 1$ we have

$$\sum_{s=0}^{\infty} \left[s^2 a_s - (s+2)(s+1)a_{s+1}\right]x^{s+1} = 0.$$

This gives the recurrence relation

$$a_{s+1} = \frac{s^2}{(s+2)(s+1)}\, a_s \qquad (s = 0, 1, \cdots).$$

Hence $a_1 = 0, a_2 = 0, \cdots$ successively. Taking $a_0 = 1$, we get as a first solution $y_1 = x^{r_1}a_0 = x$.

Second Solution. Applying reduction of order (Sec. 2.1), we substitute $y_2 = y_1 u = xu$, $y_2' = xu' + u$ and $y_2'' = xu'' + 2u'$ into the ODE, obtaining

$$(x^2 - x)(xu'' + 2u') - x(xu' + u) + xu = 0.$$

xu drops out. Division by x and simplification give

$$(x^2 - x)u'' + (x-2)u' = 0.$$

From this, using partial fractions and integrating (taking the integration constant zero), we get

$$\frac{u''}{u'} = -\frac{x-2}{x^2 - x} = -\frac{2}{x} + \frac{1}{x-1}, \qquad \ln u' = \ln\left|\frac{x-1}{x^2}\right|.$$

Taking exponents and integrating (again taking the integration constant zero), we obtain

$$u' = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \ln x + \frac{1}{x}, \qquad y_2 = xu = x\ln x + 1.$$

y_1 and y_2 are linearly independent, and y_2 has a logarithmic term. Hence y_1 and y_2 constitute a basis of solutions for positive x. ■

The Frobenius method solves the **hypergeometric equation,** whose solutions include many known functions as special cases (see the problem set). In the next section we use the method for solving Bessel's equation.

PROBLEM SET 5.4

1–17 BASIS OF SOLUTIONS BY THE FROBENIUS METHOD

Find a basis of solutions. Try to identify the series as expansions of known functions. (Show the details of your work.)

1. $xy'' + 2y' - xy = 0$
2. $(x+2)^2 y'' - 2y = 0$
3. $xy'' + 5y' + xy = 0$
4. $2xy'' + (3 - 4x)y' + (2x - 3)y = 0$
5. $x^2 y'' + 4xy' + (x^2 + 2)y = 0$
6. $4xy'' + 2y' + y = 0$
7. $(x+3)^2 y'' - 9(x+3)y' + 25y = 0$
8. $xy'' - y = 0$
9. $xy'' + (2x+1)y' + (x+1)y = 0$
10. $x^2 y'' + 2x^3 y' + (x^2 - 2)y = 0$
11. $(x^2 + x)y'' + (4x + 2)y' + 2y = 0$
12. $x^2 y'' + 6xy' + (4x^2 + 6)y = 0$
13. $2xy'' - (8x - 1)y' + (8x - 2)y = 0$
14. $xy'' + y' - xy = 0$
15. $(x-4)^2 y'' - (x-4)y' - 35y = 0$
16. $x^2 y'' + 4xy' - (x^2 - 2)y = 0$
17. $y'' + (x - 6)y = 0$

18. TEAM PROJECT. Hypergeometric Equation, Series, and Function. Gauss's **hypergeometric ODE**[5] is

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0. \tag{15}$$

Here, a, b, c are constants. This ODE is of the form $p_2 y'' + p_1 y' + p_0 y = 0$, where p_2, p_1, p_0 are polynomials of degree 2, 1, 0, respectively. These polynomials are written so that the series solution takes a most practical form, namely,

$$\begin{aligned} y_1(x) = 1 + \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 \\ + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}x^3 + \cdots . \end{aligned} \tag{16}$$

This series is called the **hypergeometric series.** Its sum $y_1(x)$ is called the **hypergeometric function** and is denoted by $F(a, b, c; x)$. Here, $c \neq 0, -1, -2, \cdots$. By choosing specific values of a, b, c we can obtain an incredibly large number of special functions as solutions of (15) [see the small sample of elementary functions in part (c)]. This accounts for the importance of (15).

(a) Hypergeometric series and function. Show that the indicial equation of (15) has the roots $r_1 = 0$ and $r_2 = 1 - c$. Show that for $r_1 = 0$ the Frobenius method gives (16). Motivate the name for (16) by showing that

$$F(1, 1, 1; x) = F(1, b, b; x) = F(a, 1, a; x) = \frac{1}{1-x}.$$

(b) Convergence. For what a or b will (16) reduce to a polynomial? Show that for any other a, b, c ($c \neq 0, -1, -2, \cdots$) the series (16) converges when $|x| < 1$.

(c) Special cases. Show that

$$(1+x)^n = F(-n, b, b; -x),$$

$$(1-x)^n = 1 - nxF(1-n, 1, 2; x),$$

$$\arctan x = xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; -x^2),$$

$$\arcsin x = xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; x^2),$$

$$\ln(1+x) = xF(1, 1, 2; -x),$$

$$\ln\frac{1+x}{1-x} = 2xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; x^2).$$

Find more such relations from the literature on special functions.

(d) Second solution. Show that for $r_2 = 1 - c$ the Frobenius method yields the following solution (where $c \neq 2, 3, 4, \cdots$):

$$\begin{aligned} y_2(x) = x^{1-c}\Big(1 + \frac{(a-c+1)(b-c+1)}{1!\,(-c+2)}x \\ + \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!\,(-c+2)(-c+3)}x^2 \\ + \cdots\Big). \end{aligned} \tag{17}$$

Show that

$$y_2(x) = x^{1-c}F(a-c+1, b-c+1, 2-c; x).$$

(e) On the generality of the hypergeometric equation. Show that

$$(t^2 + At + B)\ddot{y} + (Ct + D)\dot{y} + Ky = 0 \tag{18}$$

with $\dot{y} = dy/dt$, etc., constant A, B, C, D, K, and $t^2 + At + B = (t - t_1)(t - t_2)$, $t_1 \neq t_2$, can be reduced to the hypergeometric equation with independent variable

$$x = \frac{t - t_1}{t_2 - t_1}$$

and parameters related by $Ct_1 + D = -c(t_2 - t_1)$, $C = a + b + 1$, $K = ab$. From this you see that (15) is a "normalized form" of the more general (18) and that various cases of (18) can thus be solved in terms of hypergeometric functions.

19–24 HYPERGEOMETRIC EQUATIONS

Find a general solution in terms of hypergeometric functions.

19. $x(1-x)y'' + (\frac{1}{2} - 2x)y' - \frac{1}{4}y = 0$

20. $2x(1-x)y'' - (1+6x)y' - 2y = 0$

21. $x(1-x)y'' + \frac{1}{2}y' + 2y = 0$

22. $3t(1+t)\ddot{y} + t\dot{y} - y = 0$

23. $2(t^2 - 5t + 6)\ddot{y} + (2t - 3)\dot{y} - 8y = 0$

24. $4(t^2 - 3t + 2)\ddot{y} - 2\dot{y} + y = 0$

[5] CARL FRIEDRICH GAUSS (1777–1855), great German mathematician. He already made the first of his great discoveries as a student at Helmstedt and Göttingen. In 1807 he became a professor and director of the Observatory at Göttingen. His work was of basic importance in algebra, number theory, differential equations, differential geometry, non-Euclidean geometry, complex analysis, numeric analysis, astronomy, geodesy, electromagnetism, and theoretical mechanics. He also paved the way for a general and systematic use of complex numbers.

5.5 Bessel's Equation. Bessel Functions $J_\nu(x)$

One of the most important ODEs in applied mathematics in **Bessel's equation**,[6]

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{1}$$

Its diverse applications range from electric fields to heat conduction and vibrations (see Sec. 12.9). It often appears when a problem shows *cylindrical symmetry* (just as Legendre's equation may appear in cases of *spherical* symmetry). The parameter ν in (1) is a given number. We assume that ν is real and nonnegative.

Bessel's equation can be solved by the Frobenius method, as we mentioned at the beginning of the preceding section, where the equation is written in standard form (obtained by dividing (1) by x^2). Accordingly, we substitute the series

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \tag{2}$$

with undetermined coefficients and its derivatives into (1). This gives

$$\sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m + r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of x^{s+r} to zero. Note that this power x^{s+r} corresponds to $m = s$ in the first, second, and fourth series, and to $m = s - 2$ in the third series. Hence for $s = 0$ and $s = 1$, the third series does not contribute since $m \geqq 0$. For $s = 2, 3, \cdots$ all four series contribute, so that we get a general formula for all these s. We find

$$\begin{aligned} &\text{(a)} && r(r - 1)a_0 + ra_0 - \nu^2 a_0 = 0 && (s = 0) \\ \text{(3)}\quad &\text{(b)} && (r + 1)ra_1 + (r + 1)a_1 - \nu^2 a_1 = 0 && (s = 1) \\ &\text{(c)} && (s + r)(s + r - 1)a_s + (s + r)a_s + a_{s-2} - \nu^2 a_s = 0 && (s = 2, 3, \cdots). \end{aligned}$$

From (3a) we obtain the **indicial equation** by dropping a_0,

$$(r + \nu)(r - \nu) = 0. \tag{4}$$

The roots are $r_1 = \nu\ (\geqq 0)$ and $r_2 = -\nu$.

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[6]FRIEDRICH WILHELM BESSEL (1784–1846), German astronomer and mathematician, studied astronomy on his own in his spare time as an apprentice of a trade company and finally became director of the new Königsberg Observatory.

Formulas on Bessel functions are contained in Ref. [GR1] and the standard treatise [A13].

Coefficient Recursion for $r = r_1 = \nu$. For $r = \nu$, Eq. (3b) reduces to $(2\nu + 1)a_1 = 0$. Hence $a_1 = 0$ since $\nu \geqq 0$. Substituting $r = \nu$ in (3c) and combining the three terms containing a_s gives simply

(5) $$(s + 2\nu)sa_s + a_{s-2} = 0.$$

Since $a_1 = 0$ and $\nu \geqq 0$, it follows from (5) that $a_3 = 0$, $a_5 = 0$, $\cdots$. Hence we have to deal only with *even-numbered* coefficients a_s with $s = 2m$. For $s = 2m$, Eq. (5) becomes

$$(2m + 2\nu)2ma_{2m} + a_{2m-2} = 0.$$

Solving for a_{2m} gives the recursion formula

(6) $$a_{2m} = -\frac{1}{2^2m(\nu + m)}\,a_{2m-2}, \qquad m = 1, 2, \cdots.$$

From (6) we can now determine $a_2, a_4, \cdots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(\nu + 1)}$$

$$a_4 = -\frac{a_2}{2^2 2(\nu + 2)} = \frac{a_0}{2^4 2!\,(\nu + 1)(\nu + 2)}$$

and so on, and in general

(7) $$a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!\,(\nu + 1)(\nu + 2)\cdots(\nu + m)}, \qquad m = 1, 2, \cdots.$$

Bessel Functions $J_n(x)$ For Integer $\nu = n$

Integer values of v are denoted by n. This is standard. For $\nu = n$ the relation (7) becomes

(8) $$a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!\,(n + 1)(n + 2)\cdots(n + m)}, \qquad m = 1, 2, \cdots.$$

a_0 is still arbitrary, so that the series (2) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function. Accordingly, we have to make a choice. $a_0 = 1$ would be possible, but more practical turns out to be

(9) $$a_0 = \frac{1}{2^n n!}.$$

because then $n!(n + 1)\cdots(n + m) = (m + n)!$ in (8), so that (8) simply becomes

(10) $$a_{2m} = \frac{(-1)^m}{2^{2m+n}m!\,(n + m)!}, \qquad m = 1, 2, \cdots.$$

This simplicity of the denominator of (10) partially motivates the choice (9). With these coefficients and $r_1 = \nu = n$ we get from (2) a particular solution of (1), denoted by $J_n(x)$ and given by

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!}\,. \tag{11}$$

$J_n(x)$ is called the **Bessel function of the first kind** *of order n*. The series (11) converges for all x, as the ratio test shows. In fact, it converges very rapidly because of the factorials in the denominator.

EXAMPLE 1 **Bessel Functions $J_0(x)$ and $J_1(x)$**

For $n = 0$ we obtain from (11) the **Bessel function of order 0**

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots \tag{12}$$

which looks similar to a cosine (Fig. 107). For $n = 1$ we obtain the **Bessel function of order 1**

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + - \cdots, \tag{13}$$

which looks similar to a sine (Fig. 107). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the "waves" decreases with increasing x. Heuristically, n^2/x^2 in (1) in standard form [(1) divided by x^2] is zero (if $n = 0$) or small in absolute value for large x, and so is y'/x, so that then Bessel's equation comes close to $y'' + y = 0$, the equation of $\cos x$ and $\sin x$; also y'/x acts as a "damping term," in part responsible for the decrease in height. One can show that for large x,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{14}$$

where $\sim$ is read **"asymptotically equal"** and means that *for fixed n* the quotient of the two sides approaches 1 as $x \to \infty$.

Formula (14) is surprisingly accurate even for smaller x (> 0). For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of J_0 you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc. ■

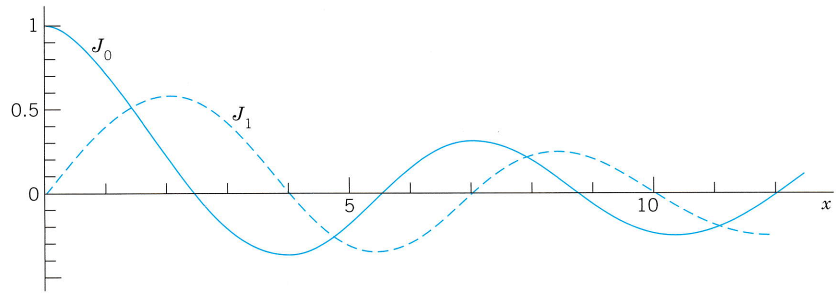

Fig. 107. Bessel functions of the first kind J_0 and J_1

Bessel Functions $J_\nu(x)$ for any $\nu \geqq 0$. Gamma Function

We now extend our discussion from integer $\nu = n$ to any $\nu \geqslant 0$. All we need is an extension of the factorials in (9) and (11) to any ν. This is done by the **gamma function** $\Gamma(\nu)$ defined by the integral

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1}\, dt \qquad (\nu > 0). \tag{15}$$

By integration by parts we obtain

$$\Gamma(\nu + 1) = \int_0^\infty e^{-t} t^{\nu}\, dt = -e^{-t} t^{\nu} \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1}\, dt.$$

The first expression on the right is zero. The integral on the right is $\Gamma(\nu)$. This yields the basic functional relation

$$\Gamma(\nu + 1) = \nu\Gamma(\nu). \tag{16}$$

Now by (15)

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t} \Big|_0^\infty = 0 - (-1) = 1.$$

From this and (16) we obtain successively $\Gamma(2) = \Gamma(1) = 1!$, $\Gamma(3) = 2\Gamma(2) = 2!$, $\cdots$ and in general

$$\Gamma(n + 1) = n! \qquad (n = 0, 1, \cdots). \tag{17}$$

This shows the ***the gamma function does in fact generalize the factorial function.***

Now in (9) we had $a_0 = 1/(2^n n!)$. This is $1/(2^n\Gamma(n + 1))$ by (17). It suggests to choose, for any ν,

$$a_0 = \frac{1}{2^\nu \Gamma(\nu + 1)}. \tag{18}$$

Then (7) becomes

$$a_{2m} = \frac{(-1)^m}{2^{2m} m!\, (\nu + 1)(\nu + 2) \cdots (\nu + m) 2^\nu \Gamma(\nu + 1)}.$$

But (16) gives in the denominator

$$(\nu + 1)\Gamma(\nu + 1) = \Gamma(\nu + 2), \qquad (\nu + 2)\Gamma(\nu + 2) = \Gamma(\nu + 3)$$

and so on, so that

$$(\nu + 1)(\nu + 2) \cdots (\nu + m)\Gamma(\nu + 1) = \Gamma(\nu + m + 1).$$

Hence because of our (standard!) choice (18) of a_0 the coefficients (7) simply are

(19) $$a_{2m} = \frac{(-1)^m}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)}.$$

With these coefficients and $r = r_1 = \nu$ we get from (2) a particular solution of (1), denoted by $J_\nu(x)$ and given by

(20) $$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)}.$$

$J_\nu(x)$ is called the **Bessel function of the first kind of order ν.** The series (20) converges for all x, as one can verify by the ratio test.

General Solution for Noninteger ν. Solution $J_{-\nu}$

For a general solution, in addition to J_ν we need a second linearly independent solution. For ν not an integer this is easy. Replacing ν by $-\nu$ in (20), we have

(21) $$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m!\,\Gamma(m - \nu + 1)}.$$

Since Bessel's equation involves ν^2, the functions J_ν and $J_{-\nu}$ are solutions of the equation for the same ν. If ν is not an integer, they are linearly independent, because the first term in (20) and the first term in (21) are finite nonzero multiples of x^ν and $x^{-\nu}$, respectively. $x = 0$ must be excluded in (21) because of the factor $x^{-\nu}$ (with $\nu > 0$). This gives

THEOREM 1

General Solution of Bessel's Equation

If ν is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is

(22) $$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x).$$

But if ν is an integer, then (22) is not a general solution because of linear dependence:

THEOREM 2

Linear Dependence of Bessel Functions J_n and J_{-n}

For integer $\nu = n$ the Bessel functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, because

(23) $$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, \cdots).$$

PROOF We use (21) and let ν approach a positive integer n. Then the gamma functions in the coefficients of the first n terms become infinite (see Fig. 552 in App. A3.1), the coefficients become zero, and the summation starts with $m = n$. Since in this case $\Gamma(m - n + 1) = (m - n)!$ by (17), we obtain

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!\,(m-n)!} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}(n+s)!\,s!} \qquad (m = n + s).$$

The last series represents $(-1)^n J_n(x)$, as you can see from (11) with m replaced by s. This completes the proof. ■

A general solution for integer n will be given in the next section, based on some further interesting ideas.

Discovery of Properties From Series

Bessel functions are a model case for showing how to discover properties and relations of functions from series by which they are *defined.* Bessel functions satisfy an incredibly large number of relationships—look at Ref. [A13] in App. 1; also, find out what your CAS knows. In Theorem 3 we shall discuss four formulas that are backbones in applications.

THEOREM 3 **Derivatives, Recursions**

The derivative of $J_\nu(x)$ with respect to x can be expressed by $J_{\nu-1}(x)$ or $J_{\nu+1}(x)$ by the formulas

(24)

$$\text{(a)} \quad [x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$$

$$\text{(b)} \quad [x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x).$$

Furthermore, $J_\nu(x)$ and its derivative satisfy the recurrence relations

(24)

$$\text{(c)} \quad J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$\text{(d)} \quad J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x).$$

PROOF **(a)** We multiply (20) by x^ν and take $x^{2\nu}$ under the summation sign. Then we have

$$x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\nu}}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)} .$$

We now differentiate this, cancel a factor 2, pull $x^{2\nu-1}$ out, and use the functional relationship $\Gamma(\nu + m + 1) = (\nu + m)\Gamma(\nu + m)$ [see (16)]. Then (20) with $\nu - 1$ instead of ν shows that we obtain the right side of (24a). Indeed,

$$(x^\nu J_\nu)' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+\nu) x^{2m+2\nu-1}}{2^{2m+\nu} m!\,\Gamma(\nu + m + 1)} = x^\nu x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu-1} m!\,\Gamma(\nu + m)} .$$

(b) Similarly, we multiply (20) by $x^{-\nu}$, so that x^ν in (20) cancels. Then we differentiate, cancel $2m$, and use $m! = m(m-1)!$. This gives, with $m = s + 1$,

$$(x^{-\nu}J_\nu)' = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m+\nu-1}(m-1)!\,\Gamma(\nu+m+1)} = \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2^{2s+\nu+1} s!\,\Gamma(\nu+s+2)}.$$

Equation (20) with $\nu + 1$ instead of ν and s instead of m shows that the expression on the right is $-x^{-\nu}J_{\nu+1}(x)$. This proves (24b).

(c), (d) We perform the differentiation in (24a). Then we do the same in (24b) and multiply the result on both sides by $x^{2\nu}$. This gives

$$\text{(a*)} \qquad \nu x^{\nu-1}J_\nu + x^\nu J_\nu' = x^\nu J_{\nu-1}$$

$$\text{(b*)} \qquad -\nu x^{\nu-1}J_\nu + x^\nu J_\nu' = -x^\nu J_{\nu+1}.$$

Substracting (b*) from (a*) and dividing the result by x^ν gives (24c). Adding (a*) and (b*) and dividing the result by x^ν gives (24d). ■

EXAMPLE 2 **Application of Theorem 3 in Evaluation and Integration**

Formula (24c) can be used recursively in the form

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$$

for calculating Bessel functions of higher order from those of lower order. For instance, $J_2(x) = 2J_1(x)/x - J_0(x)$, so that J_2 can be obtained from tables of J_0 and J_1 (in App. 5 or, more accurately, in Ref. [GR1] in App. 1).

To illustrate how Theorem 3 helps in integration, we use (24b) with $\nu = 3$ integrated on both sides. This evaluates, for instance, the integral

$$I = \int_1^2 x^{-3}J_4(x)\,dx = -x^{-3}J_3(x)\Big|_1^2 = -\frac{1}{8}J_3(2) + J_3(1).$$

A table of J_3 (on p. 398 of Ref. [GR1]) or your CAS will give you

$$-\tfrac{1}{8}\cdot 0.128943 + 0.019563 = 0.003445.$$

Your CAS (or a human computer in precomputer times) obtains J_3 from (24), first using (24c) with $\nu = 2$, that is, $J_3 = 4x^{-1}J_2 - J_1$, then (24c) with $\nu = 1$, that is, $J_2 = 2x^{-1}J_1 - J_0$. Together,

$$\begin{aligned} I &= x^{-3}(4x^{-1}(2x^{-1}J_1 - J_0) - J_1)\Big|_1^2 \\ &= -\tfrac{1}{8}[2J_1(2) - 2J_0(2) - J_1(2)] + [8J_1(1) - 4J_0(1) - J_1(1)] \\ &= -\tfrac{1}{8}J_1(2) + \tfrac{1}{4}J_0(2) + 7J_1(1) - 4J_0(1). \end{aligned}$$

This is what you get, for instance, with Maple if you type int(· · ·). And if you type evalf(int(· · ·)), you obtain 0.003445448, in agreement with the result near the beginning of the example. ■

In the theory of special functions it often happens that for certain values of a parameter a higher function becomes elementary. We have seen this in the last problem set, and we now show this for J_ν.

THEOREM 4

Elementary J_ν for Half-Integer Order ν

Bessel functions J_ν of orders $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$ are elementary; they can be expressed by finitely many cosines and sines and powers of x. In particular,

$$\text{(25)} \qquad \text{(a)} \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad \text{(b)} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

PROOF When $\nu = \frac{1}{2}$, then (20) is

$$J_{1/2}(x) = \sqrt{x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+1/2}m!\,\Gamma(m+\frac{3}{2})} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}m!\,\Gamma(m+\frac{3}{2})}.$$

To simplify the denominator, we first write it out as a product AB, where

$$A = 2^m m! = 2m(2m-2)(2m-4)\cdots 4\cdot 2$$

and [use (16)]

$$B = 2^{m+1}\Gamma(m+\tfrac{3}{2}) = 2^{m+1}(m+\tfrac{1}{2})(m-\tfrac{1}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= (2m+1)(2m-1)\cdots 3\cdot 1\cdot\sqrt{\pi};$$

here we used

$$\text{(26)} \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

We see that the product of the two right sides of A and B is simply $(2m+1)!\sqrt{\pi}$, so that $J_{1/2}$ becomes

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\sin x,$$

as claimed. Differentiation and the use of (24a) with $\nu = \frac{1}{2}$ now gives

$$[\sqrt{x}J_{1/2}(x)]' = \sqrt{\frac{2}{\pi}}\cos x = x^{1/2}J_{-1/2}(x).$$

This proves (25b). From (25) follow further formulas successively by (24c), used as in Example 2. This completes the proof. ■

EXAMPLE 3 **Further Elementary Bessel Functions**

From (24c) with $\nu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$ and (25) we obtain

$$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-3/2}(x) = -\frac{1}{x}J_{-1/2}(x) - J_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

respectively, and so on. ■

We hope that our study has not only helped you to become acquainted with Bessel functions but has also convinced you that series can be quite useful in obtaining various properties of the corresponding functions.

PROBLEM SET 5.5

1. **(Convergence)** Show that the series in (11) converges for all x. Why is the convergence very rapid?
2. **(Approximation)** Show that for small $|x|$ we have $J_0 \approx 1 - 0.25x^2$. From this compute $J_0(x)$ for $x = 0, 0.1, 0.2, \cdots, 1.0$ and determine the error by using Table A1 in App. 5 or your CAS.
3. **("Large" values)** Using (14), compute $J_0(x)$ for $x = 1.0, 2.0, 3.0, \cdots, 8.0$, determine the error by Table A1 or your CAS, and comment.
4. **(Zeros)** Compute the first four positive zeros of $J_0(x)$ and $J_1(x)$ from (14). Determine the error and comment.

5–20 ODEs REDUCIBLE TO BESSEL'S EQUATION

Using the indicated substitutions, find a general solution in terms of J_ν and $J_{-\nu}$ or indicate when this is not possible. (This is just a sample of various ODEs reducible to Bessel's equation. Some more follow in the next problem set. Show the details of your work.)

5. **(ODE with two parameters)** $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \quad (\lambda x = z)$
6. $x^2y'' + xy' + (x^2 - \frac{1}{16})y = 0$
7. $x^2y'' + xy' + \frac{1}{4}(x - \nu^2)y = 0 \quad (\sqrt{x} = z)$
8. $(2x + 1)^2y'' + 2(2x + 1)y' + 16x(x + 1)y = 0 \quad (2x + 1 = z)$
9. $xy'' - y' + 4xy = 0 \quad (y = xu,\ 2x = z)$
10. $x^2y'' + xy' + \frac{1}{4}(x^2 - 1)y = 0 \quad (x = 2z)$
11. $xy'' + (2\nu + 1)y' + xy = 0 \quad (y = x^{-\nu}u)$
12. $x^2y'' + xy' + 4(x^4 - \nu^2)y = 0 \quad (x^2 = z)$
13. $x^2y'' + xy' + 9(x^6 - \nu^2)y = 0 \quad (x^3 = z)$
14. $y'' + (e^{2x} - \frac{1}{9})y = 0 \quad (e^x = z)$
15. $xy'' + y = 0 \quad (y = \sqrt{x}u,\ 2\sqrt{x} = z)$
16. $16x^2y'' + 8xy' + (x^{1/2} + \frac{15}{16})y = 0 \quad (y = x^{1/4}u,\ x^{1/4} = z)$
17. $36x^2y'' + 18xy' + \sqrt{x}\,y = 0 \quad (y = x^{1/4}u,\ \frac{2}{3}x^{1/4} = z)$
18. $x^2y'' + xy' + \sqrt{x}\,y = 0 \quad (4x^{1/4} = z)$
19. $x^2y'' + \frac{1}{5}xy' + \sqrt{x}\,y = 0 \quad (y = x^{2/5}u,\ 4x^{1/4} = z)$
20. $x^2y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0 \quad (y = x^\nu u,\ x^\nu = z)$

21–28 APPLICATION OF (24): DERIVATIVES, INTEGRALS

Use the powerful formulas (24) to do Probs. 21–28. (Show the details of your work.)

21. **(Derivatives)** Show that $J_0'(x) = -J_1(x)$, $J_1'(x) = J_0(x) - J_1(x)/x$, $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$.
22. **(Interlacing of zeros)** Using (24) and Rolle's theorem, show that between two consecutive zeros of $J_0(x)$ there is precisely one zero of $J_1(x)$.
23. **(Interlacing of zeros)** Using (24) and Rolle's theorem, show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$.
24. **(Bessel's equation)** Derive (1) from (24).
25. **(Basic integral formulas)** Show that

$$\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + c,$$

$$\int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu} J_\nu(x) + c,$$

$$\int J_{\nu+1}(x)\,dx = \int J_{\nu-1}(x)\,dx - 2J_\nu(x).$$

26. **(Integration)** Evaluate $\int x^{-1}J_4(x)\,dx$. (Use Prob. 25; integrate by parts.)
27. **(Integration)** Show that $\int x^2J_0(x)\,dx = x^2J_1(x) + xJ_0(x) - \int J_0(x)\,dx$. (The last integral is nonelementary; tables exist, e.g. in Ref. [A13] in App. 1.)
28. **(Integration)** Evaluate $\int J_5(x)\,dx$.
29. **(Elimination of first derivative)** Show that $y = uv$ with $v(x) = \exp(-\frac{1}{2}\int p(x)\,dx)$ gives from the ODE $y'' + p(x)y' + q(x)y = 0$ the ODE

$$u'' + \left[q(x) - \tfrac{1}{4}p(x)^2 - \tfrac{1}{2}p'(x)\right]u = 0$$

no longer containing the first derivative. Show that for the Bessel equation the substitution is $y = ux^{-1/2}$ and gives

$$x^2u'' + (x^2 + \tfrac{1}{4} - \nu^2)u = 0. \tag{27}$$

30. (Elementary Bessel functions) Derive (25) in Theorem 4 from (27).

31. CAS EXPERIMENT. Change of Coefficient. Find and graph (on common axes) the solutions of

$$y'' + kx^{-1}y' + y = 0,\ y(0) = 1,\ y'(0) = 0,$$

for $k = 0, 1, 2, \cdots, 10$ (or as far as you get useful graphs). For what k do you get elementary functions? Why? Try for noninteger k, particularly between 0 and 2, to see the continuous change of the curve. Describe the change of the location of the zeros and of the extrema as k increases from 0. Can you interpret the ODE as a model in mechanics, thereby explaining your observations?

32. TEAM PROJECT. Modeling a Vibrating Cable (Fig. 108). A flexible cable, chain, or rope of length L and density (mass per unit length) ρ is fixed at the upper end ($x = 0$) and allowed to make small vibrations (small angles α in the horizontal displacement $u(x, t)$, t = time) in a vertical plane.

(a) Show the following. The weight of the cable below a point x is $W(x) = \rho g(L - x)$. The restoring force is $F(x) = W \sin \alpha \approx Wu_x$, $u_x = \partial u/\partial x$. The difference in force between x and $x + \Delta x$ is $\Delta x\,(Wu_x)_x$. Newton's second law now gives

$$\rho\, \Delta x\, u_{tt} = \Delta x\, \rho g[(L - x)u_x]_x.$$

For the expected periodic motion $u(x, t) = y(x) \cos(\omega t + \delta)$ the model of the problem is the ODE

$$(L - x)y'' - y' + \lambda^2 y = 0, \qquad \lambda^2 = \omega^2/g.$$

(b) Transform this ODE to $\ddot{y} + s^{-1}\dot{y} + y = 0$, $\dot{y} = dy/ds$, $s = 2\lambda z^{1/2}$, $z = L - x$, so that the solution is

$$y(x) = J_0(2\omega\sqrt{(L - x)/g}).$$

(c) Conclude that possible frequencies $\omega/2\pi$ are those for which $s = 2\omega\sqrt{L/g}$ is a zero of J_0. The corresponding solutions are called **normal modes.** Figure 108 shows the first of them. What does the second normal mode look like? The third? What is the frequency (cycles/min) of a cable of length 2 m? Of length 10 m?

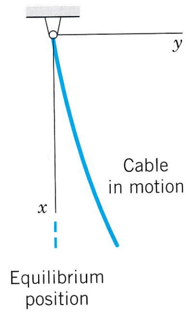

Fig. 108. Vibrating cable in Team Project 32

33. CAS EXPERIMENT. Bessel Functions for Large x.

(a) Graph $J_n(x)$ for $n = 0, \cdots, 5$ on common axes.

(b) Experiment with (14) for integer n. Using graphs, find out from which $x = x_n$ on the curves of (11) and (14) practically coincide. How does x_n change with n?

(c) What happens in (b) if $n = \pm\frac{1}{2}$? (Our usual notation in this case would be ν.)

(d) How does the error of (14) behave as function of x for fixed n? [Error = exact value minus approximation (14).]

(e) Show from the graphs that $J_0(x)$ has extrema where $J_1(x) = 0$. Which formula proves this? Find further relations between zeros and extrema.

(f) Raise and answer questions of your own, for instance, on the zeros of J_0 and J_1. How accurately are they obtained from (14)?

5.6 Bessel Functions of the Second Kind $Y_\nu(x)$

From the last section we know that J_ν and $J_{-\nu}$ form a basis of solutions of Bessel's equation, provided ν is not an integer. But when ν is an integer, these two solutions are linearly dependent on any interval (see Theorem 2 in Sec. 5.5). Hence to have a general solution also when $\nu = n$ is an integer, we need a second linearly independent solution besides J_n. This solution is called a **Bessel function of the second kind** and is denoted by Y_n. We shall now derive such a solution, beginning with the case $n = 0$.

$n = 0$: Bessel Function of the Second Kind $Y_0(x)$

When $n = 0$, Bessel's equation can be written

$$xy'' + y' + xy = 0. \qquad (1)$$

Then the indicial equation (4) in Sec. 5.5 has a double root $r = 0$. This is Case 2 in Sec. 5.4. In this case we first have only one solution, $J_0(x)$. From (8) in Sec. 5.4 we see that the desired second solution must be of the form

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} A_m x^m. \tag{2}$$

We substitute y_2 and its derivatives

$$y_2' = J_0'\ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} mA_m x^{m-1}$$

$$y_2'' = J_0''\ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}$$

into (1). Then the sum of the three logarithmic terms $xJ_0''\ln x$, $J_0'\ln x$, and $xJ_0\ln x$ is zero because J_0 is a solution of (1). The terms $-J_0/x$ and J_0/x (from xy'' and y') cancel. Hence we are left with

$$2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.$$

Addition of the first and second series gives $\Sigma m^2 A_m x^{m-1}$. The power series of $J_0'(x)$ is obtained from (12) in Sec. 5.5 and the use of $m!/m = (m-1)!$ in the form

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!\,(m-1)!}.$$

Together with $\Sigma m^2 A_m x^{m-1}$ and $\Sigma A_m x^{m+1}$ this gives

$$\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!\,(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0. \tag{3*}$$

First, we show that the A_m with odd subscripts are all zero. The power x^0 occurs only in the second series, with coefficient A_1. Hence $A_1 = 0$. Next, we consider the even powers x^{2s}. The first series contains none. In the second series, $m - 1 = 2s$ gives the term $(2s+1)^2 A_{2s+1} x^{2s}$. In the third series, $m + 1 = 2s$. Hence by equating the sum of the coefficients of x^{2s} to zero we have

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0, \qquad s = 1, 2, \cdots.$$

Since $A_1 = 0$, we thus obtain $A_3 = 0$, $A_5 = 0, \cdots$, successively.

We now equate the sum of the coefficients of x^{2s+1} to zero. For $s = 0$ this gives

$$-1 + 4A_2 = 0, \qquad \text{thus} \qquad A_2 = \tfrac{1}{4}.$$

For the other values of s we have in the first series in (3*) $2m - 1 = 2s + 1$, hence $m = s + 1$, in the second $m - 1 = 2s + 1$, and in the third $m + 1 = 2s + 1$. We thus obtain

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!\,s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0.$$

For $s = 1$ this yields

$$\frac{1}{8} + 16A_4 + A_2 = 0, \qquad \text{thus} \qquad A_4 = -\frac{3}{128}$$

and in general

(3) $$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right), \qquad m = 1, 2, \cdots.$$

Using the short notations

(4) $$h_1 = 1 \qquad h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m} \qquad m = 2, 3, \cdots$$

and inserting (4) and $A_1 = A_3 = \cdots = 0$ into (2), we obtain the result

(5) $$\begin{aligned} y_2(x) &= J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \\ &= J_0(x) \ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - + \cdots. \end{aligned}$$

Since J_0 and y_2 are linearly independent functions, they form a basis of (1) for $x > 0$. Of course, another basis is obtained if we replace y_2 by an independent particular solution of the form $a(y_2 + bJ_0)$, where $a\ (\neq 0)$ and b are constants. It is customary to choose $a = 2/\pi$ and $b = \gamma - \ln 2$, where the number $\gamma = 0.577\,215\,664\,90 \cdots$ is the so-called **Euler constant,** which is defined as the limit of

$$1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s$$

as s approaches infinity. The standard particular solution thus obtained is called the **Bessel function of the second kind** *of order zero* (Fig. 109) or **Neumann's function** *of order zero* and is denoted by $Y_0(x)$. Thus [see (4)]

(6) $$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}\right].$$

For small $x > 0$ the function $Y_0(x)$ behaves about like $\ln x$ (see Fig. 109, why?), and $Y_0(x) \to -\infty$ as $x \to 0$.

Bessel Functions of the Second Kind $Y_n(x)$

For $\nu = n = 1, 2, \cdots$ a second solution can be obtained by manipulations similar to those for $n = 0$, starting from (10), Sec 5.4. It turns out that in these cases the solution also contains a logarithmic term.

The situation is not yet completely satisfactory, because the second solution is defined differently, depending on whether the order ν is an integer or not. To provide uniformity of formalism, it is desirable to adopt a form of the second solution that is valid for all values of the order. For this reason we introduce a standard second solution $Y_\nu(x)$ defined for all ν by the formula

$$\text{(7)}\quad \text{(a)}\quad Y_\nu(x) = \frac{1}{\sin \nu\pi}\,[J_\nu(x)\cos \nu\pi - J_{-\nu}(x)]$$

$$\text{(b)}\quad Y_n(x) = \lim_{\nu \to n} Y_\nu(x).$$

This function is called the **Bessel function of the second kind** *of order* ν or **Neumann's function**[7] *of order* ν. Figure 109 shows $Y_0(x)$ and $Y_1(x)$.

Let us show that J_ν and Y_ν are indeed linearly independent for all ν (and $x > 0$).

For noninteger order ν, the function $Y_\nu(x)$ is evidently a solution of Bessel's equation because $J_\nu(x)$ and $J_{-\nu}(x)$ are solutions of that equation. Since for those ν the solutions J_ν and $J_{-\nu}$ are linearly independent and Y_ν involves $J_{-\nu}$, the functions J_ν and Y_ν are linearly independent. Furthermore, it can be shown that the limit in (7b) exists and Y_n is a solution of Bessel's equation for integer order; see Ref. [A13] in App. 1. We shall see that the series development of $Y_n(x)$ contains a logarithmic term. Hence $J_n(x)$ and $Y_n(x)$ are linearly independent solutions of Bessel's equation. The series development of $Y_n(x)$ can be obtained if we insert the series (20) and (21), Sec. 5.5, for $J_\nu(x)$ and $J_{-\nu}(x)$ into (7a) and then let ν approach n; for details see Ref. [A13]. The result is

$$\text{(8)}\quad \begin{aligned} Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln \frac{x}{2} + \gamma\right) &+ \frac{x^n}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n} m!\,(m+n)!} x^{2m} \\ &- \frac{x^{-n}}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n} m!} x^{2m} \end{aligned}$$

where $x > 0$, $n = 0, 1, \cdots$, and [as in (4)] $h_0 = 0$, $h_1 = 1$,

$$h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \qquad h_{m+n} = 1 + \frac{1}{2} + \cdots + \frac{1}{m+n}.$$

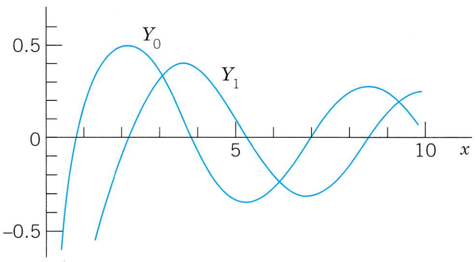

Fig. 109. Bessel functions of the second kind Y_0 and Y_1. (For a small table, see App. 5.)

[7]CARL NEUMANN (1832–1925), German mathematician and physicist. His work on potential theory sparked the development in the field of integral equations by VITO VOLTERRA (1860–1940) of Rome, ERIC IVAR FREDHOLM (1866–1927) of Stockholm, and DAVID HILBERT (1862–1943) of Göttingen (see the footnote in Sec. 7.9).

The solutions $Y_\nu(x)$ are sometimes denoted by $N_\nu(x)$; in Ref. [A13] they are called **Weber's functions;** Euler's constant in (6) is often denoted by C or $\ln \gamma$.

For $n = 0$ the last sum in (8) is to be replaced by 0 [giving agreement with (6)].
Furthermore, it can be shown that

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

Our main result may now be formulated as follows.

THEOREM 1

General Solution of Bessel's Equation

A general solution of Bessel's equation for all values of ν (and $x > 0$) is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x). \tag{9}$$

We finally mention that there is a practical need for solutions of Bessel's equation that are complex for real values of x. For this purpose the solutions

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \end{aligned} \tag{10}$$

are frequently used. These linearly independent functions are called **Bessel functions of the third kind** *of order ν* or *first and second* **Hankel functions**[8] *of order ν*.

This finishes our discussion on Bessel functions, except for their "orthogonality," which we explain in Sec. 5.7. Applications to vibrations follow in Sec. 12.9.

PROBLEM SET 5.6

1–10 SOME FURTHER ODEs REDUCIBLE TO BESSEL'S EQUATIONS

(See also Sec. 5.5.)
Using the indicated substitutions, find a general solution in terms of J_ν and Y_ν. Indicate whether you could also use $J_{-\nu}$ instead of Y_ν. (Show the details of your work.)

1. $x^2y'' + xy' + (x^2 - 25)y = 0$
2. $x^2y'' + xy' + (9x^2 - \frac{1}{9})y = 0 \quad (3x = z)$
3. $4xy'' + 4y' + y = 0 \quad (\sqrt{x} = z)$
4. $xy'' + y' + 36y = 0 \quad (12\sqrt{x} = z)$
5. $x^2y'' + xy' + (4x^4 - 16)y = 0 \quad (x^2 = z)$
6. $x^2y'' + xy' + (x^6 - 1)y = 0 \quad (\frac{1}{3}x^3 = z)$
7. $xy'' + 11y' + xy = 0 \quad (y = x^{-5}u)$
8. $y'' + 4x^2y = 0 \quad (y = u\sqrt{x},\ x^2 = z)$
9. $x^2y'' - 5xy' + 9(x^6 - 8)y = 0 \quad (y = x^3u,\ x^3 = z)$
10. $xy'' + 7y' + 4xy = 0 \quad (y = x^{-3}u,\ 2x = z)$

11. **(Hankel functions)** Show that the Hankel functions (10) form a basis of solutions of Bessel's equation for any ν.

12. **CAS EXPERIMENT. Bessel Functions for Large x.** It can be shown that for large x,

$$Y_n(x) \sim \sqrt{2/(\pi x)}\,\sin\left(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right) \tag{11}$$

with $\sim$ defined as in (14) of Sec. 5.5.

(a) Graph $Y_n(x)$ for $n = 0, \cdots, 5$ on common axes. Are there relations between zeros of one function and extrema of another? For what functions?

(b) Find out from graphs from which $x = x_n$ on the curves of (8) and (11) (both obtained from your CAS) practically coincide. How does x_n change with n?

(c) Calculate the first ten zeros x_m, $m = 1, \cdots, 10$, of $Y_0(x)$ from your CAS and from (11). How does the error behave as m increases?

(d) Do (c) for $Y_1(x)$ and $Y_2(x)$. How do the errors compare to those in (c)?

[8] HERMANN HANKEL (1839–1873), German mathematician.

13. Modified Bessel functions of the first kind of order ν are defined by $I_\nu(x) = i^{-\nu}J_\nu(ix)$, $i = \sqrt{-1}$. Show that I_ν satisfies the ODE

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0 \tag{12}$$

and has the representation

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{x^{2m+\nu}}{2^{2m+\nu}m!\,\Gamma(m + \nu + 1)}. \tag{13}$$

14. (Modified Bessel functions I_ν) Show that $I_\nu(x)$ is real for all real x (and real ν), $I_\nu(x) \neq 0$ for all real $x \neq 0$, and $I_{-n}(x) = I_n(x)$, where n is any integer.

15. Modified Bessel functions of the third kind (sometimes called *of the second kind*) are defined by the formula (14) below. Show that they satisfy the ODE (12).

$$K_\nu(x) = \frac{\pi}{2 \sin \nu\pi}\,[I_{-\nu}(x) - I_\nu(x)] \tag{14}$$

5.7 Sturm–Liouville Problems. Orthogonal Functions

So far we have considered initial value problems. We recall from Sec. 2.1 that such a problem consists of an ODE, say, of second order, and initial conditions $y(x_0) = K_0$, $y'(x_0) = K_1$ referring to the *same point* (initial point) $x = x_0$. We now turn to boundary value problems. A **boundary value problem** consists of an ODE and given **boundary conditions** referring to the two boundary points (endpoints) $x = a$ and $x = b$ of a given interval $a \leqq x \leqq b$. To solve such a problem means to find a solution of the ODE on the interval $a \leqq x \leqq b$ satisfying the boundary conditions.

We shall see that Legendre's, Bessel's, and other ODEs of importance in engineering can be written as a **Sturm–Liouville equation**

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \tag{1}$$

involving a parameter λ. The boundary value problem consisting of an ODE (1) and given **Sturm–Liouville boundary conditions**

(2)

$$\text{(a)}\quad k_1 y(a) + k_2 y'(a) = 0$$

$$\text{(b)}\quad l_1 y(b) + l_2 y'(b) = 0$$

is called a **Sturm–Liouville problem.**[9] We shall see further that these problems lead to useful series developments in terms of particular solutions of (1), (2). Crucial in this connection is *orthogonality* to be discussed later in this section.

In (1) we make the ***assumptions*** that p, q, r, and p' are continuous on $a \leqq x \leqq b$, and

$$r(x) > 0 \qquad (a \leqq x \leqq b).$$

In (2) we assume that k_1, k_2 are given constants, not both zero, and so are l_1, l_2, not both zero.

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[9]JACQUES CHARLES FRANÇOIS STURM (1803–1855), was born and studied in Switzerland and then moved to Paris, where he later became the successor of Poisson in the chair of mechanics at the Sorbonne (the University of Paris).

JOSEPH LIOUVILLE (1809–1882), French mathematician and professor in Paris, contributed to various fields in mathematics and is particularly known by his important work in complex analysis (Liouville's theorem; Sec. 14.4), special functions, differential geometry, and number theory.

EXAMPLE 1 Legendre's and Bessel's Equations are Sturm–Liouville Equations

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ may be written

$$[(1 - x^2)y']' + \lambda y = 0 \qquad \lambda = n(n + 1).$$

This is (1) with $p = 1 - x^2$, $q = 0$, and $r = 1$.

In Bessel's equation

$$\tilde{x}^2\ddot{y} + \tilde{x}\dot{y} + (\tilde{x}^2 - n^2)y = 0 \qquad \dot{y} = dy/d\tilde{x}, \quad \text{etc.}$$

as a model in physics or elsewhere, one often likes to have another parameter k in addition to n. For this reason we set $\tilde{x} = kx$. Then by the chain rule $\dot{y} = dy/d\tilde{x} = (dy/dx)\, dx/d\tilde{x} = y'/k$, $\ddot{y} = y''/k^2$. In the first two terms, k^2 and k drop out and we get

$$x^2y'' + xy' + (k^2x^2 - n^2)y = 0.$$

Division by x gives the Sturm–Liouville equation

$$[xy']' + \left(-\frac{n^2}{x} + \lambda x\right)y = 0 \qquad \lambda = k^2.$$

This is (1) with $p = x$, $q = -n^2/x$, and $r = x$. ■

Eigenfunctions, Eigenvalues

Clearly, $y \equiv 0$ is a solution—the **"trivial solution"**—for any λ because (1) is homogeneous and (2) has zeros on the right. This is of no interest. We want to find **eigenfunctions** $y(x)$, that is, solutions of (1) satisfying (2) without being identically zero. We call a number λ for which an eigenfunction exists an **eigenvalue** of the Sturm–Liouville problem (1), (2).

EXAMPLE 2 Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$\text{(3)} \qquad y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0.$$

This problem arises, for instance, if an elastic string (a violin string, for example) is stretched a little and then fixed at its ends $x = 0$ and $x = \pi$ and allowed to vibrate. Then $y(x)$ is the "space function" of the deflection $u(x, t)$ of the string, assumed in the form $u(x, t) = y(x)w(t)$, where t is time. (This model will be discussed in great detail in Secs. 12.2–12.4.)

Solution. From (1) and (2) we see that $p = 1$, $q = 0$, $r = 1$ in (1), and $a = 0$, $b = \pi$, $k_1 = l_1 = 1$, $k_2 = l_2 = 0$ in (2). For negative $\lambda = -\nu^2$ a general solution of the ODE in (3) is $y(x) = c_1e^{\nu x} + c_2e^{-\nu x}$. From the boundary conditions we obtain $c_1 = c_2 = 0$, so that $y \equiv 0$, which is not an eigenfunction. For $\lambda = 0$ the situation is similar. For positive $\lambda = \nu^2$ a general solution is

$$y(x) = A \cos \nu x + B \sin \nu x.$$

From the first boundary condition we obtain $y(0) = A = 0$. The second boundary condition then yields

$$y(\pi) = B \sin \nu\pi = 0, \qquad \text{thus} \qquad \nu = 0, \pm 1, \pm 2, \cdots.$$

For $\nu = 0$ we have $y \equiv 0$. For $\lambda = \nu^2 = 1, 4, 9, 16, \cdots$, taking $B = 1$, we obtain

$$y(x) = \sin \nu x \qquad (\nu = 1, 2, \cdots).$$

Hence the eigenvalues of the problem are $\lambda = \nu^2$, where $\nu = 1, 2, \cdots$, and corresponding eigenfunctions are $y(x) = \sin \nu x$, where $\nu = 1, 2, \cdots$. ■

Existence of Eigenvalues

Eigenvalues of a Sturm–Liouville problem (1), (2), even infinitely many, exist under rather general conditions on p, q, r in (1). (Sufficient are the conditions in Theorem 1, below, together with $p(x) > 0$ and $r(x) > 0$ on $a < x < b$. Proofs are complicated; see Ref. [A3] or [A11] listed in App. 1.)

Reality of Eigenvalues

Furthermore, if p, q, r, and p' in (1) are real-valued and continuous on the interval $a \leqq x \leqq b$ and r is positive throughout that interval (or negative throughout that interval), then all the eigenvalues of the Sturm–Liouville problem (1), (2) are real. (Proof in App. 4.) This is what the engineer would expect since eigenvalues are often related to frequencies, energies, or other physical quantities that must be real.

Orthogonality

The most remarkable and important property of eigenfunctions of Sturm–Liouville problems is their orthogonality, which will be crucial in series developments in terms of eigenfunctions.

DEFINITION

Orthogonality

Functions $y_1(x), y_2(x), \cdots$ defined on some interval $a \leqq x \leqq b$ are called **orthogonal** on this interval with respect to the **weight function** $r(x) > 0$ if for all m and all n different from m,

$$\int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0 \qquad (m \neq n). \tag{4}$$

The **norm** $\|y_m\|$ of y_m is defined by

$$\|y_m\| = \sqrt{\int_a^b r(x) y_m^{\,2}(x)\, dx}\,. \tag{5}$$

Note that this is the square root of the integral in (4) with $n = m$.

The functions $y_1, y_2, \cdots$ are called **orthonormal** on $a \leqq x \leqq b$ if they are orthogonal on this interval and all have norm 1.

If $r(x) = 1$, we more briefly call the functions *orthogonal* instead of orthogonal with respect to $r(x) = 1$; similarly for orthonormality. Then

$$\int_a^b y_m(x)\, y_n(x)\, dx = 0 \quad (m \neq n), \qquad \|y_m\| = \sqrt{\int_a^b y_m^{\,2}(x)\, dx}\,.$$

EXAMPLE 3 **Orthogonal Functions. Orthonormal Functions**

The functions $y_m(x) = \sin mx$, $m = 1, 2, \cdots$ form an orthogonal set on the interval $-\pi \leqq x \leqq \pi$, because for $m \neq n$ we obtain by integration [see (11) in App. A3.1]

$$\int_{-\pi}^{\pi} y_m(x) y_n(x)\, dx = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (m-n)x\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos (m+n)x\, dx = 0.$$

The norm $\|y_m\|$ equals $\sqrt{\pi}$, because

$$\|y_m\|^2 = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi \qquad (m = 1, 2, \cdots).$$

Hence the corresponding orthonormal set, obtained by division by the norm, is

$$\frac{\sin x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \quad \frac{\sin 3x}{\sqrt{\pi}}, \quad \cdots.$$ ■

Orthogonality of Eigenfunctions

THEOREM 1

Orthogonality of Eigenfunctions

Suppose that the functions p, q, r, and p' in the Sturm–Liouville equation (1) *are real-valued and continuous and $r(x) > 0$ on the interval $a \leqq x \leqq b$. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Sturm–Liouville problem* (1), (2) *that correspond to different eigenvalues λ_m and λ_n, respectively. Then y_m, y_n are orthogonal on that interval with respect to the weight function r, that is,*

$$\int_a^b r(x)y_m(x)y_n(x)\,dx = 0 \qquad (m \neq n). \tag{6}$$

If $p(a) = 0$, then (2a) *can be dropped from the problem. If $p(b) = 0$, then* (2b) *can be dropped.* [It is then required that y and y' remain bounded at such a point, and the problem is called **singular,** as opposed to a **regular problem** in which (2) is used.]

If $p(a) = p(b)$, then (2) *can be replaced by the* **"periodic boundary conditions"**

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{7}$$

The boundary value problem consisting of the Sturm–Liouville equation (1) and the periodic boundary conditions (7) is called a **periodic Sturm–Liouville problem.**

PROOF By assumption, y_m and y_n satisfy the Sturm–Liouville equations

$$(py_m')' + (q + \lambda_m r)y_m = 0$$
$$(py_n')' + (q + \lambda_n r)y_n = 0$$

respectively. We multiply the first equation by y_n, the second by $-y_m$, and add,

$$(\lambda_m - \lambda_n)ry_my_n = y_m(py_n')' - y_n(py_m')' = [(py_n')y_m - (py_m')y_n]'$$

where the last equality can be readily verified by performing the indicated differentiation of the last expression in brackets. This expression is continuous on $a \leqq x \leqq b$ since p and p' are continuous by assumption and y_m, y_n are solutions of (1). Integrating over x from a to b, we thus obtain

$$(\lambda_m - \lambda_n)\int_a^b ry_my_n\,dx = \Big[p(y_n'y_m - y_m'y_n)\Big]_a^b \qquad (a < b). \tag{8}$$

The expression on the right equals the sum of the subsequent Lines 1 and 2,

$$\begin{aligned} &p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] && \text{(Line 1)}\\ &-p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)] && \text{(Line 2).}\end{aligned} \tag{9}$$

Hence if (9) is zero, (8) with $\lambda_m - \lambda_n \neq 0$ implies the orthogonality (6). Accordingly, we have to show that (9) is zero, using the boundary conditions (2) as needed.

***Case 1.* $p(a) = p(b) = 0$.** Clearly, (9) is zero, and (2) is not needed.

***Case 2.* $p(a) \neq 0, p(b) = 0$.** Line 1 of (9) is zero. Consider Line 2. From (2a) we have

$$k_1 y_n(a) + k_2 y_n'(a) = 0,$$
$$k_1 y_m(a) + k_2 y_m'(a) = 0.$$

Let $k_2 \neq 0$. We multiply the first equation by $y_m(a)$, the last by $-y_n(a)$ and add,

$$k_2[y_n'(a) y_m(a) - y_m'(a) y_n(a)] = 0.$$

This is k_2 times Line 2 of (9), which thus is zero since $k_2 \neq 0$. If $k_2 = 0$, then $k_1 \neq 0$ by assumption, and the argument of proof is similar.

***Case 3.* $p(a) = 0, p(b) \neq 0$.** Line 2 of (9) is zero. From (2b) it follows that Line 1 of (9) is zero; this is similar to Case 2.

***Case 4.* $p(a) \neq 0, p(b) \neq 0$.** We use both (2a) and (2b) and proceed as in Cases 2 and 3.

***Case 5.* $p(a) = p(b)$.** Then (9) becomes

$$p(b)\big[y_n'(b) y_m(b) - y_m'(b) y_n(b) - y_n'(a) y_m(a) + y_m'(a) y_n(a)\big].$$

The expression in brackets $[\cdot\cdot\cdot]$ is zero, either by (2) used as before, or more directly by (7). Hence in this case, (7) can be used instead of (2), as claimed. This completes the proof of Theorem 1. ■

EXAMPLE 4 **Application of Theorem 1. Vibrating Elastic String**

The ODE in Example 2 is a Sturm–Liouville equation with $p = 1$, $q = 0$, and $r = 1$. From Theorem 1 it follows that the eigenfunctions $y_m = \sin mx$ $(m = 1, 2, \cdots)$ are orthogonal on the interval $0 \leqq x \leqq \pi$. ■

EXAMPLE 5 **Application of Theorem 1. Orthogonality of the Legendre Polynomials**

Legendre's equation is a Sturm–Liouville equation (see Example 1)

$$\big[(1 - x^2)y'\big]' + \lambda y = 0, \qquad \lambda = n(n + 1)$$

with $p = 1 - x^2$, $q = 0$, and $r = 1$. Since $p(-1) = p(1) = 0$, we need no boundary conditions, but have a ***singular Sturm—Liouville problem*** on the interval $-1 \leqq x \leqq 1$. We know that for $n = 0, 1, \cdots$, hence $\lambda = 0, 1 \cdot 2, 2 \cdot 3, \cdots$, the Legendre polynomials $P_n(x)$ are solutions of the problem. Hence these are the eigenfunctions. From Theorem 1 it follows that they are orthogonal on that interval, that is,

$$\textbf{(10)} \qquad \int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad (m \neq n). \quad ■$$

EXAMPLE 6 **Application of Theorem 1. Orthogonality of the Bessel Functions $J_n(x)$**

The Bessel function $J_n(\tilde{x})$ with fixed integer $n \geqq 0$ satisfies Bessel's equation (Sec. 5.5)

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x} \dot{J}_n(\tilde{x}) + (\tilde{x}^2 - n^2) J_n(\tilde{x}) = 0,$$

where $\dot{J}_n = dJ_n/d\tilde{x}$, $\ddot{J}_n = d^2J_n/d\tilde{x}^2$. In Example 1 we transformed this equation, by setting $\tilde{x} = kx$, into a Sturm–Liouville equation

$$[xJ_n'(kx)]' + \left(-\frac{n^2}{x} + k^2 x\right) J_n(kx) = 0$$

with $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and parameter $\lambda = k^2$. Since $p(0) = 0$, Theorem 1 implies orthogonality on an interval $0 \leqq x \leqq R$ (R given, fixed) of those solutions $J_n(kx)$ that are zero at $x = R$, that is,

$$(11) \qquad J_n(kR) = 0 \qquad (n \text{ fixed}).$$

[Note that $q(x) = -n^2/x$ is discontinuous at 0, but this does not affect the proof of Theorem 1.] It can be shown (see Ref. [A13]) that $J_n(\tilde{x})$ has infinitely many zeros, say, $\tilde{x} = \alpha_{n,1} < \alpha_{n,2} < \cdots$ (see Fig. 107 in Sec. 5.5 for $n = 0$ and 1). Hence we must have

$$(12) \qquad kR = \alpha_{n,m} \qquad \text{thus} \qquad k_{n,m} = \alpha_{n,m}/R \qquad (m = 1, 2, \cdots).$$

This proves the following orthogonality property.

THEOREM 2

Orthogonality of Bessel Functions

For each fixed nonnegative integer n the sequence of Bessel functions of the first kind $J_n(k_{n,1}x)$, $J_n(k_{n,2}x)$, $\cdots$ with $k_{n,m}$ as in (12) forms an orthogonal set on the interval $0 \leqq x \leqq R$ with respect to the weight function $r(x) = x$, that is,

$$\textbf{(13)} \qquad \int_0^R xJ_n(k_{n,m}x)J_n(k_{n,j}x)\,dx = 0 \qquad (j \neq m,\ n \text{ fixed}).$$

Hence we have obtained ***infinitely many orthogonal sets***, each corresponding to one of the ***fixed*** values n. This also illustrates the importance of the zeros of the Bessel functions. ■

EXAMPLE 7

Eigenvalues from Graphs

Solve the Sturm–Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(\pi) - y'(\pi) = 0$.

Solution. A general solution and its derivative are

$$y = A\cos kx + B\sin kx \qquad \text{and} \qquad y' = -Ak\sin kx + Bk\cos kx, \qquad k = \sqrt{\lambda}.$$

The first boundary condition gives $y(0) + y'(0) = A + Bk = 0$, hence $A = -Bk$. The second boundary condition and substitution of $A = -Bk$ give

$$\begin{aligned} y(\pi) - y'(\pi) &= A\cos\pi k + B\sin\pi k + Ak\sin\pi k - Bk\cos\pi k \\ &= -Bk\cos\pi k + B\sin\pi k - Bk^2\sin\pi k - Bk\cos\pi k = 0. \end{aligned}$$

We must have $B \neq 0$ since otherwise $B = A = 0$, hence $y = 0$, which is not an eigenfunction. Division by $B\cos\pi k$ gives

$$-k + \tan\pi k - k^2\tan\pi k - k = 0, \qquad \text{thus} \qquad \tan\pi k = \frac{-2k}{k^2 - 1}.$$

The graph in Fig. 110 now shows us where to look for eigenvalues. These correspond to the k-values of the points of intersection of $\tan\pi k$ and the right side $-2k/(k^2 - 1)$ of the last equation. The eigenvalues are $\lambda_m = k_m{}^2$, where $\lambda_0 = 0$ with eigenfunction $y_0 = 1$ and the other λ_m are located near $2^2, 3^2, 4^2, \cdots$, with eigenfunctions $\cos k_m x$ and $\sin k_m x$, $m = 1, 2, \cdots$. The precise numeric determination of the eigenvalues would require a root-finding method (such as those given in Sec. 19.2). ■

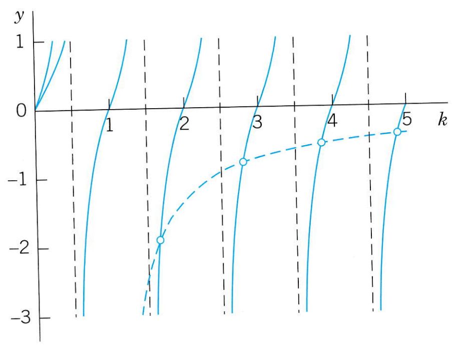


Fig. 110. Example 7. Circles mark the intersections of $\tan\pi k$ and $-2k/(k^2 - 1)$

PROBLEM SET 5.7

1. **(Proof of Theorem 1)** Carry out the details in Cases 3 and 4.
2. **Normalization of eigenfunctions** y_m of (1), (2) means that we multiply y_m by a nonzero constant c_m such that $c_m y_m$ has norm 1. Show that $z_m = cy_m$ with *any* $c \neq 0$ is an eigenfunction for the eigenvalue corresponding to y_m.
3. **(Change of x)** Show that if the functions $y_0(x)$, $y_1(x)$, $\cdots$ form an orthogonal set on an interval $a \leqq x \leqq b$ (with $r(x) = 1$), then the functions $y_0(ct + k)$, $y_1(ct + k)$, $\cdots$, $c > 0$, form an orthogonal set on the interval $(a - k)/c \leqq t \leqq (b - k)/c$.
4. **(Change of x)** Using Prob. 3, derive the orthogonality of 1, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$, $\cdots$ on $-1 \leqq x \leqq 1$ $(r(x) = 1)$ from that of 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$ on $-\pi \leqq x \leqq \pi$.
5. **(Legendre polynomials)** Show that the functions $P_n(\cos \theta)$, $n = 0, 1, \cdots$, form an orthogonal set on the interval $0 \leqq \theta \leqq \pi$ with respect to the weight function $\sin \theta$.
6. **(Tranformation to Sturm–Liouville form)** Show that $y'' + fy' + (g + \lambda h)y = 0$ takes the form (1) if you set $p = \exp(\int f\, dx)$, $q = pg$, $r = hp$. Why would you do such a transformation?

7–19 STURM–LIOUVILLE PROBLEMS

Write the given ODE in the form (1) if it is in a different form. (Use Prob. 6.) Find the eigenvalues and eigenfunctions. Verify orthogonality. (Show the details of your work.)

7. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(5) = 0$
8. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$
9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$
10. $y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$
11. $y'' + \lambda y = 0, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$
12. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) + y'(1) = 0$
13. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$
14. $(xy')' + \lambda x^{-1} y = 0, \quad y(1) = 0, \quad y'(e) = 0$. (Set $x = e^t$.)
15. $(x^{-1}y')' + (\lambda + 1)x^{-3} y = 0, \quad y(1) = 0, \quad y(e^{\pi}) = 0$. (Set $x = e^t$.)
16. $y'' - 2y' + (\lambda + 1)y = 0, \quad y(0) = 0, \quad y(1) = 0$
17. $y'' + 8y' + (\lambda + 16)y = 0, \quad y(0) = 0, \quad y(\pi) = 0$
18. $xy'' + 2y' + \lambda xy = 0, \quad y(\pi) = 0, \quad y(2\pi) = 0$. (Use a CAS or set $y = x^{-1}u$.)
19. $y'' - 2x^{-1}y' + (k^2 + 2x^{-2})y = 0$, $y(1) = 0$, $y(2) = 0$. (Use a CAS or set $y = xu$.)
20. **TEAM PROJECT. Special Functions. Orthogonal polynomials** play a great role in applications. For this reason, Legendre polynomials and various other orthogonal polynomials have been studied extensively; see Refs. [GR1], [GR10] in App. 1. Consider some of the most important ones as follows.

(a) Chebyshev polynomials[10] of the first and second kind are defined by

$$T_n(x) = \cos(n \arccos x)$$

$$U_n(x) = \frac{\sin[(n + 1) \arccos x]}{\sqrt{1 - x^2}}$$

respectively, where $n = 0, 1, \cdots$. Show that

$$T_0 = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$U_0 = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1,$$
$$U_3(x) = 8x^3 - 4x.$$

Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $-1 \leqq x \leqq 1$ with respect to the weight function $r(x) = 1/\sqrt{1 - x^2}$. (*Hint.* To evaluate the integral, set $\arccos x = \theta$.) Verify that $T_n(x)$, $n = 0, 1, 2, 3$, satisfy the **Chebyshev equation**

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

(b) Orthogonality on an infinite interval: Laguerre polynomials[11] are defined by $L_0 = 1$, and

$$L_n(x) = \frac{e^x}{n!} \frac{d^n(x^n e^{-x})}{dx^n}, \qquad n = 1, 2, \cdots.$$

Show that

$$L_1(x) = 1 - x, \qquad L_2(x) = 1 - 2x + x^2/2,$$

$$L_3(x) = 1 - 3x + 3x^2/2 - x^3/6.$$

Prove that the Laguerre polynomials are orthogonal on the positive axis $0 \leqq x < \infty$ with respect to the weight function $r(x) = e^{-x}$. *Hint.* Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x} x^k L_n\, dx = 0$ for $k < n$. Do this by k integrations by parts.

[10] PAFNUTI CHEBYSHEV (1821–1894), Russian mathematician, is known for his work in approximation theory and the theory of numbers. Another transliteration of the name is TCHEBICHEF.

[11] EDMOND LAGUERRE (1834–1886), French mathematician, who did research work in geometry and in the theory of infinite series.

5.8 Orthogonal Eigenfunction Expansions

Orthogonal functions (obtained from Sturm–Liouville problems or otherwise) yield important series developments of given functions, as we shall see. This includes the famous *Fourier series* (to which we devote Chaps. 11 and 12), the daily bread of the physicist and engineer for solving problems in heat conduction, mechanical and electrical vibrations, etc. Indeed, orthogonality is one of the most useful ideas ever introduced in applied mathematics.

Standard Notation for Orthogonality and Orthonormality

The integral (4) in Sec. 5.7 defining orthogonality is denoted by (y_m, y_n). This is standard. Also, **Kronecker's delta**[12] δ_{mn} is defined by $\delta_{mn} = 0$ if $m \neq n$ and $\delta_{mn} = 1$ if $m = n$ (thus $\delta_{nn} = 1$). Hence for orthonormal functions $y_0, y_1, y_2, \cdots$ with respect to weight $r(x)$ (> 0) on $a \leqq x \leqq b$ we can now simply write $(y_m, y_n) = \delta_{mn}$, written out

(1) $$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x)\, dx = \delta_{mn} = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = n. \end{cases}$$

Also, for the norm we can now write

(2) $$\|y\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x) y_m{}^2(x)\, dx}\,.$$

Write down a few examples of your own, to get used to this practical short notation.

Orthogonal Series

Now comes the instant that shows why orthogonality is a fundamental concept. Let $y_0, y_1, y_2, \cdots$ be an orthogonal set with respect to weight $r(x)$ on an interval $a \leqq x \leqq b$. Let $f(x)$ be a function that can be represented by a convergent series

(3) $$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots.$$

This is called an **orthogonal expansion** or **generalized Fourier series.** If the y_m are eigenfunctions of a Sturm–Liouville problem, we call (3) an **eigenfunction expansion.** In (3) we use again m for summation since n will be used as a fixed order of Bessel functions.

Given $f(x)$, we have to determine the coefficients in (3), called the **Fourier constants** *of $f(x)$ with respect to $y_0, y_1, \cdots$*. Because of the orthogonality this is simple. All we have to do is to multiply both sides of (3) by $r(x)y_n(x)$ (n ***fixed***) and then integrate on both sides from a to b. We assume that term-by-term integration is permissible. (This is justified, for instance, in the case of "uniform convergence," as is shown in Sec. 15.5.) Then we obtain

$$(f, y_n) = \int_a^b r f y_n\, dx = \int_a^b r \left(\sum_{m=0}^{\infty} a_m y_m \right) y_n\, dx = \sum_{m=0}^{\infty} a_m (y_m, y_n).$$

[12] LEOPOLD KRONECKER (1823–1891). German mathematician at Berlin University, who made important contributions to algebra, group theory, and number theory.

Because of the orthogonality all the integrals on the right are zero, except when $m = n$. Hence the whole infinite series reduces to the single term

$$a_n(y_n, y_n) = a_n \|y_n\|^2.$$

Assuming that all the functions y_n have nonzero norm, we can divide by $\|y_n\|^2$; writing again m for n, to be in agreement with (3), we get the desired formula for the Fourier constants

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x)\, dx \qquad (m = 0, 1, \cdots). \tag{4}$$

EXAMPLE 1 **Fourier Series**

A most important class of eigenfunction expansions is obtained from the periodic Sturm–Liouville problem

$$y'' + \lambda y = 0, \qquad y(\pi) = y(-\pi), \qquad y'(\pi) = y'(-\pi).$$

A general solution of the ODE is $y = A \cos kx + B \sin kx$, where $k = \sqrt{\lambda}$. Substituting y and its derivative into the boundary conditions, we obtain

$$A \cos k\pi + B \sin k\pi = A \cos(-k\pi) + B \sin(-k\pi)$$

$$-kA \sin k\pi + kB \cos k\pi = -kA \sin(-k\pi) + kB \cos(-k\pi).$$

Since $\cos(-\alpha) = \cos \alpha$, the cosine terms cancel, so that these equations give no condition for these terms. Since $\sin(-\alpha) = -\sin \alpha$, the equations gives the condition $\sin k\pi = 0$, hence $k\pi = m\pi$, $k = m = 0, 1, 2, \cdots$, so that the eigenfunctions are

$$\cos 0 = 1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos mx, \quad \sin mx, \cdots$$

corresponding pairwise to the eigenvalues $\lambda = k^2 = 0, 1, 4, \cdots, m^2, \cdots$. ($\sin 0 = 0$ is not an eigenfunction.)

By Theorem 1 in Sec. 5.7, any two of these belonging to different eigenvalues are orthogonal on the interval $-\pi \leqq x \leqq \pi$ (note that $r(x) = 1$ for the present ODE). The orthogonality of $\cos mx$ and $\sin mx$ for the same m follows by integration,

$$\int_{-\pi}^{\pi} \cos mx \sin mx\, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin 2mx\, dx = 0.$$

For the norms we get $\|1\| = \sqrt{2\pi}$, and $\sqrt{\pi}$ for all the others, as you may verify by integrating 1, $\cos^2 x$, $\sin^2 x$, etc., from $-\pi$ to π. This gives the series (with a slight extension of notation since we have two functions for each eigenvalue 1, 4, 9, $\cdots$)

$$f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx). \tag{5}$$

According to (4) the coefficients (with $m = 1, 2, \cdots$) are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\, dx. \tag{6}$$

The series (5) is called the **Fourier series** of $f(x)$. Its coefficients are called the **Fourier coefficients** of $f(x)$, as given by the so-called **Euler formulas** (6) (not to be confused with the Euler formula (11) in Sec. 2.2).

For instance, for the **"periodic rectangular wave"** in Fig. 111, given by

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x),$$

we get from (6) the values $a_0 = 0$ and

$$a_m = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos mx\,dx + \int_{0}^{\pi} 1\cdot\cos mx\,dx\right] = 0,$$

$$b_m = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin mx\,dx + \int_{0}^{\pi} 1\cdot\sin mx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos mx}{m}\Bigg|_{-\pi}^{0} - \frac{\cos mx}{m}\Bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi m}[1 - 2\cos m\pi + 1] = \begin{cases} 4/(\pi m) & \text{if } m = 1, 3, \cdots, \\ 0 & \text{if } m = 2, 4 \cdots. \end{cases}$$

Hence the Fourier series of the periodic rectangular wave is

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

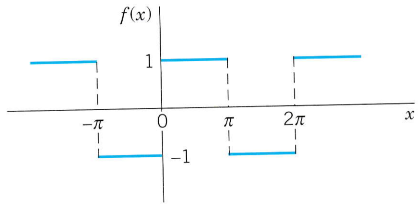


Fig. 111. Periodic rectangular wave in Example 1

Fourier series are by far the most important eigenfunction expansions, so important to the engineer that we shall devote two chapters (11 and 12) to them and their applications, and discuss numerous examples.

Did it surprise you that a series of continuous functions (sine functions) can represent a discontinuous function? More on this in Chap. 11.

EXAMPLE 2 Fourier–Legendre Series

A **Fourier–Legendre series** is an eigenfunction expansion

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) + \cdots = a_0 + a_1 x + a_2(\tfrac{3}{2}x^2 - \tfrac{1}{2}) + \cdots$$

in terms of Legendre polynomials (Sec. 5.3). The latter are the eigenfunctions of the Sturm–Liouville problem in Example 5 of Sec. 5.7 on the interval $-1 \leqq x \leqq 1$. We have $r(x) = 1$ for Legendre's equation, and (4) gives

(7) $$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx, \qquad m = 0, 1, \cdots$$

because the norm is

(8) $$\|P_m\| = \sqrt{\int_{-1}^{1} P_m(x)^2\,dx} = \sqrt{\frac{2}{2m+1}} \qquad (m = 0, 1, \cdots)$$

as we state without proof. (The proof is tricky; it uses Rodrigues's formula in Problem Set 5.3 and a reduction of the resulting integral to a quotient of gamma functions.)

For instance, let $f(x) = \sin \pi x$. Then we obtain the coefficients

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} (\sin \pi x) P_m(x)\, dx, \qquad \text{thus} \qquad a_1 = \frac{3}{2} \int_{-1}^{1} x \sin \pi x \, dx = \frac{3}{\pi} = 0.95493, \qquad \text{etc.}$$

Hence the Fourier–Legendre series of $\sin \pi x$ is

$$\sin \pi x = 0.95493P_1(x) - 1.15824P_3(x) + 0.21429P_5(x) - 0.01664P_7(x) + 0.00068P_9(x) - 0.00002P_{11}(x) + \cdots .$$

The coefficient of P_{13} is about $3 \cdot 10^{-7}$. The sum of the first three nonzero terms gives a curve that practically coincides with the sine curve. Can you see why the even-numbered coefficients are zero? Why a_3 is the absolutely biggest coefficient? ■

EXAMPLE 3 Fourier–Bessel Series

In Example 6 of Sec. 5.7 we obtained infinitely many orthogonal sets of Bessel functions, one for each of J_0, J_1, J_2, $\cdots$. Each set is orthogonal on an interval $0 \leqq x \leqq R$ with a fixed positive R of our choice and with respect to the weight x. The orthogonal set for J_n is $J_n(k_{n,1}x)$, $J_n(k_{n,2}x)$, $J_n(k_{n,3}x)$, $\cdots$, where n is ***fixed*** and $k_{n,m}$ is given in (12), Sec. 5.7. The corresponding Fourier–Bessel series is

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) = a_1 J_n(k_{n,1}x) + a_2 J_n(k_{n,2}x) + a_3 J_n(k_{n,3}x) + \cdots \qquad (n \text{ fixed}). \tag{9}$$

The coefficients are (with $\alpha_{n,m} = k_{n,m}R$)

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x\, f(x)\, J_n(k_{n,m}x)\, dx, \qquad m = 1, 2, \cdots \tag{10}$$

because the square of the norm is

$$\|J_n(k_{n,m}x)\|^2 = \int_0^R x J_n{}^2(k_{n,m}x)\, dx = \frac{R^2}{2} J_{n+1}^2(k_{n,m}R) \tag{11}$$

as we state without proof (which is tricky; see the discussion beginning on p. 576 of [A13]).

For instance, let us consider $f(x) = 1 - x^2$ and take $R = 1$ and $n = 0$ in the series (9), simply writing λ for $\alpha_{0,m}$. Then $k_{n,m} = \alpha_{0,m} = \lambda = 2.405, 5.520, 8.654, 11.792$, etc. (use a CAS or Table A1 in App. 5). Next we calculate the coefficients a_m by (10),

$$a_m = \frac{2}{J_1{}^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx.$$

This can be integrated by a CAS or by formulas as follows. First use $[xJ_1(\lambda x)]' = \lambda x J_0(\lambda x)$ from Theorem 3 in Sec. 5.5 and then integration by parts,

$$a_m = \frac{2}{J_1{}^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx = \frac{2}{J_1{}^2(\lambda)} \left[\frac{1}{\lambda} (1 - x^2) x J_1(\lambda x) \Big|_0^1 - \frac{1}{\lambda} \int_0^1 x J_1(\lambda x)(-2x)\, dx \right].$$

The integral-free part is zero. The remaining integral can be evaluated by $\left[x^2 J_2(\lambda x)\right]' = \lambda x^2 J_1(\lambda x)$ from Theorem 3 in Sec. 5.5. This gives

$$a_m = \frac{4J_2(\lambda)}{\lambda^2 J_1{}^2(\lambda)} \qquad (\lambda = \alpha_{0,m}).$$

Numeric values can be obtained from a CAS (or from the table on p. 409 of Ref. [GR1] in App. 1, together with the formula $J_2 = 2x^{-1}J_1 - J_0$ in Theorem 3 of Sec. 5.5). This gives the eigenfunction expansion of $1 - x^2$ in terms of Bessel functions J_0, that is,

$$1 - x^2 = 1.1081J_0(2.405x) - 0.1398J_0(5.520x) + 0.0455J_0(8.654x) - 0.0210J_0(11.792x) + \cdots .$$

A graph would show that the curve of $1 - x^2$ and that of the sum of the first three terms practically coincide. ■

Mean Square Convergence. Completeness of Orthonormal Sets

The remaining part of this section will give an introduction to a convergence suitable in connection with orthogonal series and quite different from the convergence used in calculus for Taylor series.

In practice, one uses only orthonormal sets that consist of "sufficiently many" functions, so that one can represent large classes of functions by a generalized Fourier series (3)—certainly all continuous functions on an interval $a \leqq x \leqq b$, but also functions that do "not have too many" discontinuities (see Example 1). Such orthonormal sets are called "complete" (in the set of functions considered; definition below). For instance, the orthonormal sets corresponding to Examples 1–3 are complete in the set of functions continuous on the intervals considered (or even in more general sets of functions; see Ref. [GR7], Secs. 3.4–3.7, listed in App. 1, where "complete sets" bear the more modern name "total sets").

In this connection, convergence is **convergence in the norm,** also called **mean-square convergence;** that is, a sequence of functions f_k is called **convergent** *with the limit* f if

$$\lim_{k\to\infty} \|f_k - f\| = 0; \tag{12*}$$

written out by (2) (where we can drop the square root, as this does not affect the limit)

$$\lim_{k\to\infty} \int_a^b r(x)[f_k(x) - f(x)]^2\,dx = 0. \tag{12}$$

Accordingly, the series (3) converges and represents f if

$$\lim_{k\to\infty} \int_a^b r(x)[s_k(x) - f(x)]^2\,dx = 0 \tag{13}$$

where s_k is the kth partial sum of (3),

$$s_k(x) = \sum_{m=0}^{k} a_m y_m(x). \tag{14}$$

By definition, an orthonormal set $y_0, y_1, \cdots$ on an interval $a \leqq x \leqq b$ is **complete** *in a set of functions* S defined on $a \leqq x \leqq b$ if we can approximate every f belonging to S arbitrarily closely by a linear combination $a_0y_0 + a_1y_1 + \cdots + a_ky_k$, that is, technically, if for every $\epsilon > 0$ we can find constants $a_0, \cdots, a_k$ (with k large enough) such that

$$\|f - (a_0y_0 + \cdots + a_ky_k)\| < \epsilon. \tag{15}$$

An interesting and basic consequence of the integral in (13) is obtained as follows. Performing the square and using (14), we first have

$$\int_a^b r(x)[s_k(x) - f(x)]^2\,dx = \int_a^b r{s_k}^2\,dx - 2\int_a^b rfs_k\,dx + \int_a^b rf^2\,dx$$

$$= \int_a^b r\left[\sum_{m=0}^{k} a_m y_m\right]^2 dx - 2\sum_{m=0}^{k} a_m \int_a^b rfy_m\,dx + \int_a^b rf^2\,dx.$$

The first integral on the right equals $\Sigma\, {a_m}^2$ because $\int ry_my_l\,dx = 0$ for $m \neq l$, and $\int r{y_m}^2\,dx = 1$. In the second sum on the right, the integral equals a_m, by (4) with

$\|y_m\|^2 = 1$. Hence the first term on the right cancels half of the second term, so that the right side reduces to

$$-\sum_{m=0}^{k} a_m^{\,2} + \int_a^b rf^2\,dx.$$

This is nonnegative because in the previous formula the integrand on the left is nonnegative (recall that the weight $r(x)$ is positive!) and so is the integral on the left. This proves the important **Bessel's inequality**

$$\sum_{m=0}^{k} a_m^{\,2} \leqq \|f\|^2 = \int_a^b r(x)f(x)^2\,dx \qquad (k = 1, 2, \cdots). \tag{16}$$

Here we can let $k \to \infty$, because the left sides form a monotone increasing sequence that is bounded by the right side, so that we have convergence by the familiar Theorem 1 in App. A3.3. Hence

$$\sum_{m=0}^{\infty} a_m^{\,2} \leqq \|f\|^2. \tag{17}$$

Furthermore, if $y_0, y_1, \cdots$ is complete in a set of functions S, then (13) holds for every f belonging to S. By (15) this implies equality in (16) with $k \to \infty$. Hence in the case of completeness every f in S satisfies the so-called **Parseval's equality**

$$\sum_{m=0}^{\infty} a_m^{\,2} = \|f\|^2 = \int_a^b r(x)f(x)^2\,dx. \tag{18}$$

As a consequence of (18) we prove that in the case of *completeness* there is no function orthogonal to *every* function of the orthonormal set, with the trivial exception of a function of zero norm:

THEOREM 1

Completeness

Let $y_0, y_1, \cdots$ be a complete orthonormal set on $a \leqq x \leqq b$ in a set of functions S. Then if a function f belongs to S and is orthogonal to every y_m, it must have norm zero. In particular, if f is continuous, then f must be identically zero.

PROOF Since f is orthogonal to every y_m, the left side of (18) must be zero. If f is continuous, then $\|f\| = 0$ implies $f(x) \equiv 0$, as can be seen directly from (2) with f instead of y_m because $r(x) > 0$. ■

EXAMPLE 4 **Fourier Series**

The orthonormal set in Example 1 is complete in the set of continuous functions on $-\pi \leqq x \leqq \pi$. Verify directly that $f(x) \equiv 0$ is the only continuous function orthogonal to all the functions of that set.

Solution. Lef f be any continuous function. By the orthogonality (we can omit $\sqrt{2\pi}$ and $\sqrt{\pi}$),

$$\int_{-\pi}^{\pi} 1 \cdot f(x)\,dx = 0, \qquad \int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0, \qquad \int_{-\pi}^{\pi} f(x)\sin mx\,dx = 0.$$

Hence $a_m = 0$ and $b_m = 0$ in (6) for all m, so that (3) reduces to $f(x) \equiv 0$. ■

This is the end of Chap. 5 on the power series method and the Frobenius method, which are indispensable in solving linear ODEs with variable coefficients, some of the most important of which we have discussed and solved. We have also seen that the latter are important sources of special functions having orthogonality properties that make them suitable for orthogonal series representations of given functions.

PROBLEM SET 5.8

1–4 FOURIER–LEGENDRE SERIES

Showing the details of your calculations, develop:

1. $7x^4 - 6x^2$ **2.** $(x + 1)^2$

3. $x^3 - x^2 + x - 1$ **4.** $1, x, x^2, x^3$

5. Prove that if $f(x)$ in Example 2 is even [that is, $f(x) = f(-x)$], its series contains only $P_m(x)$ with even m.

6–16 CAS EXPERIMENTS. FOURIER–LEGENDRE SERIES

Find and graph (on common axes) the partial sums up to that S_{m_0} whose graph practically coincides with that of $f(x)$ within graphical accuracy. State what m_0 is. On what does the size of m_0 seem to depend?

6. $f(x) = \sin \pi x$ **7.** $f(x) = \sin 2\pi x$

8. $f(x) = \cos \pi x$ **9.** $f(x) = \cos 2\pi x$

10. $f(x) = \cos 3\pi x$ **11.** $f(x) = e^x$

12. $f(x) = e^{-x^2}$ **13.** $f(x) = 1/(1 + x^2)$

14. $f(x) = J_0(\alpha_{0,1}x)$, where $\alpha_{0,1}$ is the first positive zero of J_0

15. $f(x) = J_0(\alpha_{0,2}x)$, where $\alpha_{0,2}$ is the second positive zero of J_0

16. $f(x) = J_1(\alpha_{1,1}x)$, where $\alpha_{1,1}$ is the first positive zero of J_1

17. CAS EXPERIMENT. Fourier–Bessel Series. Use Example 3 and again take $n = 10$ and $R = 1$, so that you get the series

$$f(x) = a_1 J_0(\alpha_{0,1}x) + a_2 J_0(\alpha_{0,2}x) + a_3 J_0(\alpha_{0,3}x) + \cdots \tag{19}$$

with the zeros $\alpha_{0,1}$ $\alpha_{0,2}, \cdots$ from your CAS (see also Table A1 in App. 5).

(a) Graph the terms $J_0(\alpha_{0,1}x), \cdots, J_0(\alpha_{0,10}x)$ for $0 \leqq x \leqq 1$ on common axes.

(b) Write a program for calculating partial sums of (19). Find out for what $f(x)$ your CAS can evaluate the integrals. Take two such $f(x)$ and comment empirically on the speed of convergence by observing the decrease of the coefficients.

(c) Take $f(x) = 1$ in (19) and evaluate the integrals for the coefficients analytically by (24a), Sec. 5.5, with $\nu = 1$. Graph the first few partial sums on common axes.

18. TEAM PROJECT. Orthogonality on the Entire Real Axis. Hermite Polynomials.[13] These orthogonal polynomials are defined by $He_0(1) = 1$ and

$$He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}), \quad n = 1, 2, \cdots.$$

REMARK. As is true for many special functions, the literature contains more than one notation, and one sometimes defines as Hermite polynomials the functions

$$H_0^* = 1, \qquad H_n^*(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

This differs from our definition, which is preferred in applications.

(a) Small Values of n. Show that

$$He_1(x) = x, \qquad He_2(x) = x^2 - 1,$$
$$He_3(x) = x^3 - 3x, \qquad He_4(x) = x^4 - 6x^2 + 3.$$

(b) Generating Function. A generating function of the Hermite polynomials is

$$e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x)t^n \tag{20}$$

because $He_n(x) = n!a_n(x)$. Prove this. *Hint:* Use the formula for the coefficients of a Maclaurin series and note that $tx - \frac{1}{2}t^2 = \frac{1}{2}x^2 - \frac{1}{2}(x - t)^2$.

(c) Derivative. Differentiating the generating function with respect to x, show that

$$He_n'(x) = nHe_{n-1}(x). \tag{21}$$

[13]CHARLES HERMITE (1822–1901), French mathematician, is known for his work in algebra and number theory. The great HENRI POINCARÉ (1854–1912) was one of his students.

(d) Orthogonality on the x-Axis needs a weight function that goes to zero sufficiently fast as $x \to \pm\infty$. (Why?) Show that the Hermite polynomials are orthogonal on $-\infty < x < \infty$ with respect to the weight function $r(x) = e^{-x^2/2}$. *Hint.* Use integration by parts and (21).

(e) ODEs. Show that

(22) $$He_n'(x) = xHe_n(x) - He_{n+1}(x).$$

Using this with $n - 1$ instead of n and (21), show that $y = He_n(x)$ satisfies the ODE

(23) $$y'' - xy' + ny = 0.$$

Show that $w = e^{-x^2/4}y$ is a solution of **Weber's equation**[14]

(24) $$w'' + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)w = 0 \qquad (n = 0, 1, \cdots).$$

19. WRITING PROJECT. Orthogonality. Write a short report (2–3 pages) about the most important ideas and facts related to orthogonality and orthogonal series and their applications.

CHAPTER 5 REVIEW QUESTIONS AND PROBLEMS

1. What is a power series? Can it contain negative or fractional powers? How would you test for convergence?

2. Why could we use the power series method for Legendre's equation but needed the Frobenius method for Bessel's equation?

3. Why did we introduce two kinds of Bessel functions, J and Y?

4. What is the hypergeometric equation and why did Gauss introduce it?

5. List the three cases of the Frobenius method, giving examples of your own.

6. What is the difference between an initial value problem and a boundary value problem?

7. What does orthogonality of functions mean and how is it used in series expansions? Give examples.

8. What is the Sturm–Liouville theory and its practical importance?

9. What do you remember about the orthogonality of the Legendre polynomials? Of Bessel functions?

10. What is completeness of orthogonal sets? Why is it important?

11–20 SERIES SOLUTIONS

Find a basis of solutions. Try to identify the series as expansions of known functions. (Show the details of your work.)

11. $y'' - 9y = 0$

12. $(1 - x)^2y'' + (1 - x)y' - 3y = 0$

13. $xy'' - (x + 1)y' + y = 0$

14. $x^2y'' - 3xy' + 4y = 0$

15. $y'' + 4xy' + (4x^2 + 2)y = 0$

16. $x^2y'' - 4xy' + (x^2 + 6)y = 0$

17. $xy'' + (2x + 1)y' + (x + 1)y = 0$

18. $(x^2 - 1)y'' - 2xy' + 2y = 0$

19. $(x^2 - 1)y'' + 4xy' + 2y = 0$

20. $x^2y'' + xy' + (4x^4 - 1)y = 0$

21–25 BESSEL'S EQUATION

Find a general solution in terms of Bessel functions. (Use the indicated transformations and show the details.)

21. $x^2y'' + xy' + (36x^2 - 2)y = 0 \qquad (6x = z)$

22. $x^2y'' + 5xy' + (x^2 - 12)y = 0 \qquad (y = u/x^2)$

23. $x^2y'' + xy' + 4(x^4 - 1)y = 0 \qquad (x^2 = z)$

24. $4x^2y'' - 20xy' + (4x^2 + 35)y = 0 \qquad (y = x^3u)$

25. $y'' + k^2x^2y = 0 \qquad (y = u\sqrt{x},\ \tfrac{1}{2}kx^2 = z)$

26–30 BOUNDARY VALUE PROBLEMS

Find the eigenvalues and eigenfunctions.

26. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$

27. $y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$

28. $(xy')' + \lambda x^{-1}y = 0, \quad y(1) = 0, \quad y(e) = 0.$ (Set $x = e^t$.)

29. $x^2y'' + xy' + (\lambda x^2 - 1)y = 0, \quad y(0) = 0, \quad y(1) = 0$

30. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(2\pi) = 0$

31–35 CAS PROBLEMS

Write a program, develop in a Fourier–Legendre series, and graph the first five partial sums on common axes, together with the given function. Comment on accuracy.

31. $e^{2x} \quad (-1 \leqq x \leqq 1)$

32. $\sin(\pi x^2) \quad (-1 \leqq x \leqq 1)$

33. $1/(1 + |x|) \quad (-1 \leqq x \leqq 1)$

34. $|\cos \pi x| \quad (-1 \leqq x \leqq 1)$

35. x if $0 \leqq x \leqq 1$, 0 if $-1 \leqq x < 0$

[14] HEINRICH WEBER (1842–1913), German mathematician.

SUMMARY OF CHAPTER 5

Series Solution of ODEs. Special Functions

The **power series method** gives solutions of linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

with ***variable coefficients*** p and q in the form of a power series (with any center x_0, e.g., $x_0 = 0$)

$$y(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots. \tag{2}$$

Such a solution is obtained by substituting (2) and its derivatives into (1). This gives a **recurrence formula** for the coefficients. You may program this formula (or even obtain and graph the whole solution) on your CAS.

If p and q are **analytic** at x_0 (that is, representable by a power series in powers of $x - x_0$ with positive radius of convergence; Sec. 5.2), then (1) has solutions of this form (2). The same holds if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$ in

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0$$

are analytic at x_0 and $\tilde{h}(x_0) \neq 0$, so that we can divide by $\tilde{h}$ and obtain the standard form (1). **Legendre's equation** is solved by the power series method in Sec. 5.3.

The **Frobenius method** (Sec. 5.4) extends the power series method to ODEs

$$y'' + \frac{a(x)}{x - x_0}y' + \frac{b(x)}{(x - x_0)^2}y = 0 \tag{3}$$

whose coefficients are **singular** (i.e., not analytic) at x_0, but are "not too bad," namely, such that a and b are analytic at x_0. Then (3) has at least one solution of the form

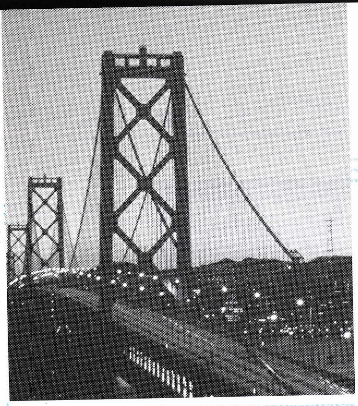
$$y(x) = (x - x_0)^r \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0(x - x_0)^r + a_1(x - x_0)^{r+1} + \cdots \tag{4}$$

where r can be any real (or even complex) number and is determined by substituting (4) into (3) from the **indicial equation** (Sec. 5.4), along with the coefficients of (4). A second linearly independent solution of (3) may be of a similar form (with different r and a_m's) or may involve a logarithmic term. **Bessel's equation** is solved by the Frobenius method in Secs. 5.5 and 5.6.

"Special functions" is a common name for higher functions, as opposed to the usual functions of calculus. Most of them arise either as nonelementary integrals [see (24)–(44) in App. 3.1] or as solutions of (1) or (3). They get a name and notation and are included in the usual CASs if they are important in application or in theory.

Of this kind, and particularly useful to the engineer and physicist, are **Legendre's equation and polynomials** P_0, P_1, $\cdots$ (Sec. 5.3), **Gauss's hypergeometric equation and functions** $F(a, b, c; x)$ (Sec. 5.4), and **Bessel's equation and functions** J_ν **and** Y_ν (Secs. 5.5, 5.6).

Modeling involving ODEs usually leads to initial value problems (Chaps. 1–3) or **boundary value problems.** Many of the latter can be written in the form of **Sturm–Liouville problems** (Sec. 5.7). These are **eigenvalue problems** involving a parameter λ that is often related to frequencies, energies, or other physical quantities. Solutions of Sturm–Liouville problems, called **eigenfunctions,** have many general properties in common, notably the highly important **orthogonality** (Sec. 5.7), which is useful in **eigenfunction expansions** (Sec. 5.8) in terms of cosine and sine (*"Fourier series"*, the topic of Chap. 11), Legendre polynomials, Bessel functions (Sec. 5.8), and other eigenfunctions.



CHAPTER 6

Laplace Transforms

The Laplace transform method is a powerful method for solving linear ODEs and corresponding initial value problems, as well as systems of ODEs arising in engineering. The process of solution consists of three steps (see Fig. 112).

Step 1. The given ODE is transformed into an algebraic equation **("subsidiary equation")**.

Step 2. The subsidiary equation is solved by purely algebraic manipulations.

Step 3. The solution in Step 2 is transformed back, resulting in the solution of the given problem.

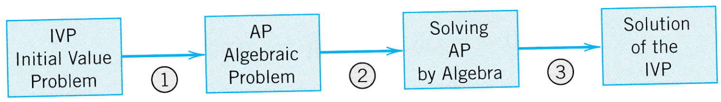


Fig. 112. Solving an IVP by Laplace transforms

Thus solving an ODE is reduced to an *algebraic* problem (plus those transformations). This switching from calculus to algebra is called **operational calculus.** The Laplace transform method is the most important operational method to the engineer. This method has two main advantages over the usual methods of Chaps. 1–4:

A. Problems are solved more directly, initial value problems without first determining a general solution, and nonhomogeneous ODEs without first solving the corresponding homogeneous ODE.

B. More importantly, the use of the **unit step function (Heaviside function** in Sec. 6.3) and **Dirac's delta** (in Sec. 6.4) make the method particularly powerful for problems with inputs (driving forces) that have discontinuities or represent short impulses or complicated periodic functions.

In this chapter we consider the Laplace transform and its application to engineering problems involving ODEs. PDEs will be solved by the Laplace transform in Sec. 12.11. **General formulas** are listed in Sec. 6.8, **transforms and inverses** in Sec. 6.9. The usual **CASs** can handle most Laplace transforms.

Prerequisite: Chap. 2
Sections that may be omitted in a shorter course: 6.5, 6.7
References and Answers to Problems: App. 1 Part A, App. 2.

6.1 Laplace Transform. Inverse Transform. Linearity. s-Shifting

If $f(t)$ is a function defined for all $t \geqq 0$, its **Laplace transform**[1] is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$; thus

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

Here we must assume that $f(t)$ is such that the integral exists (that is, has some finite value). This assumption is usually satisfied in applications—we shall discuss this near the end of the section.

Not only is the result $F(s)$ called the Laplace transform, but the operation just described, which yields $F(s)$ from a given $f(t)$, is also called the **Laplace transform.** It is an **"integral transform"**

$$F(s) = \int_0^\infty k(s, t) f(t)\, dt$$

with **"kernel"** $k(s, t) = e^{-st}$.

Furthermore, the given function $f(t)$ in (1) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$; that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F). \tag{1*}$$

Note that (1) and (1*) together imply $\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and $\mathcal{L}(\mathcal{L}^{-1}(F)) = F$.

Notation

Original functions depend on t and their transforms on s—keep this in mind! Original functions are denoted by ***lowercase letters*** and their transforms by the same ***letters in capital,*** so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

EXAMPLE 1 Laplace Transform

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad (s > 0).$$

---

[1] PIERRE SIMON MARQUIS DE LAPLACE (1749–1827), great French mathematician, was a professor in Paris. He developed the foundation of potential theory and made important contributions to celestial mechanics, astronomy in general, special functions, and probability theory. Napoléon Bonaparte was his student for a year. For Laplace's interesting political involvements, see Ref. [GR2], listed in App. 1.

The powerful practical Laplace transform techniques were developed over a century later by the English electrical engineer OLIVER HEAVISIDE (1850–1925) and were often called "Heaviside calculus."

We shall drop variables when this simplifies formulas without causing confusion. For instance, in (1) we wrote $\mathcal{L}(f)$ instead of $\mathcal{L}(f)(s)$ and in (1*) $\mathcal{L}^{-1}(F)$ instead of $\mathcal{L}^{-1}(F)(t)$.

Our notation is convenient, but we should say a word about it. The interval of integration in (1) is infinite. Such an integral is called an **improper integral** and, by definition, is evaluated according to the rule

$$\int_0^{\infty} e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^{T} e^{-st} f(t)\, dt.$$

Hence our convenient notation means

$$\int_0^{\infty} e^{-st}\, dt = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^T = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^{0} \right] = \frac{1}{s} \qquad (s > 0).$$

We shall use this notation throughout this chapter. ■

EXAMPLE 2 Laplace Transform $\mathcal{L}(e^{at})$ of the Exponential Function e^{at}

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathcal{L}(f)$.

Solution. Again by (1),

$$\mathcal{L}(e^{at}) = \int_0^{\infty} e^{-st} e^{at}\, dt = \frac{1}{a-s} e^{-(s-a)t} \Big|_0^{\infty};$$

hence, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}. \qquad ■$$

Must we go on in this fashion and obtain the transform of one function after another directly from the definition? The answer is no. And the reason is that new transforms can be found from known ones by the use of the many general properties of the Laplace transform. Above all, the Laplace transform is a "linear operation," just as differentiation and integration. By this we mean the following.

THEOREM 1 Linearity of the Laplace Transform

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

PROOF By the definition in (1),

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^{\infty} e^{-st}[af(t) + bg(t)]\, dt$$

$$= a\int_0^{\infty} e^{-st} f(t)\, dt + b\int_0^{\infty} e^{-st} g(t)\, dt = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}. \qquad ■$$

EXAMPLE 3 Application of Theorem 1: Hyperbolic Functions

Find the transforms of cosh at and sinh at.

Solution. Since $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$, we obtain from Example 2 and Theorem 1

$$\mathcal{L}(\cosh at) = \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}. \qquad ■$$

EXAMPLE 4 **Cosine and Sine**

Derive the formulas

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Solution by Calculus. We write $L_c = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain

$$L_c = \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{1}{s} - \frac{\omega}{s} L_s,$$

$$L_s = \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{e^{-st}}{-s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{\omega}{s} L_c.$$

By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \qquad L_c = \frac{s}{s^2 + \omega^2},$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \qquad L_s = \frac{\omega}{s^2 + \omega^2}.$$

Solution by Transforms Using Derivatives. See next section.

Solution by Complex Methods. In Example 2, if we set $a = i\omega$ with $i = \sqrt{-1}$, we obtain

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{(s - i\omega)(s + i\omega)} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2}.$$

Now by Theorem 1 and $e^{i\omega t} = \cos \omega t + i \sin \omega t$ [see (11) in Sec. 2.2 with ωt instead of t] we have

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos \omega t + i \sin \omega t) = \mathcal{L}(\cos \omega t) + i\mathcal{L}(\sin \omega t).$$

If we equate the real and imaginary parts of this and the previous equation, the result follows. (This formal calculation can be justified in the theory of complex integration.) ■

Basic transforms are listed in Table 6.1. We shall see that from these almost all the others can be obtained by the use of the general properties of the Laplace transform. Formulas 1–3 are special cases of formula 4, which is proved by induction. Indeed, it is true for $n = 0$ because of Example 1 and $0! = 1$. We make the induction hypothesis that it holds for any integer $n \geqq 0$ and then get it for $n + 1$ directly from (1). Indeed, integration by parts first gives

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1} \, dt = -\frac{1}{s} e^{-st} t^{n+1} \Big|_0^\infty + \frac{n+1}{s} \int_0^\infty e^{-st} t^n \, dt.$$

Now the integral-free part is zero and the last part is $(n + 1)/s$ times $\mathcal{L}(t^n)$. From this and the induction hypothesis,

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s} \mathcal{L}(t^n) = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}.$$

This proves formula 4.

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s - a}{(s-a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$

$\Gamma(a + 1)$ in formula 5 is the so-called *gamma function* [(15) in Sec. 5.5 or (24) in App. A3.1]. We get formula 5 from (1), setting $st = x$:

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, dx$$

where $s > 0$. The last integral is precisely that defining $\Gamma(a + 1)$, so we have $\Gamma(a + 1)/s^{a+1}$, as claimed. (**CAUTION!** $\Gamma(a + 1)$ has x^a in the integral, not x^{a+1}.)

Note the formula 4 also follows from 5 because $\Gamma(n + 1) = n!$ for integer $n \geqq 0$.

Formulas 6–10 were proved in Examples 2–4. Formulas 11 and 12 will follow from 7 and 8 by "shifting," to which we turn next.

s-Shifting: Replacing s by $s - a$ in the Transform

The Laplace transform has the very useful property that if we know the transform of $f(t)$, we can immediately get that of $e^{at} f(t)$, as follows.

THEOREM 2

First Shifting Theorem, s-Shifting

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at} f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathcal{L}^{-1}\{F(s - a)\}.$$

PROOF We obtain $F(s-a)$ by replacing s with $s-a$ in the integral in (1), so that

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{-st}[e^{at} f(t)]\, dt = \mathcal{L}\{e^{at} f(t)\}.$$

If $F(s)$ exists (i.e., is finite) for s greater than some k, then our first integral exists for $s - a > k$. Now take the inverse on both sides of this formula to obtain the second formula in the theorem. (**CAUTION!** $-a$ in $F(s-a)$ but $+a$ in $e^{at} f(t)$.) ■

EXAMPLE 5 ***s*-Shifting: Damped Vibrations. Completing the Square**

From Example 4 and the first shifting theorem we immediately obtain formulas 11 and 12 in Table 6.1,

$$\mathcal{L}\{e^{at}\cos \omega t\} = \frac{s-a}{(s-a)^2+\omega^2}, \qquad \mathcal{L}\{e^{at}\sin \omega t\} = \frac{\omega}{(s-a)^2+\omega^2}.$$

For instance, use these formulas to find the inverse of the transform

$$\mathcal{L}(f) = \frac{3s-137}{s^2+2s+401}.$$

Solution. Applying the inverse transform, using its linearity (Prob. 28), and completing the square, we obtain

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration (Fig. 113)

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t).$$ ■

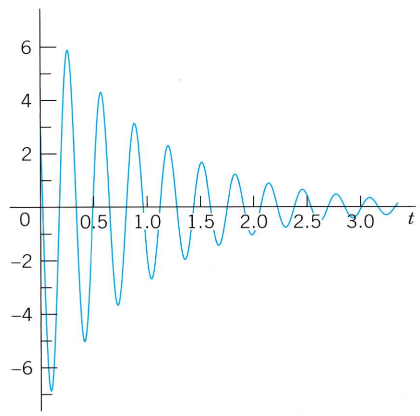


Fig. 113. Vibrations in Example 5

Existence and Uniqueness of Laplace Transforms

This is not a big *practical* problem because in most cases we can check the solution of an ODE without too much trouble. Nevertheless we should be aware of some basic facts.

A function $f(t)$ has a Laplace transform if it does not grow too fast, say, if for all $t \geqq 0$ and some constants M and k it satisfies the **"growth restriction"**

(2) $$|f(t)| \leqq Me^{kt}.$$

(The growth restriction (2) is sometimes called "growth of exponential order," which may be misleading since it hides that the exponent must be kt, not kt^2 or similar.)

$f(t)$ need not be continuous, but it should not be too bad. The technical term (generally used in mathematics) is *piecewise continuity*. $f(t)$ is **piecewise continuous** on a finite interval $a \leqq t \leqq b$ where f is defined, if this interval can be divided into *finitely many* subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior. This then gives **finite jumps** as in Fig. 114 as the only possible discontinuities, but this suffices in most applications, and so does the following theorem.

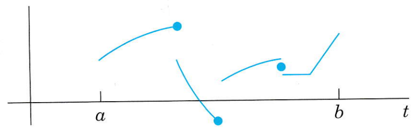


Fig. 114. Example of a piecewise continuous function $f(t)$. (The dots mark the function values at the jumps.)

THEOREM 3

Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) for all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.

PROOF Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. From (2), assuming that $s > k$ (to be needed for the existence of the last of the following integrals), we obtain the proof of the existence of $\mathcal{L}(f)$ from

$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leqq \int_0^\infty |f(t)| e^{-st}\, dt \leqq \int_0^\infty M e^{kt} e^{-st}\, dt = \frac{M}{s-k}. \quad \blacksquare$$

Note that (2) can be readily checked. For instance, $\cosh t < e^t$, $t^n < n! e^t$ (because $t^n/n!$ is a single term of the Maclaurin series), and so on. A function that does not satisfy (2) for any M and k is e^{t^2} (take logarithms to see it). We mention that the conditions in Theorem 3 are sufficient rather than necessary (see Prob. 22).

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points (see Ref. [A14] in App. 1). Hence we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical.

PROBLEM SET 6.1

1–20 LAPLACE TRANSFORMS

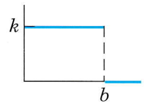
Find the Laplace transforms of the following functions. Show the details of your work. (a, b, k, ω, θ are constants.)

1. $t^2 - 2t$
2. $(t^2 - 3)^2$
3. $\cos 2\pi t$
4. $\sin^2 4t$
5. $e^{2t} \cosh t$
6. $e^{-t} \sinh 5t$
7. $\cos(\omega t + \theta)$
8. $\sin(3t - \frac{1}{2})$
9. e^{3a-2bt}
10. $-8 \sin 0.2t$

11. $\sin t \cos t$

12. $(t + 1)^3$

13.



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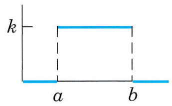

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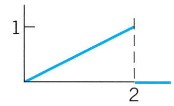

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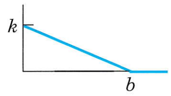

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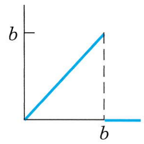

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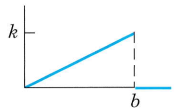

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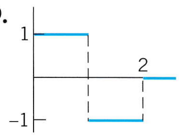
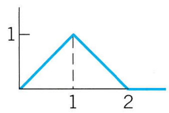


20.



21. Using $\mathcal{L}(f)$ in Prob. 13, find $\mathcal{L}(f_1)$, where $f_1(t) = 0$ if $t \leqq 2$ and $f_1(t) = 1$ if $t > 2$.

22. (Existence) Show that $\mathcal{L}(1/\sqrt{t}) = \sqrt{\pi/s}$. [Use (30) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ in App. 3.1.] Conclude from this that the conditions in Theorem 3 are sufficient but not necessary for the existence of a Laplace transform.

23. (Change of scale) If $\mathcal{L}(f(t)) = F(s)$ and c is any positive constant, show that $\mathcal{L}(f(ct)) = F(s/c)/c$. (*Hint:* Use (1).) Use this to obtain $\mathcal{L}(\cos \omega t)$ from $\mathcal{L}(\cos t)$.

24. (Nonexistence) Show that e^{t^2} does not satisfy a condition of the form (2).

25. (Nonexistence) Give simple examples of functions (defined for all $x \geqq 0$) that have no Laplace transform.

26. (Table 6.1) Derive formula 6 from formulas 9 and 10.

27. (Table 6.1) Convert Table 6.1 from a table for finding transforms to a table for finding inverse transforms (with obvious changes, e.g., $\mathcal{L}^{-1}(1/s^n) = t^{n-1}/(n-1)!$, etc.).

28. (Inverse transform) Prove that $\mathcal{L}^{-1}$ is linear. *Hint.* Use the fact that $\mathcal{L}$ is linear.

29–40 INVERSE LAPLACE TRANSFORMS

Given $F(s) = \mathcal{L}(f)$, find $f(t)$. Show the details. (L, n, k, a, b are constants.)

29. $\dfrac{4s - 3\pi}{s^2 + \pi^2}$

30. $\dfrac{2s + 16}{s^2 - 16}$

31. $\dfrac{s^4 - 3s^2 + 12}{s^5}$

32. $\dfrac{10}{2s + \sqrt{2}}$

33. $\dfrac{n\pi L}{L^2 s^2 + n^2\pi^2}$

34. $\dfrac{20}{(s-1)(s+4)}$

35. $\dfrac{8}{s^2 + 4s}$

36. $\displaystyle\sum_{k=1}^{4} \frac{(k+1)^2}{s + k^2}$

37. $\dfrac{1}{(s - \sqrt{3})(s + \sqrt{5})}$

38. $\dfrac{18s - 12}{9s^2 - 1}$

39. $\dfrac{1}{s^2 + 5} - \dfrac{1}{s + 5}$

40. $\dfrac{1}{(s + a)(s + b)}$

41–54 APPLICATIONS OF THE FIRST SHIFTING THEOREM (s-SHIFTING)

In Probs. 41–46 find the transform. In Probs. 47–54 find the inverse transform. Show the details.

41. $3.8te^{2.4t}$

42. $-3t^4 e^{-0.5t}$

43. $5e^{-at} \sin \omega t$

44. $e^{-3t} \cos \pi t$

45. $e^{-kt}(a \cos t + b \sin t)$

46. $e^{-t}(a_0 + a_1 t + \cdots + a_n t^n)$

47. $\dfrac{7}{(s-1)^3}$

48. $\dfrac{\pi}{(s+\pi)^2}$

49. $\dfrac{\sqrt{8}}{(s + \sqrt{2})^3}$

50. $\dfrac{s - 6}{(s-1)^2 + 4}$

51. $\dfrac{15}{s^2 + 4s + 29}$

52. $\dfrac{4s - 2}{s^2 - 6s + 18}$

53. $\dfrac{\pi}{s^2 + 10\pi s + 24\pi^2}$

54. $\dfrac{2s - 56}{s^2 - 4s - 12}$

6.2 Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that ***operations of calculus on functions are replaced by operations of algebra on transforms.*** Roughly, *differentiation* of $f(t)$ will correspond to *multiplication* of $\mathcal{L}(f)$ by s (see Theorems 1 and 2) and *integration* of $f(t)$ to *division* of $\mathcal{L}(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives.

THEOREM 1

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{1}$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{2}$$

Formula (1) *holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction* (2) *in Sec.* 6.1 *and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly,* (2) *holds if f and f' are continuous for all $t \geqq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.*

PROOF We prove (1) first under the *additional assumption* that f' is continuous. Then by the definition and integration by parts,

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt.$$

Since f satisfies (2) in Sec. 6.1, the integrated part on the right is zero at the upper limit when $s > k$, and at the lower limit it contributes $-f(0)$. The last integral is $\mathcal{L}(f)$. It exists for $s > k$ because of Theorem 3 in Sec. 6.1. Hence $\mathcal{L}(f')$ exists when $s > k$ and (1) holds.

If f' is merely piecewise continuous, the proof is similar. In this case the interval of integration of f' must be broken up into parts such that f' is continuous in each such part.

The proof of (2) now follows by applying (1) to f'' and then substituting (1), that is

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] = s^2\mathcal{L}(f) - sf(0) - f'(0). \quad \blacksquare$$

Continuing by substitution as in the proof of (2) and using induction, we obtain the following extension of Theorem 1.

THEOREM 2

Laplace Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \cdots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction (2) *in Sec.* 6.1. *Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies*

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{3}$$

EXAMPLE 1 **Transform of a Resonance Term (Sec. 2.8)**

Let $f(t) = t \sin \omega t$. Then $f(0) = 0$, $f'(t) = \sin \omega t + \omega t \cos \omega t$, $f'(0) = 0$, $f'' = 2\omega \cos \omega t - \omega^2 t \sin \omega t$. Hence by (2),

$$\mathcal{L}(f'') = 2\omega \frac{s}{s^2 + \omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f), \qquad \text{thus} \qquad \mathcal{L}(f) = \mathcal{L}(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega^2)^2}. \quad \blacksquare$$

EXAMPLE 2 Formulas 7 and 8 in Table 6.1, Sec. 6.1

This is a third derivation of $\mathcal{L}(\cos \omega t)$ and $\mathcal{L}(\sin \omega t)$; cf. Example 4 in Sec. 6.1. Let $f(t) = \cos \omega t$. Then $f(0) = 1$, $f'(0) = 0$, $f''(t) = -\omega^2 \cos \omega t$. From this and (2) we obtain

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s = -\omega^2\mathcal{L}(f). \qquad \text{By algebra,} \qquad \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Similarly, let $g = \sin \omega t$. Then $g(0) = 0$, $g' = \omega \cos \omega t$. From this and (1) we obtain

$$\mathcal{L}(g') = s\mathcal{L}(g) = \omega\mathcal{L}(\cos \omega t). \qquad \text{Hence} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s}\mathcal{L}(\cos \omega t) = \frac{\omega}{s^2 + \omega^2}.$$ ■

Laplace Transform of the Integral of a Function

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

THEOREM 3 Laplace Transform of Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction (2), Sec. 6.1. Then, for $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}. \tag{4}$$

PROOF Denote the integral in (4) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (2), Sec. 6.1, gives

$$|g(t)| = \left|\int_0^t f(\tau)\, d\tau\right| \leqq \int_0^t |f(\tau)|\, d\tau \leqq M\int_0^t e^{k\tau}\, d\tau = \frac{M}{k}(e^{kt} - 1) \leqq \frac{M}{k}e^{kt} \qquad (k > 0).$$

This shows that $g(t)$ also satisfies a growth restriction. Also, $g'(t) = f(t)$, except at points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous on each finite interval and, by Theorem 1, since $g(0) = 0$ (the integral from 0 to 0 is zero)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}.$$

Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides. ■

EXAMPLE 3 Application of Theorem 3: Formulas 19 and 20 in the Table of Sec. 6.9

Using Theorem 3, find the inverse of $\dfrac{1}{s(s^2 + \omega^2)}$ and $\dfrac{1}{s^2(s^2 + \omega^2)}$.

Solution. From Table 6.1 in Sec. 6.1 and the integration in (4) (second formula with the sides interchanged) we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + \omega^2)}\right\} = \int_0^t \frac{\sin \omega\tau}{\omega}\, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

This is formula 19 in Sec. 6.9. Integrating this result again and using (4) as before, we obtain formula 20 in Sec. 6.9:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin\omega t}{\omega^3}.$$

It is typical that results such as these can be found in several ways. In this example, try partial fraction reduction. ■

Differential Equations, Initial Value Problems

We shall now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$\text{(5)} \qquad y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are constant. Here $r(t)$ is the given **input** (*driving force*) applied to the mechanical or electrical system and $y(t)$ is the **output** (*response to the input*) to be obtained. In Laplace's method we do three steps:

Step 1. Setting up the subsidiary equation. This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming (5) by means of (1) and (2), namely,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2. Solution of the subsidiary equation by algebra. We divide by $s^2 + as + b$ and use the so-called **transfer function**

$$\text{(6)} \qquad Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}.$$

(Q is often denoted by H, but we need H much more frequently for other purposes.) This gives the solution

$$\text{(7)} \qquad Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; hence

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

and this explains the name of Q. Note that ***Q depends neither on $r(t)$ nor on the initial conditions*** (but only on a and b).

Step 3. Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$. We reduce (7) (usually by ***partial fractions*** as in calculus) to a sum of terms whose inverses can be found from the tables (e.g., in Sec. 6.1 or Sec. 6.9) or by a CAS, so that we obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ *of* (5).

EXAMPLE 4 **Initial Value Problem: The Basic Laplace Steps**

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution. ***Step 1.*** From (2) and Table 6.1 we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2. The transfer function is $Q = 1/(s^2 - 1)$, and (7) becomes

$$Y = (s + 1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}.$$

Simplification and **partial fraction expansion** gives

$$Y = \frac{1}{s - 1} + \left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

The diagram in Fig. 115 summarizes our approach. ■

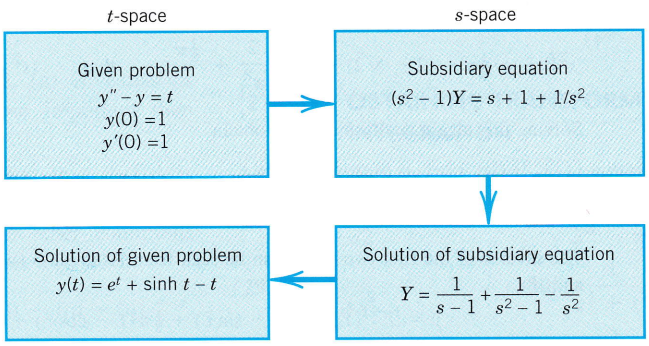


Fig. 115. Laplace transform method

EXAMPLE 5 **Comparison with the Usual Method**

Solve the initial value problem

$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

Solution. From (1) and (2) we see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, \qquad \text{thus} \qquad (s^2 + s + 9)Y = 0.16(s + 1).$$

The solution is

$$Y = \frac{0.16(s+1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2}) + 0.08}{(s + \frac{1}{2})^2 + \frac{35}{4}}.$$

Hence by the first shifting theorem and the formulas for cos and sin in Table 6.1 we obtain

$$y(t) = \mathcal{L}^{-1}(Y) = e^{-t/2}\left(0.16 \cos \sqrt{\frac{35}{4}}\,t + \frac{0.08}{\frac{1}{2}\sqrt{35}} \sin \sqrt{\frac{35}{4}}\,t\right)$$
$$= e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t).$$

This agrees with Example 2, Case (III) in Sec. 2.4. The work was less. ■

Advantages of the Laplace Method

1. *Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.* See Example 4.
2. *Initial values are automatically taken care of.* See Examples 4 and 5.
3. *Complicated inputs* $r(t)$ (right sides of linear ODEs) *can be handled very efficiently,* as we show in the next sections.

EXAMPLE 6 Shifted Data Problems

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \qquad \tilde{y}(0) = \tfrac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s\cdot\tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{thus} \qquad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

$$\tilde{y} = \mathcal{L}^{-1}(\tilde{Y}) = 2(\tilde{t} - \sin\tilde{t}) + \tfrac{1}{2}\pi(1 - \cos\tilde{t}) + \tfrac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t}$$
$$= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2}\sin\tilde{t}.$$

Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin\tilde{t} = \dfrac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t. \qquad \blacksquare$$

PROBLEM SET 6.2

1–8 OBTAINING TRANSFORMS BY DIFFERENTIATION

Using (1) or (2), find $\mathcal{L}(f)$ if $f(t)$ equals:

1. te^{kt}
2. $t\cos 5t$
3. $\sin^2 \omega t$
4. $\cos^2 \pi t$
5. $\sinh^2 at$
6. $\cosh^2 \frac{1}{2}t$
7. $t\sin\frac{1}{2}\pi t$
8. $\sin^4 t$ (Use Prob. 3.)

9. (Derivation by different methods) It is typical that various transforms can be obtained by several methods. Show this for Prob. 1. Show it for $\mathcal{L}(\cos^2 \frac{1}{2}t)$ (a) by expressing $\cos^2 \frac{1}{2}t$ in terms of $\cos t$, (b) by using Prob. 3.

10–24 INITIAL VALUE PROBLEMS

Solve the following initial value problems by the Laplace transform. (If necessary, use partial fraction expansion as in Example 4. Show all details.)

10. $y' + 4y = 0, \quad y(0) = 2.8$

11. $y' + \frac{1}{2}y = 17\sin 2t, \quad y(0) = -1$

12. $y'' - y' - 6y = 0, \quad y(0) = 6, \quad y'(0) = 13$

13. $y'' - \frac{1}{4}y = 0, \quad y(0) = 4, \quad y'(0) = 0$

14. $y'' - 4y' + 4y = 0, \quad y(0) = 2.1,$ $y'(0) = 3.9$

15. $y'' + 2y' + 2y = 0, \quad y(0) = 1,$ $y'(0) = -3$

16. $y'' + ky' - 2k^2y = 0, \quad y(0) = 2,$ $y'(0) = 2k$

17. $y'' + 7y' + 12y = 21e^{3t}, \quad y(0) = 3.5,$ $y'(0) = -10$

18. $y'' + 9y = 10e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$

19. $y'' + 3y' + 2.25y = 9t^3 + 64, \quad y(0) = 1,$ $y'(0) = 31.5$

20. $y'' - 6y' + 5y = 29 \cos 2t, \quad y(0) = 3.2,$ $y'(0) = 6.2$

21. **(Shifted data)** $y' - 6y = 0, \quad y(2) = 4$

22. $y'' - 2y' - 3y = 0, \quad y(1) = -3,$ $y'(1) = -17$

23. $y'' + 3y' - 4y = 6e^{2t-2}, \quad y(1) = 4,$ $y'(1) = 5$

24. $y'' + 2y' + 5y = 50t - 150, \quad y(3) = -4,$ $y'(3) = 14$

25. **PROJECT. Comments on Sec. 6.2.** **(a)** Give reasons why Theorems 1 and 2 are more important than Theorem 3.

(b) Extend Theorem 1 by showing that if $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at some $t = a$ (> 0), the other conditions remaining as in Theorem 1, then (see Fig. 116)

$$(1^*) \quad \mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a + 0) - f(a - 0)]e^{-as}.$$

(c) Verify (1*) for $f(t) = e^{-t}$ if $0 < t < 1$ and 0 if $t > 1$.

(d) Verify (1*) for two more complicated functions of your choice.

(e) Compare the Laplace transform of solving ODEs with the method in Chap. 2. Give examples of your own to illustrate the advantages of the present method (to the extent we have seen them so far).

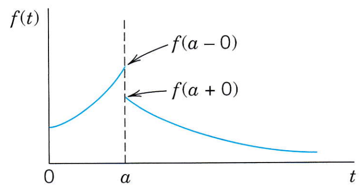


Fig. 116. Formula (1*)

26. **PROJECT. Further Results by Differentiation.** Proceeding as in Example 1, obtain

$$\text{(a)} \quad \mathcal{L}(t \cos \omega t) = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

and from this and Example 1: (b) formula 21, (c) 22, (d) 23 in Sec. 6.9,

$$\text{(e)} \quad \mathcal{L}(t \cosh at) = \frac{s^2 + a^2}{(s^2 - a^2)^2},$$

$$\text{(f)} \quad \mathcal{L}(t \sinh at) = \frac{2as}{(s^2 - a^2)^2}.$$

27–34 OBTAINING TRANSFORMS BY INTEGRATION

Using Theorem 3, find $f(t)$ if $\mathcal{L}(f)$ equals:

27. $\dfrac{1}{s^2 + s/2}$ **28.** $\dfrac{10}{s^3 - \pi s^2}$

29. $\dfrac{1}{s^3 - ks^2}$ **30.** $\dfrac{1}{s^4 + s^2}$

31. $\dfrac{5}{s^3 - 5s}$ **32.** $\dfrac{2}{s^3 + 9s}$

33. $\dfrac{1}{s^4 - 4s^2}$ **34.** $\dfrac{1}{s^4 + \pi^2 s^2}$

35. **(Partial fractions)** Solve Probs. 27, 29, and 31 by using partial fractions.

6.3 Unit Step Function. *t*-Shifting

This section and the next one are extremely important because we shall now reach the point where the Laplace transform method shows its real power in applications and its superiority over the classical approach of Chap. 2. The reason is that we shall introduce two auxiliary functions, the *unit step function* or *Heaviside function* $u(t - a)$ (below) and *Dirac's delta* $\delta(t - a)$ (in Sec. 6.4). These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant (hammerblows, for example).

Unit Step Function (Heaviside Function) $u(t - a)$

The **unit step function** or **Heaviside function** $u(t - a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geqq 0). \tag{1}$$

Figure 117 shows the special case $u(t)$, which has its jump at zero, and Fig. 118 the general case $u(t - a)$ for an arbitrary positive a. (For Heaviside see Sec. 6.1.)

The transform of $u(t - a)$ follows directly from the defining integral in Sec. 6.1,

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt = \int_a^\infty e^{-st} \cdot 1\, dt = -\frac{e^{-st}}{s}\Bigg|_{t=a}^{\infty};$$

here the integration begins at $t = a$ ($\geqq 0$) because $u(t - a)$ is 0 for $t < a$. Hence

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad (s > 0). \tag{2}$$

The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either "off" or "on." Multiplying functions $f(t)$ with $u(t - a)$, we can produce all sorts of effects. The simple basic idea is illustrated in Figs. 119 and 120. In Fig. 119 the given function is shown in (A). In (B) it is switched off between $t = 0$ and $t = 2$ (because $u(t - 2) = 0$ when $t < 2$) and is switched on beginning at $t = 2$. In (C) it is shifted to the right by 2 units, say, for instance, by 2 secs, so that it begins 2 secs later in the same fashion as before. More generally we have the following.

Let $f(t) = 0$ for all negative t. Then $f(t - a)u(t - a)$ with $a > 0$ is $f(t)$ ***shifted*** *(translated) to the right by the amount a.*

Figure 120 shows the effect of many unit step functions, three of them in (A) and infinitely many in (B) when continued periodically to the right; this is the effect of a rectifier that clips off the negative half-waves of a sinusoidal voltage. CAUTION! Make sure that you fully understand these figures, in particular the difference between parts (B) and (C) of Figure 119. Figure 119(C) will be applied next.

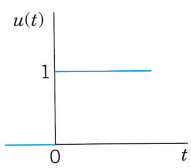


Fig. 117. Unit step function $u(t)$

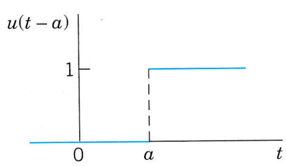


Fig. 118. Unit step function $u(t - a)$

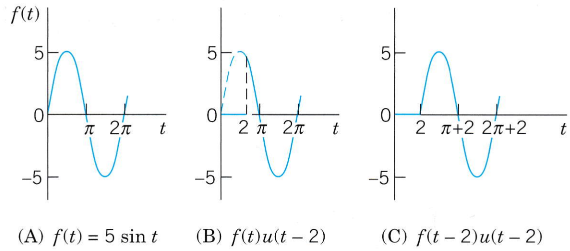


Fig. 119. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

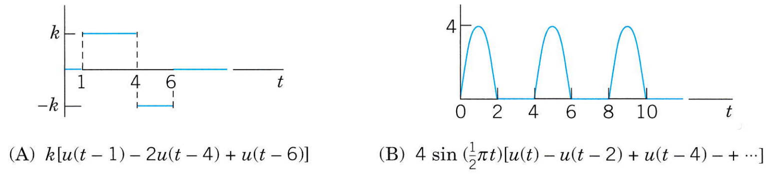


Fig. 120. Use of many unit step functions.

Time Shifting (t-Shifting): Replacing t by $t - a$ in $f(t)$

The first shifting theorem ("s-shifting") in Sec. 6.1 concerned transforms $F(s) = \mathcal{L}\{f(t)\}$ and $F(s - a) = \mathcal{L}\{e^{at}f(t)\}$. The second shifting theorem will concern functions $f(t)$ and $f(t - a)$. Unit step functions are just tools, and the theorem will be needed to apply them in connection with any other functions.

THEOREM 1

Second Shifting Theorem; Time Shifting

If $f(t)$ has the transform $F(s)$, then the **"shifted function"**

(3) $$\tilde{f}(t) = f(t - a)u(t - a) = \begin{cases} 0 & \text{if } t < a \\ f(t - a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}\{f(t)\} = F(s)$, then

(4) $$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s).$$

Or, if we take the inverse on both sides, we can write

(4*) $$f(t - a)u(t - a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$

Practically speaking, if we know $F(s)$, we can obtain the transform of (3) by multiplying $F(s)$ by e^{-as}. In Fig. 119, the transform of 5 sin t is $F(s) = 5/(s^2 + 1)$, hence the shifted function 5 sin $(t - 2)$ $u(t - 2)$ shown in Fig. 119(C) has the transform

$$e^{-2s}F(s) = 5e^{-2s}/(s^2 + 1).$$

PROOF We prove Theorem 1. In (4) on the right we use the definition of the Laplace transform, writing τ for t (to have t available later). Then, taking e^{-as} inside the integral, we have

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau.$$

Substituting $\tau + a = t$, thus $\tau = t - a$, $d\tau = dt$, in the integral (CAUTION, the lower limit changes!), we obtain

$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)\,dt.$$

To make the right side into a Laplace transform, we must have an integral from 0 to ∞, not from a to ∞. But this is easy. We multiply the integrand by $u(t-a)$. Then for t from 0 to a the integrand is 0, and we can write, with $\tilde{f}$ as in (3),

$$e^{-as}F(s) = \int_0^\infty e^{-st}f(t-a)u(t-a)\,dt = \int_0^\infty e^{-st}\tilde{f}(t)\,dt.$$

(Do you now see why $u(t-a)$ appears?) This integral is the left side of (4), the Laplace transform of $\tilde{f}(t)$ in (3). This completes the proof. ■

EXAMPLE 1 **Application of Theorem 1. Use of Unit Step Functions**

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } \quad t > \frac{1}{2}\pi. \end{cases} \qquad \text{(Fig. 121)}$$

Solution. ***Step 1.*** In terms of unit step functions,

$$f(t) = 2(1 - u(t-1)) + \tfrac{1}{2}t^2(u(t-1) - u(t - \tfrac{1}{2}\pi)) + (\cos t)u(t - \tfrac{1}{2}\pi).$$

Indeed, $2(1 - u(t-1))$ gives $f(t)$ for $0 < t < 1$, and so on.

Step 2. To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t-a)u(t-a)$. Thus, $2(1-u(t-1))$ remains as it is and gives the transform $2(1-e^{-s})/s$. Then

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{L}\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\bigg\} = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{\frac{1}{2}\left(t - \frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t - \frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right)u\left(t - \frac{1}{2}\pi\right)\bigg\}$$

$$= \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t - \frac{1}{2}\pi\right)\right)u\left(t - \frac{1}{2}\pi\right)\right\} = -\frac{1}{s^2+1}e^{-\pi s/2}.$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

If the conversion of $f(t)$ to $f(t-a)$ is inconvenient, replace it by

$$\textbf{(4**)}\qquad \mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

(4**) follows from (4) by writing $f(t-a) = g(t)$, hence $f(t) = g(t+a)$ and then again writing f for g. Thus,

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2 + t + \frac{1}{2}\right\} = e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)$$

as before. Similarly for $\mathcal{L}\{\frac{1}{2}t^2u(t-\frac{1}{2}\pi)\}$. Finally, by (4**),

$$\mathcal{L}\left\{\cos t\, u\left(t-\frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t+\frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2+1}.$$ ■

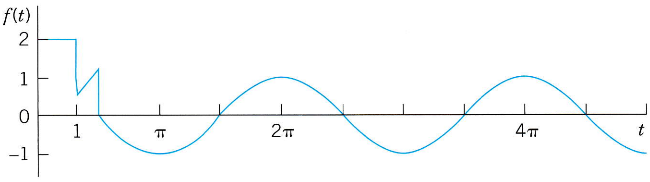


Fig. 121. $f(t)$ in Example 1

EXAMPLE 2 Application of Both Shifting Theorems. Inverse Transform

Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}.$$

Solution. Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $(\sin \pi t)/\pi$, $(\sin \pi t)/\pi$, and te^{-2t} because $1/s^2$ has the inverse t, so that $1/(s+2)^2$ has the inverse te^{-2t} by the first shifting theorem in Sec. 6.1. Hence by the second shifting theorem (t-shifting),

$$f(t) = \frac{1}{\pi}\sin(\pi(t-1))\,u(t-1) + \frac{1}{\pi}\sin(\pi(t-2))\,u(t-2) + (t-3)e^{-2(t-3)}\,u(t-3).$$

Now $\sin(\pi t - \pi) = -\sin \pi t$ and $\sin(\pi t - 2\pi) = \sin \pi t$, so that the second and third terms cancel each other when $t > 2$. Hence we obtain $f(t) = 0$ if $0 < t < 1$, $-(\sin \pi t)/\pi$ if $1 < t < 2$, 0 if $2 < t < 3$, and $(t-3)e^{-2(t-3)}$ if $t > 3$. See Fig. 122. ■

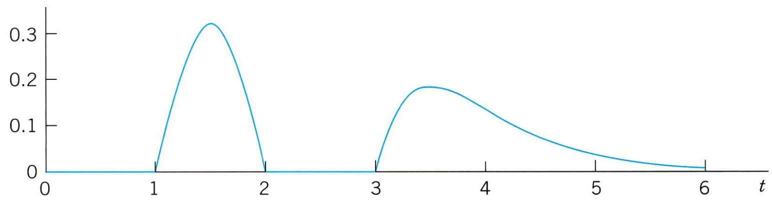


Fig. 122. $f(t)$ in Example 2

EXAMPLE 3 Response of an RC-Circuit to a Single Rectangular Wave

Find the current $i(t)$ in the RC-circuit in Fig. 123 if a single rectangular wave with voltage V_0 is applied. The circuit is assumed to be quiescent before the wave is applied.

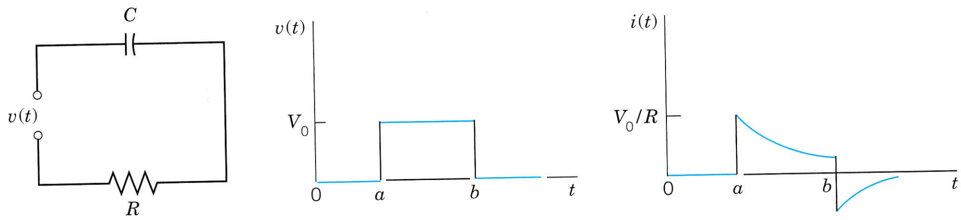


Fig. 123. *RC*-circuit, electromotive force *v*(*t*), and current in Example 3

Solution. The input is $V_0[u(t-a) - u(t-b)]$. Hence the circuit is modeled by the integro-differential equation (see Sec. 2.9 and Fig. 123)

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t) = V_0[u(t-a) - u(t-b)].$$

Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}\,[e^{-as} - e^{-bs}].$$

Solving this equation algebraically for $I(s)$, we get

$$I(s) = F(s)(e^{-as} - e^{-bs}) \qquad \text{where} \qquad F(s) = \frac{V_0/R}{s + 1/(RC)} \qquad \text{and} \qquad \mathcal{L}^{-1}(F) = \frac{V_0}{R}\,e^{-t/(RC)},$$

the last expression being obtained from Table 6.1 in Sec. 6.1. Hence Theorem 1 yields the solution (Fig. 123)

$$i(t) = \mathcal{L}^{-1}(I) = \mathcal{L}^{-1}\{e^{-as}F(s)\} - \mathcal{L}^{-1}\{e^{-bs}F(s)\} = \frac{V_0}{R}\left[e^{-(t-a)/(RC)}u(t-a) - e^{-(t-b)/(RC)}u(t-b)\right];$$

that is, $i(t) = 0$ if $t < a$, and

$$i(t) = \begin{cases} K_1 e^{-t/(RC)} & \text{if } a < t < b \\ (K_1 - K_2)e^{-t/(RC)} & \text{if } a > b \end{cases}$$

where $K_1 = V_0 e^{a/(RC)}/R$ and $K_2 = V_0 e^{b/(RC)}/R$. ■

EXAMPLE 4 Response of an *RLC*-Circuit to a Sinusoidal Input Acting Over a Time Interval

Find the response (the current) of the *RLC*-circuit in Fig. 124, where $E(t)$ is sinusoidal, acting for a short time interval only, say,

$$E(t) = 100 \sin 400t \quad \text{if } 0 < t < 2\pi \qquad \text{and} \qquad E(t) = 0 \text{ if } t > 2\pi$$

and current and charge are initially zero.

Solution. The electromotive force $E(t)$ can be represented by $(100 \sin 400t)(1 - u(t - 2\pi))$. Hence the model for the current $i(t)$ in the circuit is the integro-differential equation (see Sec. 2.9)

$$0.1i' + 11i + 100\int_0^t i(\tau)\,d\tau = (100 \sin 400t)(1 - u(t - 2\pi)), \qquad i(0) = 0, \quad i'(0) = 0.$$

From Theorems 2 and 3 in Sec. 6.2 we obtain the subsidiary equation for $I(s) = \mathcal{L}(i)$

$$0.1sI + 11I + 100\,\frac{I}{s} = \frac{100 \cdot 400s}{s^2 + 400^2}\left(\frac{1}{s} - \frac{e^{-2\pi s}}{s}\right).$$

Solving it algebraically and noting that $s^2 + 110s + 1000 = (s + 10)(s + 100)$, we obtain

$$I(s) = \frac{1000 \cdot 400}{(s + 10)(s + 100)} \left(\frac{s}{s^2 + 400^2} - \frac{se^{-2\pi s}}{s^2 + 400^2} \right).$$

For the first term in the parentheses $(\cdots)$ times the factor in front of them we use the partial fraction expansion

$$\frac{400\,000s}{(s + 10)(s + 100)(s^2 + 400^2)} = \frac{A}{s + 10} + \frac{B}{s + 100} + \frac{Ds + K}{s^2 + 400^2}.$$

Now determine A, B, D, K by your favorite method or by a CAS or as follows. Multiplication by the common denominator gives

$$400\,000s = A(s + 100)(s^2 + 400^2) + B(s + 10)(s^2 + 400^2) + (Ds + K)(s + 10)(s + 100).$$

We set $s = -10$ and -100 and then equate the sums of the s^3 and s^2 terms to zero, obtaining (all values rounded)

$(s = -10)$	$-4\,000\,000 = 90(10^2 + 400^2)A,$	$A = -0.27760$
$(s = -100)$	$-40\,000\,000 = -90(100^2 + 400^2)B,$	$B = 2.6144$
(s^3-terms)	$0 = A + B + D,$	$D = -2.3368$
(s^2-terms)	$0 = 100A + 10B + 110D + K,$	$K = 258.66.$

Since $K = 258.66 = 0.6467 \cdot 400$, we thus obtain for the first term I_1 in $I = I_1 - I_2$

$$I_1 = -\frac{0.2776}{s + 10} + \frac{2.6144}{s + 100} - \frac{2.3368s}{s^2 + 400^2} + \frac{0.6467 \cdot 400}{s^2 + 400^2}.$$

From Table 6.1 in Sec. 6.1 we see that its inverse is

$$i_1(t) = -0.2776e^{-10t} + 2.6144e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

This is the current $i(t)$ when $0 < t < 2\pi$. It agrees for $0 < t < 2\pi$ with that in Example 1 of Sec. 2.9 (except for notation), which concerned the same *RLC*-circuit. Its graph in Fig. 62 in Sec. 2.9 shows that the exponential terms decrease very rapidly. Note that the present amount of work was substantially less.

The second term I_1 of I differs from the first term by the factor $e^{-2\pi s}$. Since $\cos 400(t - 2\pi) = \cos 400t$ and $\sin 400(t - 2\pi) = \sin 400t$, the second shifting theorem (Theorem 1) gives the inverse $i_2(t) = 0$ if $0 < t < 2\pi$, and for $> 2\pi$ it gives

$$i_2(t) = -0.2776e^{-10(t-2\pi)} + 2.6144e^{-100(t-2\pi)} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

Hence in $i(t)$ the cosine and sine terms cancel, and the current for $t > 2\pi$ is

$$i(t) = -0.2776(e^{-10t} - e^{-10(t-2\pi)}) + 2.6144(e^{-100t} - e^{-100(t-2\pi)}).$$

It goes to zero very rapidly, practically within 0.5 sec. ■

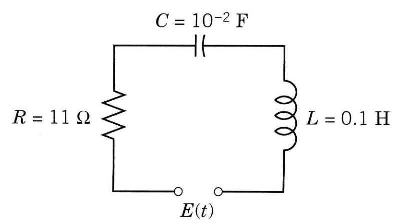


Fig. 124. *RLC*-circuit in Example 4

PROBLEM SET 6.3

1. WRITING PROJECT. Shifting Theorem. Explain and compare the different roles of the two shifting theorems, using your own formulations and examples.

2–13 UNIT STEP FUNCTION AND SECOND SHIFTING THEOREM

Sketch or graph the given function (which is assumed to be zero outside the given interval). Represent it using unit step functions. Find its transform. Show the details of your work.

2. $t \ (0 < t < 1)$
3. $e^t \ (0 < t < 2)$
4. $\sin 3t \ (0 < t < \pi)$
5. $t^2 \ (1 < t < 2)$
6. $t^2 \ (t > 3)$
7. $\cos \pi t \ (1 < t < 4)$
8. $1 - e^{-t} \ (0 < t < \pi)$
9. $t \ (5 < t < 10)$
10. $\sin \omega t \ (t > 6\pi/\omega)$
11. $20 \cos \pi t \ (3 < t < 6)$
12. $\sinh t \ (0 < t < 2)$
13. $e^{\pi t} \ (2 < t < 4)$

14–22 INVERSE TRANSFORMS BY THE SECOND SHIFTING THEOREM

Find and sketch or graph $f(t)$ if $\mathcal{L}(f)$ equals:

14. $se^{-s}/(s^2 + \omega^2)$
15. e^{-4s}/s^2
16. $s^{-2} - (s^{-2} + s^{-1})e^{-s}$
17. $(e^{-2\pi s} - e^{-8\pi s})/(s^2 + 1)$
18. $e^{-\pi s}/(s^2 + 2s + 2)$
19. e^{-2s}/s^5
20. $(1 - e^{-s+k})/(s - k)$
21. $se^{-3s}/(s^2 - 4)$
22. $2.5(e^{-3.8s} - e^{-2.6s})/s$

23–34 INITIAL VALUE PROBLEMS, SOME WITH DISCONTINUOUS INPUTS

Using the Laplace transform and showing the details, solve:

23. $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

24. $9y'' - 6y' + y = 0, \quad y(0) = 3, \quad y'(0) = 1$

25. $y'' + 4y' + 13y = 145 \cos 2t, \quad y(0) = 10, \quad y'(0) = 14$

26. $y'' + 10y' + 24y = 144t^2, \quad y(0) = \frac{19}{12}, \quad y'(0) = -5$

27. $y'' + 9y = r(t)$, $r(t) = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0$, $y'(0) = 4$

28. $y'' + 3y' + 2y = r(t)$, $r(t) = 1$ if $0 < t < 1$ and 0 if $t > 1$; $y(0) = 0$, $y'(0) = 0$

29. $y'' + y = r(t)$, $r(t) = t$ if $0 < t < 1$ and 0 if $t > 1$; $y(0) = y'(0) = 0$

30. $y'' - 16y = r(t)$, $r(t) = 48e^{2t}$ if $0 < t < 4$ and 0 if $t > 4$; $y(0) = 3$, $y'(0) = -4$

31. $y'' + y' - 2y = r(t)$, $r(t) = 3 \sin t - \cos t$ if $0 < t < 2\pi$ and $3 \sin 2t - \cos 2t$ if $t > 2\pi$; $y(0) = 1$, $y'(0) = 0$

32. $y'' + 8y' + 15y = r(t)$, $r(t) = 35e^{2t}$ if $0 < t < 2$ and 0 if $t > 2$; $y(0) = 3$, $y'(0) = -8$

33. (Shifted data) $y'' + 4y = 8t^2$ if $0 < t < 5$ and 0 if $t > 5$; $y(1) = 1 + \cos 2$, $y'(1) = 4 - 2 \sin 2$

34. $y'' + 2y' + 5y = 10 \sin t$ if $0 < t < 2\pi$ and 0 if $t > 2\pi$; $y(\pi) = 1$, $y'(\pi) = 2e^{-\pi} - 2$

MODELS OF ELECTRIC CIRCUITS

35. (Discharge) Using the Laplace transform, find the charge $q(t)$ on the capacitor of capacitance C in Fig. 125 if the capacitor is charged so that its potential is V_0 and the switch is closed at $t = 0$.

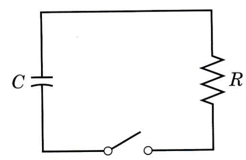


Fig. 125. Problem 35

36–38 RC-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 126 with $R = 10\ \Omega$ and $C = 10^{-2}$ F, where the current at $t = 0$ is assumed to be zero, and:

36. $v(t) = 100$ V if $0.5 < t < 0.6$ and 0 otherwise. Why does $i(t)$ have jumps?

37. $v = 0$ if $t < 2$ and $100(t - 2)$ V if $t > 2$

38. $v = 0$ if $t < 4$ and $14 \cdot 10^6 e^{-3t}$ V if $t > 4$

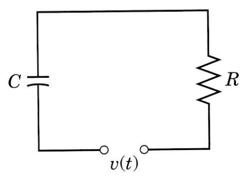


Fig. 126. Problems 36–38

39–41 RL-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 127, assuming $i(0) = 0$ and:

39. $R = 10\ \Omega$, $L = 0.5$ H, $v = 200t$ V if $0 < t < 2$ and 0 if $t > 2$

40. $R = 1\ \text{k}\Omega\ (= 1000\ \Omega)$, $L = 1$ H, $v = 0$ if $0 < t < \pi$, and $40 \sin t$ V if $t > \pi$

41. $R = 25\ \Omega$, $L = 0.1$ H, $v = 490e^{-5t}$ V if $0 < t < 1$ and 0 if $t > 1$

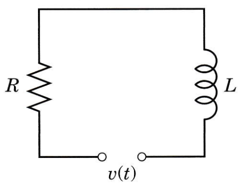


Fig. 127. Problems 39–41

42–44 **LC-CIRCUIT**

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 128, assuming zero initial current and charge on the capacitor and:

42. $L = 1$ H, $C = 0.25$ F, $v = 200(t - \frac{1}{3}t^3)$ V if $0 < t < 1$ and 0 if $t > 1$

43. $L = 1$ H, $C = 10^{-2}$ F, $v = -9900 \cos t$ V if $\pi < t < 3\pi$ and 0 otherwise

44. $L = 0.5$ H, $C = 0.05$ F, $v = 78 \sin t$ V if $0 < t < \pi$ and 0 if $t > \pi$

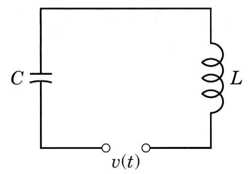


Fig. 128. Problems 42–44

45–47 **RLC-CIRCUIT**

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 129, assuming zero initial current and charge and:

45. $R = 2\ \Omega$, $L = 1$ H, $C = 0.5$ F, $v(t) = 1$ kV if $0 < t < 2$ and 0 if $t > 2$

46. $R = 4\ \Omega$, $L = 1$ H, $C = 0.05$ F, $v = 34e^{-t}$ V if $0 < t < 4$ and 0 if $t > 4$

47. $R = 2\ \Omega$, $L = 1$ H, $C = 0.1$ F, $v = 255 \sin t$ V if $0 < t < 2\pi$ and 0 if $t > 2\pi$

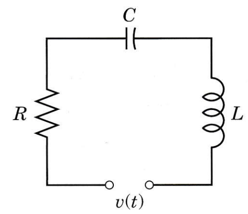


Fig. 129. Problems 45–47

6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

Phenomena of an impulsive nature, such as the action of forces or voltages over short intervals of time, arise in various applications, for instance, if a mechanical system is hit by a hammerblow, an airplane makes a "hard" landing, a ship is hit by a single high wave, or we hit a tennisball by a racket, and so on. Our goal is to show how such problems are modeled by "Dirac's delta function" and can be solved very efficiently by the Laplace transform.

To model situations of that type, we consider the function

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leqq t \leqq a+k \\ 0 & \text{otherwise} \end{cases} \qquad (1) \qquad \text{(Fig. 130)}$$

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a \leqq t \leqq a + k$ is called the **impulse** of the force; similarly for electromotive forces $E(t)$ acting on circuits. Since the blue rectangle in Fig. 130 has area 1, the impulse of f_k in (1) is

$$I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1. \tag{2}$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k \to 0$ $(k > 0)$. This limit is denoted by $\delta(t-a)$, that is,

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a).$$

$\delta(t-a)$ is called the **Dirac delta function**[2] or the **unit impulse function.**

$\delta(t-a)$ is not a function in the ordinary sense as used in calculus, but a so-called *generalized function.*[2] To see this, we note that the impulse I_k of f_k is 1, so that from (1) and (2) by taking the limit as $k \to 0$ we obtain

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-a)\,dt = 1, \tag{3}$$

but from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Nevertheless, in impulse problems it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function. In particular, for a *continuous* function $g(t)$ one uses the property [often called the **sifting property** of $\delta(t-a)$, not to be confused with *shifting*]

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a) \tag{4}$$

which is plausible by (2).

To obtain the Laplace transform of $\delta(t-a)$, we write

$$f_k(t-a) = \frac{1}{k}\,[u(t-a) - u(t-(a+k))]$$

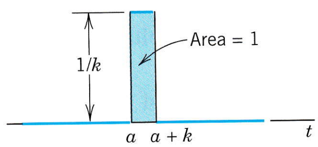


Fig. 130. The function $f_k(t-a)$ in (1)

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[2]PAUL DIRAC (1902–1984), English physicist, was awarded the Nobel Prize [jointly with the Austrian ERWIN SCHRÖDINGER (1887–1961)] in 1933 for his work in quantum mechanics.

Generalized functions are also called **distributions.** Their theory was created in 1936 by the Russian mathematician SERGEI L'VOVICH SOBOLEV (1908–1989), and in 1945, under wider aspects, by the French mathematician LAURENT SCHWARTZ (1915–2002).

and take the transform [see (2)]

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\,\frac{1-e^{-ks}}{ks}\,.$$

We now take the limit as $k \to 0$. By l'Hôpital's rule the quotient on the right has the limit 1 (differentiate the numerator and the denominator separately with respect to k, obtaining se^{-ks} and s, respectively, and use $se^{-ks}/s \to 1$ as $k \to 0$). Hence the right side has the limit e^{-as}. This suggests defining the transform of $\delta(t-a)$ by this limit, that is,

(5) $$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

The unit step and unit impulse functions can now be used on the right side of ODEs modeling mechanical or electrical systems, as we illustrate next.

EXAMPLE 1 **Mass–Spring System Under a Square Wave**

Determine the response of the damped mass–spring system (see Sec. 2.8) under a square wave, modeled by (see Fig. 131)

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution. From (1) and (2) in Sec. 6.2 and (2) and (4) in this section we obtain the subsidiary equation

$$s^2Y + 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}). \qquad \text{Solution} \qquad Y(s) = \frac{1}{s(s^2+3s+2)}(e^{-s} - e^{-2s}).$$

Using the notation $F(s)$ and partial fractions, we obtain

$$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\,.$$

From Table 6.1 in Sec. 6.1, we see that the inverse is

$$f(t) = \mathcal{L}^{-1}(F) = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}.$$

Therefore, by Theorem 1 in Sec. 6.3 (t-shifting) we obtain the square-wave response shown in Fig. 131,

$$\begin{aligned} y &= \mathcal{L}^{-1}(F(s)e^{-s} - F(s)e^{-2s}) \\ &= f(t-1)u(t-1) - f(t-2)u(t-2) \\ &= \begin{cases} 0 & (0 < t < 1) \\ \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)} & (1 < t < 2) \\ -e^{-(t-1)} + e^{-(t-2)} + \tfrac{1}{2}e^{-2(t-1)} - \tfrac{1}{2}e^{-2(t-2)} & (t > 2). \end{cases} \end{aligned}$$ ■

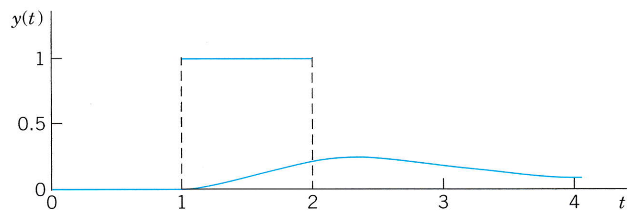

Fig. 131. Square wave and response in Example 1

EXAMPLE 2 **Hammerblow Response of a Mass–Spring System**

Find the response of the system in Example 1 with the square wave replaced by a unit impulse at time $t = 1$.

Solution. We now have the ODE and the subsidiary equation

$$y'' + 3y' + 2y = \delta(t - 1), \qquad \text{and} \qquad (s^2 + 3s + 2)Y = e^{-s}.$$

Solving algebraically gives

$$Y(s) = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}.$$

By Theorem 1 the inverse is

$$y(t) = \mathcal{L}^{-1}(Y) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } \quad t > 1. \end{cases}$$

$y(t)$ is shown in Fig. 132. Can you imagine how Fig. 131 approaches Fig. 132 as the wave becomes shorter and shorter, the area of the rectangle remaining 1? ■

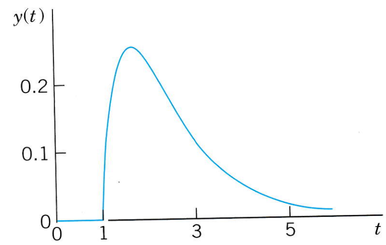

Fig. 132. Response to a hammerblow in Example 2

EXAMPLE 3 **Four-Terminal *RLC*-Network**

Find the output voltage response in Fig. 133 if $R = 20\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, the input is $\delta(t)$ (a unit impulse at time $t = 0$), and current and charge are zero at time $t = 0$.

Solution. To understand what is going on, note that the network is an *RLC*-circuit to which two wires at A and B are attached for recording the voltage $v(t)$ on the capacitor. Recalling from Sec. 2.9 that current $i(t)$ and charge $q(t)$ are related by $i = q' = dq/dt$, we obtain the model

$$Li' + Ri + \frac{q}{C} = Lq'' + Rq' + \frac{q}{C} = q'' + 20q' + 10\,000q = \delta(t).$$

From (1) and (2) in Sec. 6.2 and (5) in this section we obtain the subsidiary equation for $Q(s) = \mathcal{L}(q)$

$$(s^2 + 20s + 10\,000)Q = 1. \qquad \text{Solution} \qquad Q = \frac{1}{(s+10)^2 + 9900}.$$

By the first shifting theorem in Sec. 6.1 we obtain from Q damped oscillations for q and v; rounding $9900 \approx 99.50^2$, we get (Fig. 133)

$$q = \mathcal{L}^{-1}(Q) = \frac{1}{99.50}e^{-10t}\sin 99.50t \qquad \text{and} \qquad v = \frac{q}{C} = 100.5e^{-10t}\sin 99.50t. \qquad ■$$

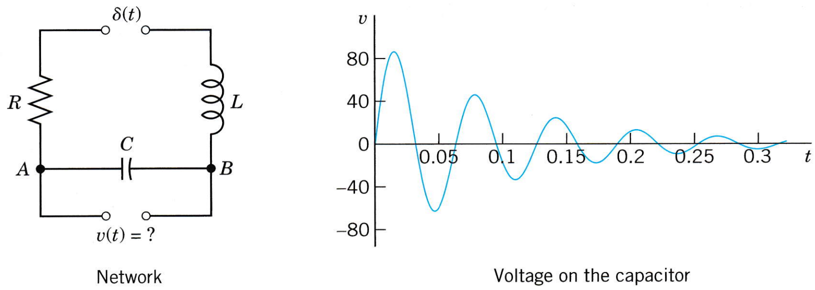

Fig. 133. Network and output voltage in Example 3

More on Partial Fractions

We have seen that the solution Y of a subsidiary equation usually appears as a quotient of polynomials $Y(s) = F(s)/G(s)$, so that a partial fraction representation leads to a sum of expressions whose inverses we can obtain from a table, aided by the first shifting theorem (Sec. 6.1). These representations are sometimes called **Heaviside expansions.**

An *unrepeated factor* $s - a$ in $G(s)$ requires a single partial fraction $A/(s - a)$. See Examples 1 and 2 on pp. 243, 244. *Repeated real factors* $(s - a)^2$, $(s - a)^3$, etc., require partial fractions

$$\frac{A_2}{(s-a)^2} + \frac{A_1}{s-a}, \qquad \frac{A_3}{(s-a)^3} + \frac{A_2}{(s-a)^2} + \frac{A_1}{s-a}, \quad \text{etc.,}$$

The inverses are $(A_2 t + A_1)e^{at}$, $(\frac{1}{2}A_3 t^2 + A_2 t + A_1)e^{at}$, etc.

Unrepeated complex factors $(s - a)(s - \bar{a})$, $a = \alpha + i\beta$, $\bar{a} = \alpha - i\beta$, require a partial fraction $(As + B)/[(s - \alpha)^2 + \beta^2]$. For an application, see Example 4 in Sec. 6.3. A further one is the following.

EXAMPLE 4 **Unrepeated Complex Factors. Damped Forced Vibrations**

Solve the initial value problem for a damped mass–spring system acted upon by a sinusoidal force for some time interval (Fig. 134),

$$y'' + 2y' + 2y = r(t), \quad r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi; \qquad y(0) = 1, \quad y'(0) = -5.$$

Solution. From Table 6.1, (1), (2) in Sec. 6.2, and the second shifting theorem in Sec. 6.3, we obtain the subsidiary equation

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10 \frac{2}{s^2 + 4}(1 - e^{-\pi s}).$$

We collect the Y-terms, $(s^2 + 2s + 2)Y$, take $-s + 5 - 2 = -s + 3$ to the right, and solve,

$$Y = \frac{20}{(s^2 + 4)(s^2 + 2s + 2)} - \frac{20e^{-\pi s}}{(s^2 + 4)(s^2 + 2s + 2)} + \frac{s - 3}{s^2 + 2s + 2}. \tag{6}$$

For the last fraction we get from Table 6.1 and the first shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{s + 1 - 4}{(s + 1)^2 + 1}\right\} = e^{-t}(\cos t - 4 \sin t). \tag{7}$$

In the first fraction in (6) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{s^2+4} + \frac{Ms+N}{s^2+2s+2}.$$

Multiplication by the common denominator gives

$$20 = (As+B)(s^2+2s+2) + (Ms+N)(s^2+4).$$

We determine A, B, M, N. Equating the coefficients of each power of s on both sides gives the four equations

$$\text{(a)}\quad [s^3]:\quad 0 = A + M \qquad\qquad \text{(b)}\quad [s^2]:\quad 0 = 2A + B + N$$

$$\text{(c)}\quad [s]:\quad 0 = 2A + 2B + 4M \qquad\qquad \text{(d)}\quad [s^0]:\quad 20 = 2B + 4N.$$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$, $B = -2$, $M = 2$, $N = 6$, and the first fraction in (6) has the representation

$$\text{(8)}\qquad \frac{-2s-2}{s^2+4} + \frac{2(s+1)+6-2}{(s+1)^2+1}. \qquad \text{Inverse transform:}\quad -2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t).$$

The sum of this and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$\text{(9)}\qquad y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi.$$

In the second fraction in (6) taken with the minus sign we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform

$$+2\cos(2t - 2\pi) + \sin(2t - 2\pi) - e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)]$$

$$= 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t).$$

The sum of this and (9) is the solution for $t > \pi$,

$$\text{(10)}\qquad y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t] \qquad \text{if } t > \pi.$$

Figure 134 shows (9) (for $0 < t < \pi$) and (10) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$. ■

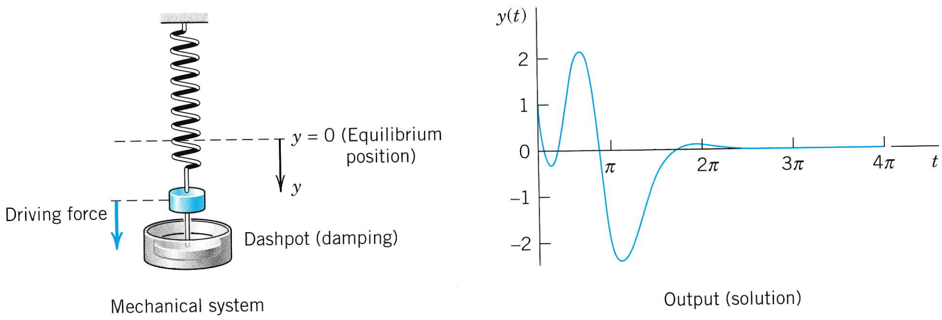

Fig. 134. Example 4

The case of repeated complex factors $[(s - a)(s - \bar{a})]^2$, which is important in connection with resonance, will be handled by "convolution" in the next section.

PROBLEM SET 6.4

1–12 EFFECT OF DELTA FUNCTION ON VIBRATING SYSTEMS

Showing the details, find, graph, and discuss the solution.

1. $y'' + y = \delta(t - 2\pi)$, $y(0) = 10$, $y'(0) = 0$
2. $y'' + 2y' + 2y = e^{-t} + 5\,\delta(t - 2)$, $y(0) = 0$, $y'(0) = 1$
3. $y'' - y = 10\delta(t - \frac{1}{2}) - 100\delta(t - 1)$, $y(0) = 10$, $y'(0) = 1$
4. $y'' + 3y' + 2y = 10(\sin t + \delta(t - 1))$, $y(0) = 1$, $y'(0) = -1$
5. $y'' + 4y' + 5y = [1 - u(t - 10)]e^t - e^{10}\,\delta(t - 10)$, $y(0) = 0$, $y'(0) = 1$
6. $y'' + 2y' - 3y = 100\delta(t - 2) + 100\delta(t - 3)$, $y(0) = 1$, $y'(0) = 0$
7. $y'' + 2y' + 10y = 10[1 - u(t - 4)] - 10\delta(t - 5)$, $y(0) = 1$, $y'(0) = 1$
8. $y'' + 5y' + 6y = \delta(t - \frac{1}{2}\pi) + u(t - \pi)\cos t$, $y(0) = 0$, $y'(0) = 0$
9. $y'' + 2y' + 5y = 25t - 100\delta(t - \pi)$, $y(0) = -2$, $y'(0) = 5$
10. $y'' + 5y = 25t - 100\delta(t - \pi)$, $y(0) = -2$, $y'(0) = 5$. (Compare with Prob. 9.)
11. $y'' + 3y' - 4y = 2e^t - 8e^2\delta(t - 2)$, $y(0) = 2$, $y'(0) = 0$
12. $y'' + y = -2\sin t + 10\delta(t - \pi)$, $y(0) = 0$, $y'(0) = 1$

13. CAS PROJECT. Effect of Damping. Consider a vibrating system of your choice modeled by

$$y'' + cy' + ky = r(t)$$

with $r(t)$ involving a δ-function. **(a)** Using graphs of the solution, describe the effect of continuously decreasing the damping to 0, keeping k constant.

(b) What happens if c is kept constant and k is continuously increased, starting from 0?

(c) Extend your results to a system with two δ-functions on the right, acting at different times.

14. CAS PROJECT. Limit of a Rectangular Wave. Effects of Impulse.

(a) In Example 1, take a rectangular wave of area 1 from 1 to $1 + k$. Graph the responses for a sequence of values of k approaching zero, illustrating that for smaller and smaller k those curves approach the curve shown in Fig. 132. *Hint:* If your CAS gives no solution for the differential equation. involving k, take specific k's from the beginning.

(b) Experiment on the response of the ODE in Example 1 (or of another ODE of your choice) to an impulse $\delta(t - a)$ for various systematically chosen a (> 0); choose initial conditions $y(0) \neq 0$, $y'(0) = 0$. Also consider the solution if no impulse is applied. Is there a dependence of the response on a? On b if you choose $b\delta(t - a)$? Would $-\delta(t - \tilde{a})$ with $\tilde{a} > a$ annihilate the effect of $\delta(t - a)$? Can you think of other questions that one could consider experimentally by inspecting graphs?

15. PROJECT. Heaviside Formulas. (a) Show that for a simple root a and fraction $A/(s - a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s - a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s - a)^m} + \frac{A_{m-1}}{(s - a)^{m-1}} + \cdots + \frac{A_1}{s - a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s - a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m - k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s - a)^m F(s)}{G(s)} \right], \quad k = 1, \cdots, m - 1.$$

16. TEAM PROJECT. Laplace Transform of Periodic Functions

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

$$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s > 0). \tag{11}$$

Prove this theorem. *Hint:* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n - 1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (11), show that the half-wave rectification of $\sin \omega t$ in Fig. 135 has the Laplace transform

$$\mathscr{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})}$$

$$= \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}.$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 136.)

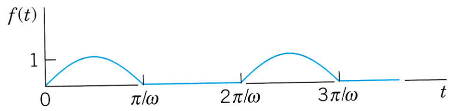

Fig. 135. Half-wave rectification

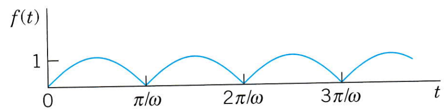

Fig. 136. Full-wave rectification

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of $\sin \omega t$ is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}.$$

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 137.

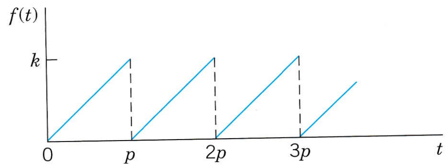

Fig. 137. Saw-tooth wave

(e) Staircase function. Find the Laplace transform of the staircase function in Fig. 138 by noting that it is the difference of kt/p and the function in (d).

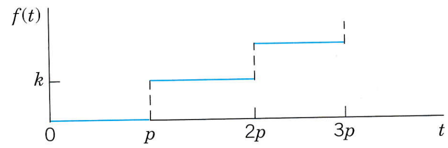

Fig. 138. Staircase function

6.5 Convolution. Integral Equations

Convolution has to do with the multiplication of transforms. The situation is as follows. *Addition* of transforms provides no problem; we know that $\mathscr{L}(f + g) = \mathscr{L}(f) + \mathscr{L}(g)$. Now **multiplication of transforms** occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $\mathscr{L}(f)$ and $\mathscr{L}(g)$ and would like to know the function whose transform is the product $\mathscr{L}(f)\mathscr{L}(g)$. We might perhaps guess that it is fg, but this is false. *The transform of a product is generally different from the product of the transforms of the factors,*

$$\mathscr{L}(fg) \neq \mathscr{L}(f)\mathscr{L}(g) \qquad \text{in general.}$$

To see this take $f = e^t$ and $g = 1$. Then $fg = e^t$, $\mathscr{L}(fg) = 1/(s-1)$, but $\mathscr{L}(f) = 1/(s-1)$ and $\mathscr{L}(1) = 1/s$ give $\mathscr{L}(f)\mathscr{L}(g) = 1/(s^2 - s)$.

According to the next theorem, the correct answer is that $\mathscr{L}(f)\mathscr{L}(g)$ is the transform of the **convolution** of f and g, denoted by the standard notation $f * g$ and defined by the integral

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau. \qquad (1)$$

THEOREM 1 **Convolution Theorem**

If two functions f and g satisfy the assumption in the existence theorem in Sec. 6.1, *so that their transforms F and G exist, the product H = FG is the transform of h given by* (1). (Proof after Example 2.)

EXAMPLE 1 **Convolution**

Let $H(s) = 1/[(s - a)s]$. Find $h(t)$.

Solution. $1/(s - a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{a\tau}$ and $g(t - \tau) \equiv 1$ we thus obtain from (1) the answer

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 \, d\tau = \frac{1}{a}(e^{at} - 1).$$

To check, calculate

$$H(s) = \mathcal{L}(h)(s) = \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{a} \cdot \frac{a}{s^2 - as} = \frac{1}{s-a} \cdot \frac{1}{s} = \mathcal{L}(e^{at})\,\mathcal{L}(1).$$

EXAMPLE 2 **Convolution**

Let $H(s) = 1/(s^2 + \omega^2)^2$. Find $h(t)$.

Solution. The inverse of $1/(s^2 + \omega^2)$ is $(\sin \omega t)/\omega$. Hence from (1) and the trigonometric formula (11) in App. 3.1 with $x = \frac{1}{2}(\omega t + \omega\tau)$ and $y = \frac{1}{2}(\omega t - \omega\tau)$ we obtain

$$\begin{aligned} h(t) = \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} &= \frac{1}{\omega^2}\int_0^t \sin \omega\tau \sin \omega(t - \tau)\, d\tau \\ &= \frac{1}{2\omega^2}\int_0^t [-\cos \omega t + \cos \omega\tau]\, d\tau \\ &= \frac{1}{2\omega^2}\left[-\tau \cos \omega t + \frac{\sin \omega\tau}{\omega}\right]_{\tau=0}^{t} \\ &= \frac{1}{2\omega^2}\left[-t \cos \omega t + \frac{\sin \omega t}{\omega}\right] \end{aligned}$$

in agreement with formula 21 in the table in Sec. 6.9.

PROOF We prove the Convolution Theorem 1. CAUTION! Note which ones are the variables of integration! We can denote them as we want, for instance, by τ and p, and write

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \qquad \text{and} \qquad G(s) = \int_0^\infty e^{-sp} g(p)\, dp.$$

We now set $t = p + \tau$, where τ is at first constant. Then $p = t - \tau$, and t varies from τ to ∞. Thus

$$G(s) = \int_\tau^\infty e^{-s(t-\tau)} g(t - \tau)\, dt = e^{s\tau}\int_\tau^\infty e^{-st} g(t - \tau)\, dt.$$

τ in F and t in G vary independently. Hence we can insert the G-integral into the F-integral. Cancellation of $e^{-s\tau}$ and $e^{s\tau}$ then gives

$$F(s)G(s) = \int_0^\infty e^{-s\tau}f(\tau)e^{s\tau}\int_\tau^\infty e^{-st}g(t-\tau)\,dt\,d\tau = \int_0^\infty f(\tau)\int_\tau^\infty e^{-st}g(t-\tau)\,dt\,d\tau.$$

Here we integrate for fixed τ over t from τ to ∞ and then over τ from 0 to ∞. This is the blue region in Fig. 139. Under the assumption on f and g the order of integration can be reversed (see Ref. [A5] for a proof using uniform convergence). We then integrate first over τ from 0 to t and then over t from 0 to ∞, that is,

$$F(s)G(s) = \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,d\tau\,dt = \int_0^\infty e^{-st}h(t)\,dt = \mathcal{L}(h) = H(s).$$

This completes the proof. ■

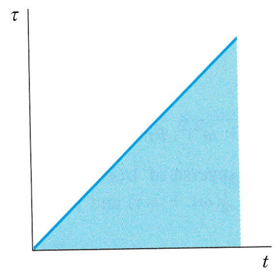

Fig. 139. Region of integration in the $t\tau$-plane in the proof of Theorem 1

From the definition it follows almost immediately that convolution has the properties

$$f * g = g * f \qquad \text{(commutative law)}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)}$$

$$(f * g) * v = f * (g * v) \qquad \text{(associative law)}$$

$$f * 0 = 0 * f = 0$$

similar to those of the multiplication of numbers. Unusual are the following two properties.

EXAMPLE 3 **Unusual Properties of Convolution**

$f * 1 \neq f$ in general. For instance,

$$t * 1 = \int_0^t \tau \cdot 1\,d\tau = \tfrac{1}{2}t^2 \neq t.$$

$(f * f)(t) \geqq 0$ may not hold. For instance, Example 2 with $\omega = 1$ gives

$$\sin t * \sin t = -\tfrac{1}{2}t\cos t + \tfrac{1}{2}\sin t \qquad \text{(Fig. 140).}$$ ■

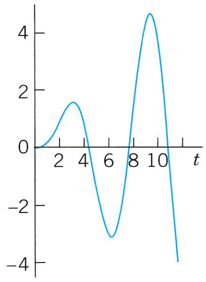

Fig. 140. Example 3

We shall now take up the case of a complex double root (left aside in the last section in connection with partial fractions) and find the solution (the inverse transform) directly by convolution.

EXAMPLE 4 Repeated Complex Factors. Resonance

In an undamped mass–spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system. Then the model is (see Sec. 2.8)

$$y'' + \omega_0{}^2 y = K \sin \omega_0 t$$

where $\omega_0{}^2 = k/m$, k is the spring constant, and m is the mass of the body attached to the spring. We assume $y(0) = 0$ and $y'(0) = 0$, for simplicity. Then the subsidiary equation is

$$s^2 Y + \omega_0{}^2 Y = \frac{K\omega_0}{s^2 + \omega_0{}^2}. \qquad \text{Its solution is} \qquad Y = \frac{K\omega_0}{(s^2 + \omega_0{}^2)^2}.$$

This is a transform as in Example 2 with $\omega = \omega_0$ and multiplied by $K\omega_0$. Hence from Example 2 we can see directly that the solution of our problem is

$$y(t) = \frac{K\omega_0}{2\omega_0{}^2}\left(-t \cos \omega_0 t + \frac{\sin \omega_0 t}{\omega_0}\right) = \frac{K}{2\omega_0{}^2}(-\omega_0 t \cos \omega_0 t + \sin \omega_0 t).$$

We see that the first term grows without bound. Clearly, in the case of resonance such a term must occur. (See also a similar kind of solution in Fig. 54 in Sec. 2.8.) ■

Application to Nonhomogeneous Linear ODEs

Nonhomogeneous linear ODEs can now be solved by a general method based on convolution by which the solution is obtained in the form of an integral. To see this, recall from Sec. 6.2 that the subsidiary equation of the ODE

$$y'' + ay' + by = r(t) \qquad (a, b \text{ constant}) \tag{2}$$

has the solution [(7) in Sec. 6.2]

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

with $R(s) = \mathcal{L}(r)$ and $Q(s) = 1/(s^2 + as + b)$ the transfer function. Inversion of the first term $[\cdots]$ provides no difficulty; depending on whether $\frac{1}{4}a^2 - b$ is positive, zero, or negative, its inverse will be a linear combination of two exponential functions, or of the form $(c_1 + c_2 t)e^{-at/2}$, or a damped oscillation, respectively. The interesting term is $R(s)Q(s)$ because $r(t)$ can have various forms of practical importance, as we shall see. If $y(0) = 0$ and $y'(0) = 0$, then $Y = RQ$, and the convolution theorem gives the solution

(3)
$$y(t) = \int_0^t q(t - \tau) r(\tau)\, d\tau.$$

EXAMPLE 5 **Response of a Damped Vibrating System to a Single Square Wave**

Using convolution, determine the response of the damped mass–spring system modeled by

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and 0 otherwise}, \qquad y(0) = y'(0) = 0.$$

This system with an **input** (a driving force) ***that acts for some time only*** (Fig. 141) has been solved by partial fraction reduction in Sec. 6.4 (Example 1).

Solution by Convolution. The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the convolution integral (3) is (except for the limits of integration)

$$y(t) = \int q(t - \tau) \cdot 1\, d\tau = \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau = e^{-(t-\tau)} - \tfrac{1}{2}e^{-2(t-\tau)}.$$

Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if $t < 1$, the integral is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ (not 0) to t. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}.$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right).$$

Figure 141 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero (why?), and finally decreases to zero in a monotone fashion. ■

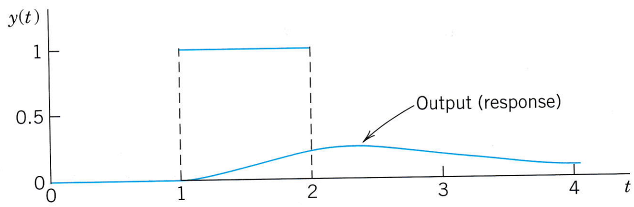

Fig. 141. Square wave and response in Example 5

Integral Equations

Convolution also helps in solving certain **integral equations,** that is, equations in which the unknown function $y(t)$ appears in an integral (and perhaps also outside of it). This concerns equations with an integral of the form of a convolution. Hence these are special and it suffices to explain the idea in terms of two examples and add a few problems in the problem set.

EXAMPLE 6 **A Volterra Integral Equation of the Second Kind**

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$$

Solution. From (1) we see that the given equation can be written as a convolution, $y - y * \sin t = t$. Writing $Y = \mathcal{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \quad \text{and gives the answer} \quad y(t) = t + \frac{t^3}{6}.$$

Check the result by a CAS or by substitution and repeated integration by parts (which will need patience). ■

EXAMPLE 7 **Another Volterra Integral Equation of the Second Kind**

Solve the Volterra integral equation

$$y(t) - \int_0^t (1+\tau)\,y(t-\tau)\,d\tau = 1 - \sinh t.$$

Solution. By (1) we can write $y - (1+t) * y = 1 - \sinh t$. Writing $Y = \mathcal{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1}, \quad \text{hence} \quad Y(s)\cdot\frac{s^2-s-1}{s^2} = \frac{s^2-1-s}{s(s^2-1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2-1} \quad \text{and the solution is} \quad y(t) = \cosh t.$$ ■

PROBLEM SET 6.5

1–8 CONVOLUTIONS BY INTEGRATION

Find by integration:

1. $1 * 1$
2. $t * t$
3. $t * e^t$
4. $e^{at} * e^{bt}$ $(a \neq b)$
5. $1 * \cos \omega t$
6. $1 * f(t)$
7. $e^{kt} * e^{-kt}$
8. $\sin t * \cos t$

9–16 INVERSE TRANSFORMS BY CONVOLUTION

Find $f(t)$ if $\mathcal{L}(f)$ equals:

9. $\dfrac{1}{(s-3)(s+5)}$
10. $\dfrac{1}{s(s-1)}$
11. $\dfrac{1}{s(s^2+4)}$
12. $\dfrac{1}{s^2(s-2)}$
13. $\dfrac{1}{s^2(s^2+1)}$
14. $\dfrac{s}{(s^2+16)^2}$
15. $\dfrac{1}{s(s^2-9)}$
16. $\dfrac{5}{(s^2+1)(s^2+25)}$

17. (Partial fractions) Solve Probs. 9, 11, and 13 by using partial fractions. Comment on the amount of work.

18–25 SOLVING INITIAL VALUE PROBLEMS

Using the convolution theorem, solve:

18. $y'' + y = \sin t, \quad y(0) = 0, \quad y'(0) = 0$
19. $y'' + 4y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0$
20. $y'' + 5y' + 4y = 2e^{-2t}, \quad y(0) = 0, \quad y'(0) = 0$

[3]If the upper limit of integration is *variable,* the equation is named after the Italian mathematician VITO VOLTERRA (1860–1940), and if that limit is *constant,* the equation is named after the Swedish mathematician IVAR FREDHOLM (1866–1927). "Of the second kind (first kind)" indicates that y occurs (does not occur) outside of the integral.

21. $y'' + 9y = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0, \quad y'(0) = 4$

22. $y'' + 3y' + 2y = 1$ if $0 < t < a$ and 0 if $t > a$; $y(0) = 0, \quad y'(0) = 0$

23. $y'' + 4y = 5u(t-1); \quad y(0) = 0, \quad y'(0) = 0$

24. $y'' + 5y' + 6y = \delta(t-3); \quad y(0) = 1, \quad y'(0) = 0$

25. $y'' + 6y' + 8y = 2\delta(t-1) + 2\delta(t-2); \quad y(0) = 1, \quad y'(0) = 0$

26. TEAM PROJECT. Properties of Convolution. Prove:

(a) Commutativity, $f * g = g * f$

(b) Associativity, $(f * g) * v = f * (g * v)$

(c) Distributivity, $f * (g_1 + g_2) = f * g_1 + f * g_2$

(d) Dirac's delta. Derive the sifting formula (4) in Sec. 6.4 by using f_k with $a = 0$ [(1), Sec. 6.4] and applying the mean value theorem for integrals.

(e) Unspecified driving force. Show that forced vibrations governed by

$$y'' + \omega^2 y = r(t), \qquad y(0) = K_1, \qquad y'(0) = K_2$$

with $\omega \neq 0$ and an unspecified driving force $r(t)$ can be written in convolution form,

$$y = \frac{1}{\omega} \sin \omega t * r(t) + K_1 \cos \omega t + \frac{K_2}{\omega} \sin \omega t.$$

27–34 INTEGRAL EQUATIONS

Using Laplace transforms and showing the details, solve:

27. $y(t) - \int_0^t y(\tau)\, d\tau = 1$

28. $y(t) + \int_0^t y(\tau) \cosh (t - \tau)\, d\tau = t + e^t$

29. $y(t) - \int_0^t y(\tau) \sin (t - \tau)\, d\tau = \cos t$

30. $y(t) + 2\int_0^t y(\tau) \cos (t - \tau)\, d\tau = \cos t$

31. $y(t) + \int_0^t (t - \tau) y(\tau)\, d\tau = 1$

32. $y(t) - \int_0^t y(\tau)(t - \tau)\, d\tau = 2 - \frac{1}{2}t^2$

33. $y(t) + 2e^t \int_0^t e^{-\tau} y(\tau)\, d\tau = te^t$

34. $y(t) + \int_0^t e^{2(t-\tau)} y(\tau)\, d\tau = t^2 - t - \frac{1}{2} + \frac{1}{2}e^{2t}$

35. CAS EXPERIMENT. Variation of a Parameter.

(a) Replace 2 in Prob. 33 by a parameter k and investigate graphically how the solution curve changes if you vary k, in particular near $k = -2$.

(b) Make similar experiments with an integral equation of your choice whose solution is oscillating.

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

The variety of methods for obtaining transforms and inverse transforms and their application in solving ODEs is surprisingly large. We have seen that they include direct integration, the use of linearity (Sec. 6.1), shifting (Secs. 6.1, 6.3), convolution (Sec. 6.5), and differentiation and integration of functions $f(t)$ (Sec. 6.2). But this is not all. In this section we shall consider operations of somewhat lesser importance, namely, differentiation and integration of *transforms* $F(s)$ and corresponding operations for functions $f(t)$, with applications to ODEs with variable coefficients.

Differentiation of Transforms

It can be shown that if a function $f(t)$ satisfies the conditions of the existence theorem in Sec. 6.1, then the derivative $F'(s) = dF/ds$ of the transform $F(s) = \mathcal{L}(f)$ can be obtained by differentiating $F(s)$ under the integral sign with respect to s (proof in Ref. [GR4] listed in App. 1). Thus, if

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \qquad \text{then} \qquad F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt.$$

Consequently, if $\mathcal{L}(f) = F(s)$, then

$$\textbf{(1)} \qquad \mathcal{L}\{tf(t)\} = -F'(s), \qquad \text{hence} \qquad \mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

where the second formula is obtained by applying $\mathcal{L}^{-1}$ on both sides of the first formula. In this way, *differentiation of the transform of a function corresponds to the multiplication of the function by* $-t$.

EXAMPLE 1 Differentiation of Transforms. Formulas 21–23 in Sec. 6.9

We shall derive the following three formulas.

	$\mathcal{L}(f)$	$f(t)$
(2)	$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
(3)	$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{t}{2\beta}\sin\beta t$
(4)	$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

Solution. From (1) and formula 8 (with $\omega = \beta$) in Table 6.1 of Sec. 6.1 we obtain by differentiation (CAUTION! Chain rule!)

$$\mathcal{L}(t\sin\beta t) = \frac{2\beta s}{(s^2+\beta^2)^2}\,.$$

Dividing by 2β and using the linearity of $\mathcal{L}$, we obtain (3).

Formulas (2) and (4) are obtained as follows. From (1) and formula 7 (with $\omega = \beta$) in Table 6.1 we find

$$\text{(5)} \qquad \mathcal{L}(t\cos\beta t) = -\frac{(s^2+\beta^2) - 2s^2}{(s^2+\beta^2)^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}\,.$$

From this and formula 8 (with $\omega = \beta$) in Table 6.1 we have

$$\mathcal{L}\left(t\cos\beta t \pm \frac{1}{\beta}\sin\beta t\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2} \pm \frac{1}{s^2+\beta^2}\,.$$

On the right we now take the common denominator. Then we see that for the plus sign the numerator becomes $s^2 - \beta^2 + s^2 + \beta^2 = 2s^2$, so that (4) follows by division by 2. Similarly, for the minus sign the numerator takes the form $s^2 - \beta^2 - s^2 - \beta^2 = -2\beta^2$, and we obtain (2). This agrees with Example 2 in Sec. 6.5. ■

Integration of Transforms

Similarly, if $f(t)$ satisfies the conditions of the existence theorem in Sec. 6.1 and the limit of $f(t)/t$, as t approaches 0 from the right, exists, then for $s > k$,

$$\textbf{(6)} \qquad \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s} \qquad \text{hence} \qquad \mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}\,.$$

In this way, *integration of the transform of a function $f(t)$ corresponds to the division of $f(t)$ by t.*

We indicate how (6) is obtained. From the definition it follows that

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t)\, dt\right] d\tilde{s},$$

and it can be shown (see Ref. [GR4] in App. 1) that under the above assumptions we may reverse the order of integration, that is,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty \left[\int_s^\infty e^{-\tilde{s}t} f(t)\, d\tilde{s}\right] dt = \int_0^\infty f(t) \left[\int_s^\infty e^{-\tilde{s}t}\, d\tilde{s}\right] dt.$$

Integration of $e^{-\tilde{s}t}$ with respect to $\tilde{s}$ gives $e^{-\tilde{s}t}/(-t)$. Here the integral over $\tilde{s}$ on the right equals e^{-st}/t. Therefore,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\} \qquad (s > k). \qquad \blacksquare$$

EXAMPLE 2 **Differentiation and Integration of Transforms**

Find the inverse transform of $\ln\left(1 + \dfrac{\omega^2}{s^2}\right) = \ln \dfrac{s^2 + \omega^2}{s^2}$.

Solution. Denote the given transform by $F(s)$. Its derivative is

$$F'(s) = \frac{d}{ds}\left(\ln(s^2 + \omega^2) - \ln s^2\right) = \frac{2s}{s^2 + \omega^2} - \frac{2s}{s^2}.$$

Taking the inverse transform and using (1), we obtain

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2 + \omega^2} - \frac{2}{s}\right\} = 2\cos\omega t - 2 = -tf(t).$$

Hence the inverse $f(t)$ of $F(s)$ is $f(t) = 2(1 - \cos\omega t)/t$. This agrees with formula 42 in Sec. 6.9.

Alternatively, if we let

$$G(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}, \qquad \text{then} \qquad g(t) = \mathcal{L}^{-1}(G) = 2(\cos\omega t - 1).$$

From this and (6) we get, in agreement with the answer just obtained,

$$\ln \frac{s^2 + \omega^2}{s^2} = \int_s^\infty G(s)\, ds = -\frac{g(t)}{t} = \frac{2}{t}(1 - \cos\omega t),$$

the minus occurring since s is the lower limit of integration.

In a similar way we obtain formula 43 in Sec. 6.9,

$$\mathcal{L}^{-1}\left\{\ln\left(1 - \frac{a^2}{s^2}\right)\right\} = \frac{2}{t}(1 - \cosh at). \qquad \blacksquare$$

Special Linear ODEs with Variable Coefficients

Formula (1) can be used to solve certain ODEs with variable coefficients. The idea is this. Let $\mathcal{L}(y) = Y$. Then $\mathcal{L}(y') = sY - y(0)$ (see Sec. 6.2). Hence by (1),

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}. \tag{7}$$

Similarly, $\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$ and by (1)

$$\mathcal{L}(ty'') = -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] = -2sY - s^2\frac{dY}{ds} + y(0). \tag{8}$$

Hence if an ODE has coefficients such as $at + b$, the subsidiary equation is a first-order ODE for Y, which is sometimes simpler than the given second-order ODE. But if the latter has coefficients $at^2 + bt + c$, then two applications of (1) would give a second-order ODE for Y, and this shows that the present method works well only for rather special ODEs with variable coefficients. An important ODE for which the method is advantageous is the following.

EXAMPLE 3 Laguerre's Equation. Laguerre Polynomials

Laguerre's ODE is

$$ty'' + (1 - t)y' + ny = 0. \tag{9}$$

We determine a solution of (9) with $n = 0, 1, 2, \cdots$. From (7)–(9) we get the subsidiary equation

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0.$$

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, integrating (with the constant of integration taken zero), and taking exponentials, we get

$$\frac{dY}{Y} = -\frac{n+1-s}{s-s^2}\,ds = \left(\frac{n}{s-1} - \frac{n+1}{s}\right)ds \quad \text{and} \quad Y = \frac{(s-1)^n}{s^{n+1}}. \tag{10*}$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and prove **Rodrigues's formula**

$$l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 1, 2, \cdots. \tag{10}$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called **Laguerre polynomials** and are usually denoted by L_n (see Problem Set 5.7, but we continue to reserve capital letters for transforms). We prove (10). By Table 6.1 and the first shifting theorem (s-shifting),

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s+1)^{n+1}}, \qquad \text{hence by (3) in Sec. 6.2} \qquad \mathcal{L}\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n!\,s^n}{(s+1)^{n+1}}$$

because the derivatives up to the order $n - 1$ are zero at 0. Now make another shift and divide by $n!$ to get [see (10) and then (10*)]

$$\mathcal{L}(l_n) = \frac{(s-1)^n}{s^{n+1}} = Y. \qquad \blacksquare$$

PROBLEM SET 6.6

1–12 TRANSFORMS BY DIFFERENTIATION

Showing the details of your work, find $\mathcal{L}(f)$ if $f(t)$ equals:

1. $4te^t$
2. $-t \cosh 2t$
3. $t \sin \omega t$
4. $t \cos (t + k)$
5. $te^{-2t} \sin t$
6. $t^2 \sin 3t$
7. $t^2 \sinh 4t$
8. $t^n e^{kt}$
9. $t^2 \sin \omega t$
10. $t \cos \omega t$
11. $t \sin (t + k)$
12. $te^{-kt} \sin t$

13–20 INVERSE TRANSFORMS

Using differentiation, integration, s-shifting, or convolution (and showing the details), find $f(t)$ if $\mathcal{L}(f)$ equals:

13. $\dfrac{6}{(s+1)^2}$

14. $\dfrac{s}{(s^2+16)^2}$

15. $\dfrac{2(s+2)}{[(s+2)^2+1]^2}$

16. $\dfrac{s}{(s^2-1)^2}$

17. $\dfrac{2}{(s-k)^3}$

18. $\ln \dfrac{s+a}{s+b}$

19. $\ln \dfrac{s}{s-1}$

20. $\operatorname{arccot} \dfrac{s}{\omega}$

21. **WRITING PROJECT. Differentiation and Integration of Functions and Transforms.** Make a short draft of these four operations from memory. Then compare your notes with the text and write a report of 2–3 pages on these operations and their significance in applications.

22. **CAS PROJECT. Laguerre Polynomials. (a)** Write a CAS program for finding $l_n(t)$ in explicit form from (10). Apply it to calculate $l_0, \cdots, l_{10}$. Verify that $l_0, \cdots, l_{10}$ satisfy Laguerre's differential equation (9).

(b) Show that

$$l_n(t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n}{m} t^m$$

and calculate $l_0, \cdots, l_{10}$ from this formula.

(c) Calculate $l_0, \cdots, l_{10}$ recursively from $l_0 = 1$, $l_1 = 1 - t$ by

$$(n+1)l_{n+1} = (2n + 1 - t)l_n - nl_{n-1}.$$

(d) Experiment with the graphs of $l_0, \cdots, l_{10}$, finding out empirically how the first maximum, first minimum, $\cdots$ is moving with respect to its location as a function of n. Write a short report on this.

(e) A **generating function** (definition in Problem Set 5.3) for the Laguerre polynomials is

$$\sum_{n=0}^{\infty} l_n(t)x^n = (1-x)^{-1}e^{tx/(x-1)}.$$

Obtain $l_0, \cdots, l_{10}$ from the corresponding partial sum of this power series in x and compare the l_n with those in (a), (b), or (c).

6.7 Systems of ODEs

The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients (as discussed in Sec. 4.1)

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned} \tag{1}$$

Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(g_1)$, $G_2 = \mathcal{L}(g_2)$, we obtain from (1) in Sec. 6.2 the subsidiary system

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s). \end{aligned}$$

By collecting the Y_1- and Y_2-terms we have

$$\begin{aligned} (a_{11} - s)Y_1 + \quad a_{12}Y_2 \quad &= -y_1(0) - G_1(s) \\ a_{21}Y_1 \quad + (a_{22} - s)Y_2 &= -y_2(0) - G_2(s). \end{aligned} \tag{2}$$

By solving this system algebraically for $Y_1(s)$, $Y_2(s)$ and taking the inverse transform we obtain the solution $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$ of the given system (1).

Note that (1) and (2) may be written in vector form (and similarly for the systems in the examples); thus, setting $\mathbf{y} = [y_1 \quad y_2]^\mathsf{T}$, $\mathbf{A} = [a_{jk}]$, $\mathbf{g} = [g_1 \quad g_2]^\mathsf{T}$, $\mathbf{Y} = [Y_1 \quad Y_2]^\mathsf{T}$, $\mathbf{G} = [G_1 \quad G_2]^\mathsf{T}$ we have

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad \text{and} \qquad (\mathbf{A} - s\mathbf{I})\mathbf{Y} = -\mathbf{y}(0) - \mathbf{G}.$$

EXAMPLE 1 **Mixing Problem Involving Two Tanks**

Tank T_1 in Fig. 142 contains initially 100 gal of pure water. Tank T_2 contains initially 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1. The outflow from T_2 is $2 + 6 = 8$ gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

Solution. The model is obtained in the form of two equations

$$\text{Time rate of change} = \text{Inflow/min} - \text{Outflow/min}$$

for the two tanks (see Sec. 4.1). Thus,

$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6, \qquad y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2.$$

The initial conditions are $y_1(0) = 0$, $y_2(0) = 150$. From this we see that the subsidiary system (2) is

$$\begin{aligned} (-0.08 - s)Y_1 + \quad 0.02Y_2 \quad &= -\frac{6}{s} \\ 0.08Y_1 \quad + (-0.08 - s)Y_2 &= -150. \end{aligned}$$

We solve this algebraically for Y_1 and Y_2 by elimination (or by Cramer's rule in Sec. 7.7), and we write the solutions in terms of partial fractions,

$$Y_1 = \frac{9s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} - \frac{62.5}{s + 0.12} - \frac{37.5}{s + 0.04}$$

$$Y_2 = \frac{150s^2 + 12s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} + \frac{125}{s + 0.12} - \frac{75}{s + 0.04}.$$

By taking the inverse transform we arrive at the solution

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$

Figure 142 shows the interesting plot of these functions. Can you give physical explanations for their main features? Why do they have the limit 100? Why is y_2 not monotone, whereas y_1 is? Why is y_1 from some time on suddenly larger than y_2? Etc. ■

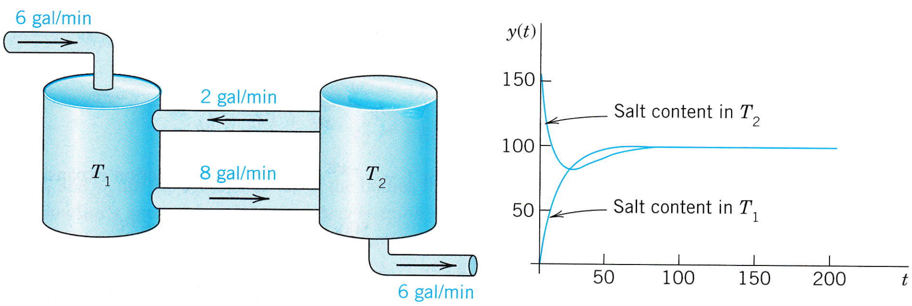

Fig. 142. Mixing problem in Example 1

Other systems of ODEs of practical importance can be solved by the Laplace transform method in a similar way, and eigenvalues and eigenvectors as we had to determine them in Chap. 4 will come out automatically, as we have seen in Example 1.

EXAMPLE 2 Electrical Network

Find the currents $i_1(t)$ and $i_2(t)$ in the network in Fig. 143 with L and R measured in terms of the usual units (see Sec. 2.9), $v(t) = 100$ volts if $0 \leqq t \leqq 0.5$ sec and 0 thereafter, and $i(0) = 0$, $i'(0) = 0$.

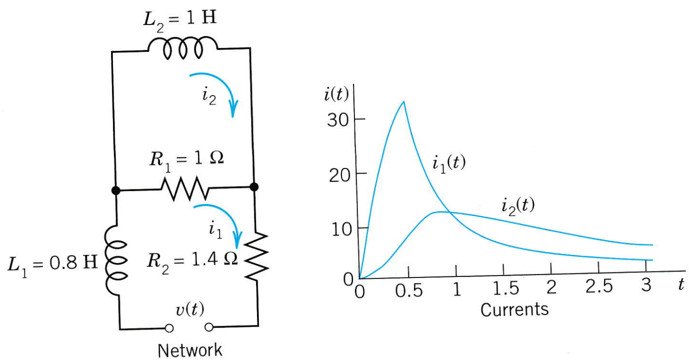

Fig. 143. Electrical network in Example 2

Solution. The model of the network is obtained from Kirchhoff's voltage law as in Sec. 2.9. For the lower circuit we obtain

$$0.8 i_1' + 1(i_1 - i_2) + 1.4 i_1 = 100[1 - u(t - \tfrac{1}{2})]$$

and for the upper

$$1 \cdot i_2' + 1(i_2 - i_1) = 0.$$

Division by 0.8 and ordering gives for the lower circuit

$$i_1' + 3 i_1 - 1.25 i_2 = 125[1 - u(t - \tfrac{1}{2})]$$

and for the upper

$$i_2' - i_1 + i_2 = 0.$$

With $i_1(0) = 0$, $i_2(0) = 0$ we obtain from (1) in Sec. 6.2 and the second shifting theorem the subsidiary system

$$(s + 3) I_1 - 1.25 I_2 = 125 \left(\frac{1}{s} - \frac{e^{-s/2}}{s}\right)$$

$$-I_1 + (s + 1) I_2 = 0.$$

Solving algebraically for I_1 and I_2 gives

$$I_1 = \frac{125(s + 1)}{s(s + \frac{1}{2})(s + \frac{7}{2})} (1 - e^{-s/2}),$$

$$I_2 = \frac{125}{s(s + \frac{1}{2})(s + \frac{7}{2})} (1 - e^{-s/2}).$$

The right sides without the factor $1 - e^{-s/2}$ have the partial fraction expansions

$$\frac{500}{7s} - \frac{125}{3(s + \frac{1}{2})} - \frac{625}{21(s + \frac{7}{2})}$$

and

$$\frac{500}{7s} - \frac{250}{3(s + \frac{1}{2})} + \frac{250}{21(s + \frac{7}{2})},$$

respectively. The inverse transform of this gives the solution for $0 \leqq t \leqq \frac{1}{2}$,

$$\begin{aligned} i_1(t) &= -\frac{125}{3}e^{-t/2} - \frac{625}{21}e^{-7t/2} + \frac{500}{7} \\ i_2(t) &= -\frac{250}{3}e^{-t/2} + \frac{250}{21}e^{-7t/2} + \frac{500}{7} \end{aligned} \qquad (0 \leqq t \leqq \tfrac{1}{2}).$$

According to the second shifting theorem the solution for $t > \frac{1}{2}$ is $i_1(t) - i_1(t - \frac{1}{2})$ and $i_2(t) - i_2(t - \frac{1}{2})$, that is,

$$\begin{aligned} i_1(t) &= -\frac{125}{3}(1 - e^{1/4})e^{-t/2} - \frac{625}{21}(1 - e^{7/4})e^{-7t/2} \\ i_2(t) &= -\frac{250}{3}(1 - e^{1/4})e^{-t/2} + \frac{250}{21}(1 - e^{7/4})e^{-7t/2} \end{aligned} \qquad (t > \tfrac{1}{2}).$$

Can you explain physically why both currents eventually go to zero, and why $i_1(t)$ has a sharp cusp whereas $i_2(t)$ has a continuous tangent direction at $t = \frac{1}{2}$? ■

Systems of ODEs of higher order can be solved by the Laplace transform method in a similar fashion. As an important application, typical of many similar mechanical systems, we consider coupled vibrating masses on springs.

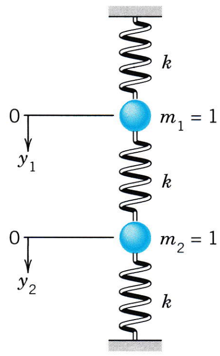

Fig. 144. Example 3

EXAMPLE 3 Model of Two Masses on Springs (Fig. 144)

The mechanical system in Fig. 144 consists of two bodies of mass 1 on three springs of the same spring constant k and of negligibly small masses of the springs. Also damping is assumed to be practically zero. Then the model of the physical system is the system of ODEs

$$\begin{aligned} y_1'' &= -ky_1 + k(y_2 - y_1) \\ y_2'' &= -k(y_2 - y_1) - ky_2. \end{aligned} \tag{3}$$

Here y_1 and y_2 are the displacements of the bodies from their positions of static equilibrium. These ODEs follow from **Newton's second law,** *Mass* × *Acceleration* = *Force,* as in Sec. 2.4 for a single body. We again regard downward forces as positive and upward as negative. On the upper body, $-ky_1$ is the force of the upper spring and $k(y_2 - y_1)$ that of the middle spring, $y_2 - y_1$ being the net change in spring length—think this over before going on. On the lower body, $-k(y_2 - y_1)$ is the force of the middle spring and $-ky_2$ that of the lower spring.

We shall determine the solution corresponding to the initial conditions $y_1(0) = 1$, $y_2(0) = 1$, $y_1'(0) = \sqrt{3k}$, $y_2'(0) = -\sqrt{3k}$. Let $Y_1 = \mathcal{L}(y_1)$ and $Y_2 = \mathcal{L}(y_2)$. Then from (2) in Sec. 6.2 and the initial conditions we obtain the subsidiary system

$$s^2Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2.$$

This system of linear algebraic equations in the unknowns Y_1 and Y_2 may be written

$$(s^2 + 2k)Y_1 - kY_2 = s + \sqrt{3k}$$
$$-kY_1 + (s^2 + 2k)Y_2 = s - \sqrt{3k}.$$

Elimination (or Cramer's rule in Sec. 7.7) yields the solution, which we can expand in terms of partial fractions,

$$Y_1 = \frac{(s + \sqrt{3k})(s^2 + 2k) + k(s - \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}$$

$$Y_2 = \frac{(s^2 + 2k)(s - \sqrt{3k}) + k(s + \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}.$$

Hence the solution of our initial value problem is (Fig. 145)

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$
$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos\sqrt{k}t - \sin\sqrt{3k}t.$$

We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" oscillation and a "rapid" oscillation. ■

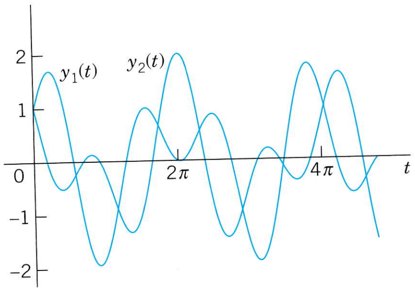

Fig. 145. Solutions in Example 3

PROBLEM SET 6.7

1–20 SYSTEMS OF ODES

Using the Laplace transform and showing the details of your work, solve the initial value problem:

1. $y_1' = -y_1 - y_2, \quad y_2' = y_1 - y_2,$
   $y_1(0) = 0, \quad y_2(0) = 1$
2. $y_1' = 5y_1 + y_2, \quad y_2' = y_1 + 5y_2,$
   $y_1(0) = 1, \quad y_2(0) = -3$
3. $y_1' = -6y_1 + 4y_2, \quad y_2' = -4y_1 + 4y_2,$
   $y_1(0) = -2, \quad y_2(0) = -7$
4. $y_1' + y_2 = 0, \quad y_1 + y_2' = 2\cos t,$
   $y_1(0) = 1, \quad y_2(0) = 0$
5. $y_1' = -4y_1 - 2y_2 + t, \quad y_2' = 3y_1 + y_2 - t,$
   $y_1(0) = 5.75, \quad y_2(0) = -6.75$
6. $y_1' = 4y_2 - 8\cos 4t, \quad y_2' = -3y_1 - 9\sin 4t,$
   $y_1(0) = 0, \quad y_2(0) = 3$

7. $y_1' = 5y_1 - 4y_2 - 9t^2 + 2t,$
$y_2' = 10y_1 - 7y_2 - 17t^2 - 2t,$
$y_1(0) = 2, \quad y_2(0) = 0$

8. $y_1' = 6y_1 + y_2, \quad y_2' = 9y_1 + 6y_2,$
$y_1(0) = -3, \quad y_2(0) = -3$

9. $y_1' = 5y_1 + 5y_2 - 15\cos t + 27\sin t,$
$y_2' = -10y_1 - 5y_2 - 150\sin t,$
$y_1(0) = 2, \quad y_2(0) = 2$

10. $y_1' = -2y_1 + 3y_2, \quad y_2' = 4y_1 - y_2,$
$y_1(0) = 4, \quad y_2(0) = 3$

11. $y_1' = y_2 + 1 - u(t-1),$
$y_2' = -y_1 + 1 - u(t-1), \quad y_1(0) = 0,$
$y_2(0) = 0$

12. $y_1' = 2y_1 + y_2, \quad y_2' = 4y_1 + 2y_2 + 64tu(t-1),$
$y_1(0) = 2, \quad y_2(0) = 0$

13. $y_1' = y_1 + 6u(t-2)e^{4t}, \quad y_2' = y_1 + 2y_2,$
$y_1(0) = 0, \quad y_2(0) = 1$

14. $y_1' = -y_2, \quad y_2' = -y_1 + 2[1 - u(t - 2\pi)]\cos t,$
$y_1(0) = 1, \quad y_2(0) = 0$

15. $y_1' = -3y_1 + y_2 + u(t-1)e^t,$
$y_2' = -4y_1 + 2y_2 + u(t-1)e^t,$
$y_1(0) = 0, \quad y_2(0) = 3$

16. $y_1'' = -2y_1 + 2y_2, \quad y_2'' = 2y_1 - 5y_2,$
$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 3, \quad y_2'(0) = 0$

17. $y_1'' = 4y_1 + 8y_2, \quad y_2'' = 5y_1 + y_2,$
$y_1(0) = 8, \quad y_1'(0) = -18, \quad y_2(0) = 5,$
$y_2'(0) = -21$

18. $y_1'' + y_2 = -101\sin 10t, \quad y_2'' + y_1 = 101\sin 10t,$
$y_1(0) = 0, \quad y_1'(0) = 6, \quad y_2(0) = 8,$
$y_2'(0) = -6$

19. $y_1' + y_2' = 2e^t + e^{-t}, \quad y_2' + y_3' = 2\sinh t,$
$y_3' + y_1' = e^t$
$y_1(0) = 0, \quad y_2(0) = 1, \quad y_3(0) = 1$

20. $4y_1' + y_2' - 2y_3' = 0, \quad -2y_1' + y_3' = 1,$
$2y_2' - 4y_3' = -16t$
$y_1(0) = 2, \quad y_2(0) = 0, \quad y_3(0) = 0$

21. TEAM PROJECT. Comparison of Methods for Linear Systems of ODEs.

(a) Models. Solve the models in Examples 1 and 2 of Sec. 4.1 by Laplace transforms and compare the amount of work with that in Sec. 4.1. (Show the details of your work.)

(b) Homogeneous Systems. Solve the systems (8), (11)–(13) in Sec. 4.3 by Laplace transforms. (Show the details.)

(c) Nonhomogeneous System. Solve the system (3) in Sec. 4.6 by Laplace transforms. (Show the details.)

FURTHER APPLICATIONS

22. (Forced vibrations of two masses) Solve the model in Example 3 with $k = 4$ and initial conditions $y_1(0) = 1$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = -1$ under the assumption that the force $11 \sin t$ is acting on the first body and the force $-11 \sin t$ on the second. Graph the two curves on common axes and explain the motion physically.

23. CAS Experiment. Effect of Initial Conditions. In Prob. 22, vary the initial conditions systematically, describe and explain the graphs physically. The great variety of curves will surprise you. Are they always periodic? Can you find empirical laws for the changes in terms of continuous changes of those conditions?

24. (Mixing problem) What will happen in Example 1 if you double all flows (in particular, an increase to 12 gal/min containing 12 lb of salt from the outside), leaving the size of the tanks and the initial conditions as before? First guess, then calculate. Can you relate the new solution to the old one?

25. (Electrical network) Using Laplace transforms, find the currents $i_1(t)$ and $i_2(t)$ in Fig. 146, where $v(t) = 390 \cos t$ and $i_1(0) = 0$, $i_2(0) = 0$. How soon will the currents practically reach their steady state?

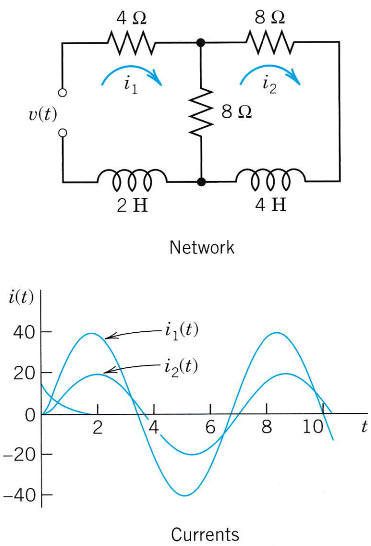

Fig. 146. Electrical network and currents in Problem 25

26. (Single cosine wave) Solve Prob. 25 when the EMF (electromotive force) is acting from 0 to 2π only. Can you do this just by looking at Prob. 25, practically without calculation?

6.8 Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$	s-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$ $= \int_0^t f(t-\tau) g(\tau)\, d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\, u(t-a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\, d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$	f Periodic with Period p	6.4 Project 16

6.9 Table of Laplace Transforms

For more extensive tables, see Ref.* [A9] *in Appendix 1.

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\dfrac{1}{s-a}$	e^{at}	6.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!} t^{n-1}e^{at}$	
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)} t^{k-1}e^{at}$	
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}(ae^{at} - be^{bt})$	
13	$\dfrac{1}{s^2 + \omega^2}$	$\dfrac{1}{\omega} \sin \omega t$	6.1
14	$\dfrac{s}{s^2 + \omega^2}$	$\cos \omega t$	
15	$\dfrac{1}{s^2 - a^2}$	$\dfrac{1}{a} \sinh at$	
16	$\dfrac{s}{s^2 - a^2}$	$\cosh at$	
17	$\dfrac{1}{(s-a)^2 + \omega^2}$	$\dfrac{1}{\omega} e^{at} \sin \omega t$	
18	$\dfrac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	
19	$\dfrac{1}{s(s^2 + \omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$	6.2
20	$\dfrac{1}{s^2(s^2 + \omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$	
21	$\dfrac{1}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	6.6

(continued)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
22	$\dfrac{s}{(s^2 + \omega^2)^2}$	$\dfrac{t}{2\omega} \sin \omega t$	6.6
23	$\dfrac{s^2}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	6.6
24	$\dfrac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2 - a^2} (\cos at - \cos bt)$	
25	$\dfrac{1}{s^4 + 4k^4}$	$\dfrac{1}{4k^3} (\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\dfrac{s}{s^4 + 4k^4}$	$\dfrac{1}{2k^2} \sin kt \sinh kt$	
27	$\dfrac{1}{s^4 - k^4}$	$\dfrac{1}{2k^3} (\sinh kt - \sin kt)$	
28	$\dfrac{s}{s^4 - k^4}$	$\dfrac{1}{2k^2} (\cosh kt - \cos kt)$	
29	$\sqrt{s - a} - \sqrt{s - b}$	$\dfrac{1}{2\sqrt{\pi t^3}} (e^{bt} - e^{at})$	
30	$\dfrac{1}{\sqrt{s + a}\sqrt{s + b}}$	$e^{-(a+b)t/2} I_0 \left(\dfrac{a - b}{2} t\right)$	5.6
31	$\dfrac{1}{\sqrt{s^2 + a^2}}$	$J_0(at)$	5.5
32	$\dfrac{s}{(s - a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}} e^{at}(1 + 2at)$	
33	$\dfrac{1}{(s^2 - a^2)^k} \quad (k > 0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)} \left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	5.6
34	e^{-as}/s	$u(t - a)$	6.3
35	e^{-as}	$\delta(t - a)$	6.4
36	$\dfrac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$	5.5
37	$\dfrac{1}{\sqrt{s}} e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}} e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\dfrac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$	
40	$\dfrac{1}{s} \ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	5.6

(*continued*)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
41	$\ln \dfrac{s-a}{s-b}$	$\dfrac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \dfrac{s^2+\omega^2}{s^2}$	$\dfrac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \dfrac{s^2-a^2}{s^2}$	$\dfrac{2}{t}(1 - \cosh at)$	
44	$\arctan \dfrac{\omega}{s}$	$\dfrac{1}{t} \sin \omega t$	
45	$\dfrac{1}{s} \operatorname{arccot} s$	$\operatorname{Si}(t)$	App. A3.1

CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

1. What do we mean by operational calculus?
2. What are the steps needed in solving an ODE by Laplace transform? What is the subsidiary equation?
3. The Laplace transform is a linear operation. What does this mean? Why is it important?
4. For what problems is the Laplace transform preferable over the usual method? Explain.
5. What are the unit step and Dirac's delta functions? Give examples.
6. What is the difference between the two shifting theorems? When do they apply?
7. Is $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
8. Can a discontinuous function have a Laplace transform? Does every continuous function have a Laplace transform? Give reasons.
9. State the transforms of a few simple functions from memory.
10. If two different continuous functions have transforms, the latter are different. Why is this practically important?

11–22 LAPLACE TRANSFORMS

Find the transform (showing the details of your work and indicating the method or formula you are using):

11. te^{3t} **12.** $e^{-t} \sin 2t$

13. $\sin^2 t$ **14.** $\cos^2 4t$

15. $tu(t - \pi)$ **16.** $u(t - 2\pi) \sin t$

17. $e^t * \cos 2t$ **18.** $(\sin \omega t) * (\cos \omega t)$

19. $\sin t + \sinh t$ **20.** $\cosh t - \cos t$

21. $e^{at} - e^{bt} \quad (a \neq b)$ **22.** $\cosh 2t - \cosh t$

23–34 INVERSE LAPLACE TRANSFORMS

Find the inverse transform (showing the details of your work and indicating the method or formula used):

23. $\dfrac{10s}{s^2 + 2}$ **24.** $\dfrac{15}{s^2 - 4}$

25. $\dfrac{12}{s^2 + 4s + 20}$ **26.** $\dfrac{3s}{s^2 - 2s + 2}$

27. $\dfrac{5s + 4}{s^2} e^{-2s}$ **28.** $\dfrac{2s - 10}{s^3} e^{-5s}$

29. $\dfrac{2s + 4}{(s^2 + 4s + 5)^2}$ **30.** $\dfrac{s^2 - 16}{(s^2 + 16)^2}$

31. $\left(\dfrac{2}{s^2} + \dfrac{2}{s^3}\right) e^{-s}$ **32.** $\dfrac{180 + 18s^2 + 3s^4}{s^7}$

33. $\dfrac{\pi}{s^2(s^2 + \omega^2)}$ **34.** $\dfrac{2}{2s^2 + 2s + 1}$

35–50 SINGLE ODEs AND SYSTEMS OF ODEs

Solve by Laplace transforms, showing the details and graphing the solution:

35. $y'' + y = u(t-1)$, $y(0) = 0$, $y'(0) = 20$

36. $y'' + 16y = 4\delta(t-\pi)$, $y(0) = -1$, $y'(0) = 0$

37. $y'' + 4y = 8\delta(t-5)$, $y(0) = 10$, $y'(0) = -1$

38. $y'' + y = u(t-2)$, $y(0) = 0$, $y'(0) = 0$

39. $y'' + 2y' + 10y = 0$, $y(0) = 7$, $y'(0) = -1$

40. $y'' + 4y' + 5y = 50t$, $y(0) = 5$, $y'(0) = -5$

41. $y'' - y' - 2y = 12u(t-\pi)\sin t$, $y(0) = 1$, $y'(0) = -1$

42. $y'' - 2y' + y = t\delta(t-1)$, $y(0) = 0$, $y'(0) = 0$

43. $y'' - 4y' + 4y = \delta(t-1) - \delta(t-2)$, $y(0) = 0$, $y'(0) = 0$

44. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$

45. $y_1' + y_2 = \sin t$, $y_2' + y_1 = -\sin t$, $y_1(0) = 1$, $y_2(0) = 0$

46. $y_1' = -3y_1 + y_2 - 12t$, $y_2' = -4y_1 + 2y_2 + 12t$, $y_1(0) = 0$, $y_2(0) = 0$

47. $y_1' = y_2$, $y_2' = -5y_1 - 2y_2$, $y_1(0) = 0$, $y_2(0) = 1$

48. $y_1' = y_2$, $y_2' = -4y_1 + \delta(t-\pi)$, $y_1(0) = 0$, $y_2(0) = 0$

49. $y_1'' = 4y_2 - 4e^t$, $y_2'' = 3y_1 + y_2$, $y_1(0) = 1$, $y_1'(0) = 2$, $y_2(0) = 2$, $y_2'(0) = 3$

50. $y_1'' = 16y_2$, $y_2'' = 16y_1$, $y_1(0) = 2$, $y_1'(0) = 12$, $y_2(0) = 6$, $y_2'(0) = 4$

MODELS OF CIRCUITS AND NETWORKS

51. **(RC-circuit)** Find and graph the current $i(t)$ in the RC-circuit in Fig. 147, where $R = 100\ \Omega$, $C = 10^{-3}$ F, $v(t) = 100t$ V if $0 < t < 2$, $v(t) = 200$ V if $t > 2$ and the initial charge on the capacitor is 0.

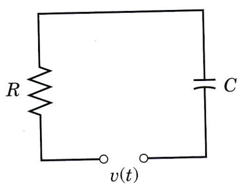

Fig. 147. RC-circuit

52. **(LC-circuit)** Find and graph the charge $q(t)$ and the current $i(t)$ in the LC-circuit in Fig. 148, where $L = 0.5$ H, $C = 0.02$ F, $v(t) = 1425 \sin 5t$ V if $0 < t < \pi$, $v(t) = 0$ if $t > \pi$, and current and charge at $t = 0$ are 0.

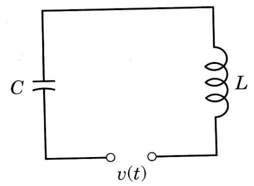

Fig. 148. LC-circuit

53. **(RLC-circuit)** Find and graph the current $i(t)$ in the RLC-circuit in Fig. 149, where $R = 1\ \Omega$, $L = 0.25$ H, $C = 0.2$ F, $v(t) = 377 \sin 20t$ V, and current and charge at $t = 0$ are 0.

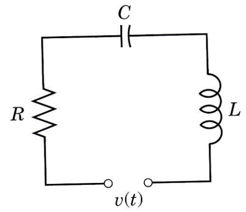

Fig. 149. RLC-circuit

54. **(Network)** Show that by Kirchhoff's voltage law (Sec. 2.9), the currents in the network in Fig. 150 are obtained from the system

$$Li_1' + R(i_1 - i_2) = v(t)$$

$$R(i_2' - i_1') + \frac{1}{C} i_2 = 0.$$

Solve this system, where $R = 1\ \Omega$, $L = 2$ H, $C = 0.5$ F, $v(t) = 90e^{-t/4}$ V, $i_1(0) = 0$, $i_2(0) = 2$ A.

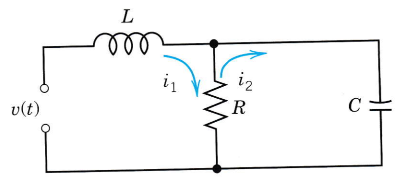

Fig. 150. Network in Problem 54

55. **(Network)** Set up the model of the network in Fig. 151 and find and graph the currents, assuming that the currents and the charge on the capacitor are 0 when the switch is closed at $t = 0$.

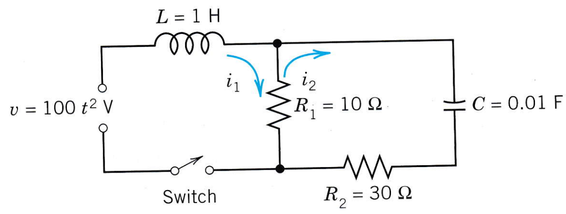

Fig. 151. Network in Problem 55

SUMMARY OF CHAPTER 6

Laplace Transforms

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = \mathcal{L}(f)$ of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{(Sec. 6.1).} \tag{1}$$

This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

$$\begin{aligned} \mathcal{L}(f') &= s\mathcal{L}(f) - f(0) \\ \mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0) \end{aligned} \qquad \text{(Sec. 6.2)} \tag{2}$$

etc. Hence by taking the transform of a given differential equation

$$y'' + ay' + by = r(t) \qquad (a,\ b \text{ constant}) \tag{3}$$

and writing $\mathcal{L}(y) = Y(s)$, we obtain the **subsidiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0). \tag{4}$$

Here, in obtaining the transform $\mathcal{L}(r)$ we can get help from the small table in Sec. 6.1 or the larger table in Sec. 6.9. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$. In the third step we determine the **inverse transform** $y(t) = \mathcal{L}^{-1}(Y)$, that is, the solution of the problem. This is generally the hardest step, and in it we may again use one of those two tables. $Y(s)$ will often be a rational function, so that we can obtain the inverse $\mathcal{L}^{-1}(Y)$ by partial fraction reduction (Sec. 6.4) if we see no simpler way.

The Laplace method avoids the determination of a general solution of the homogeneous ODE, and we also need not determine values of arbitrary constants in a general solution from initial conditions; instead, we can insert the latter directly into (4). Two further facts account for the practical importance of the Laplace transform. First, it has some basic properties and resulting techniques that simplify the determination of transforms and inverses. The most important of these properties are listed in Sec. 6.8, together with references to the corresponding sections. More on the use of unit step functions and Dirac's delta can be found in Secs. 6.3 and 6.4, and more on convolution in Sec. 6.5. Second, due to these properties, the present method is particularly suitable for handling right sides $r(t)$ given by different expressions over different intervals of time, for instance, when $r(t)$ is a square wave or an impulse or of a form such as $r(t) = \cos t$ if $0 \leqq t \leqq 4\pi$ and 0 elsewhere.

The application of the Laplace transform to systems of ODEs is shown in Sec. 6.7. (The application to PDEs follows in Sec. 12.11.)