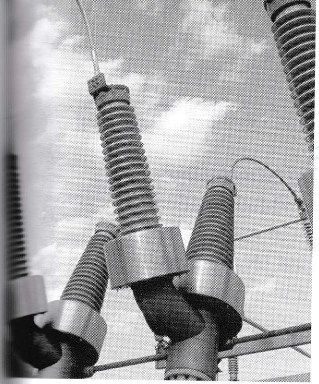

APPENDIX 1

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Software *see* at the beginning of Chaps. 19 and 24.

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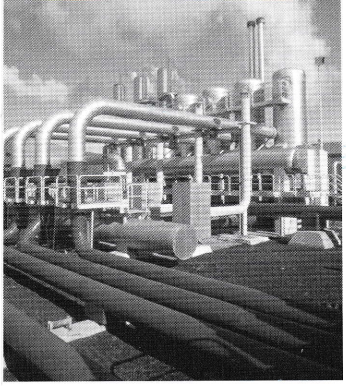

APPENDIX 2

Answers to Odd-Numbered Problems

Problem Set 1.1, page 8

1. $(\cos \pi x)/\pi + c$ **3.** $e^{x^2/2} + c$ **5.** First order

7. Second order **9.** Third order

11. $y = \frac{1}{2} \tan (2x + n\pi),\ n = 0, \pm 1, \pm 2, \cdots$

13. $y = e^{-x^2}$ **15.** (A) No. (B) No. Only $y = 0$.

17. $y'' = g,\ y' = gt,\ y = gt^2/2$

19. $y'' = k,\ y' = kt + 6,\ y = \frac{1}{2}kt^2 + 6t,\ y(60) = 1800k + 360 = 3000,\ k = 1.47,$
$y'(60) = 1.47 \cdot 60 + 6 = 94$ [m/sec] $= 210$ [mph]

21. $e^{kH} = \frac{1}{2},\ H = (\ln \frac{1}{2})/k = (10^{11} \ln 2)/1.4 = 1570$ [years]

Problem Set 1.2, page 11

11. $y = -(2/\pi) \cos \frac{1}{2}\pi x + c$ **15.** $y = x(1 - \ln x) + c$

17. Verify the general solution $y^2 + t^2 = c$. Circle of radius $3\sqrt{2}$

19. $mv' = mg - bv^2,\ v' = 9.8 - v^2,\ v(0) = 10.\ v' = 0$ gives the limit $\sqrt{9.8} = 3.1$ [meter/sec].

Problem Set 1.3, page 18

3. $\cos 2y\, dy = 2\, dx,\ y = \frac{1}{2} \arcsin (4x + c)$ **5.** $y^2 + 36x^2 = c$, ellipses

7. $dy/y = \cot \pi x\, dx,\ y = c(\sin \pi x)^{1/\pi}$ **9.** $y = \tan (c - e^{-\pi x}/\pi)$

11. $r = r_0 e^{-t^2}$ **13.** $I = I_0 e^{-Rt/L}$

15. $y = e^x/\sqrt{2x + 5}$ **17.** $y = 4 \ln x$

19. $y = \sqrt{\ln (x^2 - 2x + e)}$

21. $y' = (y - b)/(x - a),\ y - b = c(x - a)$

23. $y_0 e^k = 2y_0,\ e^k = 2$ (1 week), $e^{2k} = 2^2$ (2 weeks), $e^{4k} = 2^4$

25. $y = y_0 e^{kt} = y_0 e^{-0.0001213t} = y_0 e^{-0.0001213 \cdot 4000} = 0.62y_0$; 62%; cf. Example 2.

27. $y' = -ky,\ y = y_0 e^{-kt},\ e^{-5k} = 0.5,\ k = -(\ln 0.5)/5 = 0.139,$
$t = -(\ln 0.05)/0.139 = 22$ [min]

29. $T(0) = 10,\ T = 23 - 13e^{kt},\ T(2) = 23 - 13e^{2k} = 18,\ k = -0.478,\ T = 22.8$ gives $t = [\ln (-0.2/-13)]/(-0.478) = 8.73$ [min].

31. $h = gt^2/2,\ t = \sqrt{2h/g},\ v = gt = g\sqrt{2h/g} = \sqrt{2gh}$

33. $y' = 0 - (2/800)y,\ y = 200e^{-0.0025t},\ t = 300$ [min], $y(300) = 94.5$ [lb]

35. (A) is related to the error function and (C) concerns the Fresnel integral $C(x)$; see App. 3.1. (D) $y' = 2xy + 1,\ y(0) = 0$

Problem Set 1.4, page 25

1. Exact. $x^4 + y^4 = c$

3. Exact. $u = \cos \pi x \sinh y + k(y)$, $u_y = \cos \pi x \cosh y + k'$, $k' = 0$.
Ans. $\cos \pi x \sinh y = c$

5. Exact. $9x^2 + 4y^2 = c$

7. Exact, $M_\theta = N_r = -2e^{-2\theta}$, $u = re^{-2\theta} + k(\theta)$, $u_\theta = -2re^{-2\theta} + k'$, $k' = 0$.
Ans. $re^{-2\theta} = c$, $r = ce^{2\theta}$

9. Exact. $u = y/x + \sin 2x + k(y)$, $u_y = 1/x + k' = 1/x - 2 \sin 2y$.
Ans. $y/x + \sin 2x + \cos 2y = c$

11. Not exact. $F = 1/x^2$ by Theorem 1. $-y/x^2\, dx + 1/x\, dy = d(y/x) = 0$. *Ans.* $y = cx$

13. $-3y^2/x^4\, dx + 2y/x^3\, dy = d(y^2/x^3) = 0$. $y = cx^{3/2}$ (semicubical parabolas)

15. Exact, $u = e^{2x} \cos y + k(y)$, $u_y = -e^{2x} \sin y + k'$, $k' = 0$. *Ans.* $e^{2x} \cos y = c$, $c = 1$

17. Not exact. Try *R*. $F = e^{-x}$, $e^{-x}(\cos \omega x + \omega \sin \omega x)\, dx + dy = 0$, $u = y + l(x)$, $u_x = l' = e^{-x}(\cos \omega x + \omega \sin \omega x)$, $u = y + l = y - e^{-x} \cos \omega x = c$, $c = 0$

19. $u = e^x + k(y)$, $u_y = k' = -1 + e^y$, $k = -y + e^y$. *Ans.* $e^x - y + e^y = c$

21. $B = C$, $\frac{1}{2}Ax^2 + Cxy + \frac{1}{2}Dy^2 = c$

Problem Set 1.5, page 32

3. $y = ce^{-3.5x} + 0.8$

5. $y = 2.6e^{-1.25x} + 4$

7. $y = x + c$ (if $k = 0$). $y = ce^{-kx} + e^{2kx}/3k$ if $k \neq 0$

9. Separate. $y - 2.5 = c \cosh^4 1.5x$

11. $y = 2xe^{\cos 2x}$

13. $y = \sin 2x + c/\sin^2 2x$, $c = 1$

15. $y = e^{1/x}(x^2 + c)$, $c = 4.1$

17. $y = (c + \frac{1}{2} \cosh 10x)/x^3$. Note $(x^3y)' = 5 \sinh 10x$.

19. $y = 1/u$, $u = ce^{-5.7x} - \dfrac{6.5}{5.7}$

21. $u = y^{-2} = e^{x^2}(1 + ce^{2x})$, $c = 3$, $u(0) = 4$

23. Separate variables. $y^2 = 1 - ce^{\cos x}$, $c = -1/e$

25. $y' = Ry + k$, $y = ce^{Rt} - k/R$, $c = y_0 + k/R$. $y_0 = 1000$, $R = 0.06$, $t = 65 - 25 = 40$, $k = 1000$, $y = \$178\,076.12$. Start at 45 gives $y_0[(1 + 1/0.06)e^{0.06 \cdot 20} - 1/0.06] = 41.988732y_0 = 178\,076.12$, $y_0 = k = \$4241.05$.

27. $y' = 175(0.0001 - y/450)$, $y(0) = 450 \cdot 0.0004 = 0.18$,
$y = 0.135e^{-0.3889t} + 0.045 = 0.18/2$,
$e^{-0.3889t} = (0.09 - 0.045)/0.135 = 1/3$,
$t = (\ln 3)/0.3889 = 2.82$. *Ans.* About 3 years

29. $y' = A - ky$, $y(0) = 0$, $y = A(1 - e^{-kt})/k$

31. $y' = By^2 - Ay = By(y - A/B)$, $A > 0$, $B > 0$. Constant solutions $y = 0$, $y = A/B$.
$y' > 0$ if $y > A/B$ (unlimited growth), $y' < 0$ if $0 < y < A/B$ (extinction).
$y = A/(ce^{At} + B)$, $y(0) > A/B$ if $c < 0$, $y(0) < A/B$ if $c > 0$.

33. $y' = y - y^2 - 0.2y$, $y = 1/(1.25 - 0.75e^{-0.8t})$, limit 0.8, limit 1

35. $y' = y - 0.25y^2 - 0.1y = 0.25y(3.6 - y)$. Equilibrium harvest 3.6,
$y = 18/(5 + ce^{-0.9t})$

37. $(y_1 + y_2)' + p(y_1 + y_2) = (y_1' + py_1) + (y_2' + py_2) = 0 + 0 = 0$

39. $(y_1 + y_2)' + p(y_1 + y_2) = (y_1' + py_1) + (y_2' + py_2) = r + 0 = r$

41. Solution of $cy_1' + pcy_1 = c(y_1' + py_1) = cr$

43. CAS Experiment (a) $y = x \sin(1/x) + cx$. $c = 0$ if $y(2/\pi) = 2/\pi$. y is undefined at $x = 0$, the point at which the "waves" of $\sin(1/x)$ accumulate; the factor x makes them smaller and smaller. Experiment with various x-intervals.
(b) $y = x^n[\sin(1/x) + c]$. $y(2/\pi) = (2/\pi)^n$. n need not be an integer. Try $n = \frac{1}{2}$. Try $n = -1$ and see how the "waves" near 0 become larger and larger.

45. $y = uy^*$, $y' + py = u'y^* + uy^{*\prime} + puy^* = u'y^* + u(y^{*\prime} + py^*) = u'y^* + u \cdot 0 = r$, $u' = r/y^* = re^{\int p\,dx}$, $u = \int e^{\int p\,dx} r\,dx + c$. Thus, $y = uy_h$ gives (4). We shall see that this method extends to higher-order ODEs (Secs. 2.10 and 3.3).

Problem Set 1.6, page 36

1. $y' = 4$, $\tilde{y}' = -1/4$, $\tilde{y} = -x/4 + c^*$
3. $y/x = c$, $y'/x = y/x^2$, $y' = y/x$, $\tilde{y}' = -x/\tilde{y}$, $\tilde{y}^2 + x^2 = c^*$, circles
5. $2xy + x^2y' = 0$, $y' = -2y/x$, $\tilde{y}' = x/(2\tilde{y})$, $\tilde{y}^2 - x^2/2 = c^*$, hyperbolas
7. $ye^{-x^2/2} = c$, $y' = xy$, $\tilde{y}' = -1/(x\tilde{y})$, $\tilde{y}\tilde{y}' = -1/x$, $\tilde{y}^2/2 = -\ln|x| + c^{**}$, $x = c^*e^{-\tilde{y}^2/2}$, bell-shaped curves (with x and $\tilde{y}$ interchanged)
9. $y' = -4x/y$, $\tilde{y}' = \tilde{y}/4x$, $4 \ln|\tilde{y}| = \ln|x| + c^{**}$, $x = c^*\tilde{y}^4$, parabolas
11. $xe^{-y/4} = c$, $y' = 4/x$, $\tilde{y}' = -x/4$, $\tilde{y} = -x^2/8 + c^*$
13. Use $dy/dx = 1/(dx/dy)$. $(y - 2x)e^x = c$, $(y' - 2 + y - 2x)e^x = 0$, $y' = 2 - y + 2x$, $dx/d\tilde{y} = -2 + \tilde{y} - 2x$ is linear, $dx/d\tilde{y} + 2x = \tilde{y} - 2$, $x = c^*e^{-2y} + y/2 - 5/4$
15. $u = c$, $u_x dx + u_y dy = 0$, $y' = -u_x/u_y$. Trajectories $\tilde{y}' = u_{\tilde{y}}/u_x$. Now $v = c^*$, $v_x dx + v_y dy = 0$, $y' = -v_x/v_y$. This agrees with the trajectory ODE in u if $u_x = v_y$ (equal denominators) and $u_y = -v_x$ (equal numerators). But these are just the Cauchy-Riemann equations.
17. $2x + 2yy' = 0$, $y' = -x/y$. Trajectories $\tilde{y}' = \tilde{y}/x$, $\ln|\tilde{y}| = \ln|x| + c^{**}$, $\tilde{y} = c^*x$.
19. $y' = -4x/9y$. Trajectories $\tilde{y}' = 9\tilde{y}/4x$. $\tilde{y} = c^*x^{9/4}$ $(c^* > 0)$. Sketch or graph these curves.

Problem Set 1.7, page 41

1. In $|x - x_0| < a$; just take b in $\alpha = b/K$ large, namely, $b = aK$.
3. No. At a common point (x_1, y_1) they would both satisfy the "initial condition" $y(x_1) = y_1$, violating uniqueness.
5. $y' = f(x, y) = r(x) - p(x)y$; hence $\partial f/\partial y = -p(x)$ is continuous and is thus bounded in the closed interval $|x - x_0| \leqq a$.
7. R has sides $2a$ and $2b$ and center $(1, 1)$ since $y(1) = 1$. In R, $f = 2y^2 \leqq 2(b + 1)^2 = K$, $\alpha = b/K = b/(2(b + 1)^2)$. $d\alpha/db = 0$ gives $b = 1$, and $\alpha_{opt} = b/K = 1/8$. Solution by $dy/y^2 = 2\,dx$, etc., $y = 1/(3 - 2x)$.
9. $|1 + y^2| \leqq K = 1 + b^2$, $\alpha = b/K$, $d\alpha/db = 0$, $b = 1$, $\alpha = 1/2$.

Chapter 1 Review Questions and Problems, page 42

11. $dy/(y^2 + \frac{1}{4}) = 4\,dx$, $2 \arctan 2y = 4x + c^*$, $y = \frac{1}{2}\tan(2x + c)$
13. Logistic ODE. $y = 1/u$, $y' = -u'/u^2 = 4/u - 1/u^2$, $u = c^*e^{-4x} + \frac{1}{4}$
15. $dy/(y^2 + 1) = x^2\,dx$, $\arctan y = x^3/3 + c$, $y = \tan(x^3/3 + c)$
17. Bernoulli. $y' + xy = x/y$, $u = y^2$, $u' = 2yy' = 2x - 2xu$ linear, $u = e^{-x^2}(\int e^{x^2} 2x\,dx + c) = 1 + ce^{-x^2}$, $y = \sqrt{u}$. Or write $yy' = -x(y^2 - 1)$ and separate.

19. Linear, $y = e^{\cos x}(\int e^{-\cos x} \sin x \, dx + c) = ce^{\cos x} + 1$. Or by separation.

21. Not exact. Use Theorem 1, Sec. 1.4; $R = 2/x$, $F = x^2$; the resulting exact ODE is $3x^2 \sin 2y \, dx + 2x^3 \cos 2y \, dy = d(x^3 \sin 2y)$, $x^3 \sin 2y = c$. Or by separation, $\cot 2y \, dy = -3/(2x) \, dx$, etc., $\sin 2y = cx^{-3}$.

23. Exact. $u = \int M \, dx = \sin xy - x^2 + k$, $u_y = x \cos xy + k' = N$, $k = y^2$, $\sin xy - x^2 + y^2 = c$.

25. Not exact. $R^* = 1$ in Theorem 2, Sec. 1.4, $F^* = e^y$. Exact is
$e^y \sin(y - x) \, dx + e^y[\cos(y - x) - \sin(y - x)] \, dy = 0$.
$u = \int M \, dx = e^y \cos(y - x) + k$, $u_y = e^y(\cos(y - x) - \sin(y - x)) + k' = N$,
$e^y \cos(y - x) = c$.

27. Separation. $y^2 + x^2 = 25$

29. Separation. $y = \tan(x + c)$, $c = -\frac{1}{4}\pi$

31. Exact. $u = x^2y^2 + \cos x + 2y = c$, $c = u(0, 1) = 3$

33. $y' = x/y$. Trajectories $\tilde{y}' = -\tilde{y}/x$, $\tilde{y} = c^*/x$ by separation. Hyperbolas.

35. $y = y_0 e^{kt}$, $e^{4k} = 0.9$, $k = \frac{1}{4} \ln 0.9$, $e^{kt} = 0.5$,
$t = (\ln 0.5)/k = (\ln 0.5)/[(\ln 0.9)/4] = 26.3$ [days]

37. $e^{kt} = 0.01$, $t = (\ln 0.01)/k = 175$ [days]

39. $y' = -4x/y$. Trajectories $\tilde{y} = c_1 x^{1/4}$ or $x = c_2 \tilde{y}^4$

41. Logistic ODE $y' = Ay - By^2$, $y = 1/u$, $u' + Au = +B$, $u = ce^{-At} + B/A$

43. A = amount of incident light. A thin layer of thickness Δx absorbs $\Delta A = -kA\Delta x$ ($-k$ = constant of proportionality). Thus $\Delta A/\Delta x = -kA$. Let $\Delta x \to 0$. Then $A' = -kA$, $A = A_0 e^{-kx}$ = amount of light in a thick layer at depth x from the surface of incidence.

Problem Set 2.1, page 52

1. $y = 2.5e^{4x} + 0.5e^{-4x}$
3. $y = e^{-x} \cos x$
5. $y = 4x^2 + 7/x^2$
7. Yes
9. Yes if $a \neq 0$
11. No
13. No
15. $F(x, z, z') = 0$
17. $y = c_1 e^{kx} + c_2$
19. $y \, dz/dy = 4z$, $y = (c_1 x + c_2)^{-1/3}$
21. $(dz/dy)z = -z^3 \sin y$, $-1/z = -dx/dy = \cos y + \tilde{c}_1$, $x = -\sin y + c_1 y + c_2$
23. $y''y' = 2$, $y = \frac{4}{3}(t + 1)^{3/2} - \frac{1}{3}$, $y(3) = \frac{31}{3}$, $y'(3) = 4$
25. $y'' = ky'$, $z' = kz$, $z = c_1 e^{kx} = y'$, $c_1 = 1$, $y = (e^{kx} - 1)/k$

Problem Set 2.2, page 59

1. $y = c_1 e^{7x} + c_2 e^{-x}$
3. $y = (c_1 + c_2 x)e^{2.5x}$
5. $y = c_1 e^{0.9x} + c_2 e^{-1.1x}$
7. $y = e^{0.5x}(A \cos 1.5x + B \sin 1.5x)$
9. $y = c_1 e^{3.5x} + c_2 e^{-1.5x}$
11. $y = A \cos 3\pi x + B \sin 3\pi x$
13. $y = c_1 e^{12x} + c_2 e^{-12x}$
15. $y'' - 3y' + 2y = 0$
17. $y'' - 2\sqrt{3}\, y' + 3y = 0$
19. $y'' - 16y = 0$
21. $y = 4e^{3x} - 2e^{-x}$
23. $y = e^{-2x}(2 \cos x - \sin x)$
25. $y = 2 + e^{-\pi x}$
27. $y = (2 - 4x)e^{-0.25x}$
29. $y = e^{-0.1x}(3.2 \cos 0.2x + 1.6 \sin 0.2x)$
31. $y = 4e^{5x} - 4e^{-5x}$
33. $y_1 = e^{-x}$, $y_2 = 0.001e^{x} + e^{-x}$
35. Write $E = e^{-ax/2}$, $c = \cos \omega x$, $s = \sin \omega x$. Note that $E' = -\frac{1}{2}aE$, $c' = -\omega s$, $s' = \omega c$. Substitute, drop E, collect c-terms, then s-terms, and use $\omega^2 = b - \frac{1}{4}a^2$, to get $c(b - \frac{1}{2}a^2 + \frac{1}{4}a^2 - \omega^2) + s(-a\omega + \frac{1}{2}a\omega + \frac{1}{2}a\omega) = 0 + 0 = 0$.

Problem Set 2.3, page 61

1. $0, 0, -2\cos x$

3. $-0.8x^3 + 6x^2 + 0.4, 0, e^{0.4x}$

5. $-12x^3 + 9x^2 + 8x - 2, -28\sin 4x - 4\cos 4x, 0$

7. $y = (c_1 + c_2x)e^{-2x}$

9. $y = e^{-3x}(A\cos 2x + B\sin 2x)$

11. $y = c_1e^{-3.1x} + c_2e^{-x}$

13. $y = A\cos 4.2\omega x + B\sin 4.2\omega x$

Problem Set 2.4, page 68

1. $y = y_0\cos\omega_0 t + (v_0/\omega_0)\sin\omega_0 t$. At integer t (if $\omega_0 = \pi$), because of periodicity.

3. $mL\theta'' = -mg\sin\theta \approx -mg\theta$ (tangential component of $W = mg$), $\theta'' + \omega_0{}^2\theta = 0$, $\omega_0/(2\pi) = \sqrt{g/L}/(2\pi)$.

5. No, because the frequency depends only on k/m.

7. (i) Greater by a factor $\sqrt{3}$. (ii) Lower

9. $\omega^* = [\omega_0{}^2 - c^2/(4m^2)]^{1/2} = \omega_0[1 - c^2/(4mk)]^{1/2} \approx \omega_0(1 - c^2/8mk) = 2.9583$

11. $2\pi/\omega^*$ since Eq. (10) and $y' = 0$ give $\tan(\omega^* t - \delta) = -\alpha/\omega^*$; tan is periodic with period π/ω^*.

13. Case (II) of (5) with $c = \sqrt{4mk} = \sqrt{4 \cdot 500 \cdot 4500} = 3000$ [kg/sec], where 500 kg is the mass per wheel.

15. $y = [y_0 + (v_0 + \alpha y_0)t]e^{-\alpha t}$, $y = [1 + (v_0 + 1)t]e^{-t}$; (ii) $v_0 = -2, -3/2, -4/3, -5/4, -6/5$

17. $y = 0$ gives $c_1 = -c_2e^{-2\beta t}$, which has one or no positive zero, depending on the initial conditions.

Problem Set 2.5, page 72

1. $c_1x^3 + c_2x^{-2}$

3. $(c_1 + c_2\ln|x|)x^4$

5. $x[A\cos(\ln|x|) + B\sin(\ln|x|)]$

7. $c_1x^{1.4} + c_2x^{1.6}$

9. $c_1x^{0.1} + c_2x^{0.9}$

11. $3x^2 - 2x^3$

13. $x^{-0.5}[2\cos(10\ln|x|) - \sin(10\ln|x|)]$

15. $2x^{-3} + 10$

Problem Set 2.6, page 77

1. $y'' - 0.25y = 0,\ W = -1$

3. $y'' - 2ky' + k^2y = 0,\ W = e^{2kx}$

5. $x^2y'' + 0.5xy' + 0.0625y = 0,\ W = x^{-0.5}$

7. $x^2y'' + xy' + 4y = 0,\ W = 2/x$

9. $x^2y'' - 0.75y = 0,\ W = -2$

11. $y'' - 6.25y = 0,\ W = 2.5$

13. $y'' + 2y' + 1.64y = 0,\ W = 0.8e^{-2x}$

15. $y'' + 5y' + 6.34y = 0,\ W = 0.3e^{-5x}$

17. $y'' + 7.6\pi y' + 14.44\pi^2y = 0,\ W = e^{-7.6\pi x}$

Problem Set 2.7, page 83

1. $c_1e^{-x} + c_2e^{-2x} + 2.5e^{2x}$

3. $c_1e^{4x} + c_2e^{-4x} + 2.4xe^{4x} - 4e^x$

5. $c_1e^{2x} + c_2e^{-3x} - x^3 - 3x - 0.5$

7. $e^{-3x}(A\cos 8x + B\sin 8x) + e^x(\cos 4x + \frac{1}{2}\sin 4x)$

9. $c_1e^{-0.4x} + c_2e^{0.4x} + 20xe^{0.4x} - 20xe^{-0.4x}$

11. $c_1\cos 1.2x + c_2\sin 1.2x + 10x\sin 1.2x$

13. $e^{-2x}(A\cos x + B\sin x) + 5x^2 - 8x + 4.4 - 1.6\cos 2x + 0.2\sin 2x$

15. $4x\sin 2x$

17. $e^{-0.1x}(1.5 \cos 0.5x - \sin 0.5x) + 2e^{0.5x}$

19. $2e^{-3x} + 3e^{4x} - 12x^3 + 3x^2 - 6.5x$

Problem Set 2.8, page 90

1. $-0.4 \cos 3t + 7.2 \sin 3t$

3. $-12.8 \cos 4.5t + 3.6 \sin 4.5t$

5. $0.16 \cos 2t + 0.12 \sin 2t$

7. $\frac{7}{45} \cos 3t - \frac{1}{45} \sin 3t$

9. $c_1 e^{-t/2} + c_2 e^{-3t/2} - \frac{32}{5} \cos t - \frac{4}{5} \sin t$

11. $(c_1 + c_2 t)e^{-3t/2} - \frac{4}{3} \cos 3t - \sin 3t$

13. $e^{-1.5t}(A \cos t + B \sin t) + 4 + 0.8 \cos 2t - 6.4 \sin 2t$

15. $0.32e^{-t} \cos 5t + 0.68 \cos 3t + 0.24 \sin 3t$

17. $5e^{-4t} - 4e^{-2t} - 0.3 \cos 2t + 0.1 \sin 2t$

19. $e^{-1.5t}(0.2 \cos t - 1.1 \sin t) + 0.8 \cos t + 0.4 \sin t$

Problem Set 2.9, page 97

1. $LI' + RI = E$, $I = (E/R) + ce^{-Rt/L} = 2.4 + ce^{-50t}$

3. $RI' + I/C = 0$, $I = ce^{-t/(RC)}$

5. $I = 5(\cos t - \cos 10t)/99$

7. I_0 is maximum when $S = 0$; thus $C = 1/(\omega^2 L)$.

9. $R > R_{\text{crit}} = 2\sqrt{L/C}$ is Case I, etc.

11. 0

13. $c_1 e^{-20t} + c_2 e^{-10t} + 16.5 \sin 10t + 5.5 \cos 10t$

15. $E' = -e^{-4t}(7.605 \cos \frac{1}{2}t + 1.95 \sin \frac{1}{2}t)$, $I = e^{-0.1t}(A \cos \frac{1}{2}t + B \sin \frac{1}{2}t) - e^{-4t} \cos \frac{1}{2}t$

17. $E(0) = 600$, $I'(0) = 600$, $I = e^{-3t}(-100 \cos 4t + 75 \sin 4t) + 100 \cos t$

19. (*b*) $R = 2\ \Omega$, $L = 1$ H, $C = 1/12$ F, $E = 4.4 \sin 10t$ V

Problem Set 2.10, page 101

1. $A \cos x + B \sin x - x \cos x + (\sin x) \ln |\sin x|$

3. $c_1 x + c_2 x^2 - x \cos x$

5. $(\cos x)(c_1 + \sin x - \ln |\sec x + \tan x|) + (\sin x)(c_2 - \cos x) = (c_1 - \ln |\sec x + \tan x|) \cos x + c_2 \sin x$

7. $(c_1 + \frac{3}{2}x) \sin x + (c_2 + \ln |\cos x|) \cos x$

9. $(c_1 + c_2 x)e^x + x^2 + 4x + 6 - e^x(\ln |x| + 1)$

11. $c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}x \cosh 2x$

13. $c_1 x + c_2 x^2 - x \sin x$

15. $A \cos x + B \sin x + y_{p1} + y_{p2}$, y_{p1} as y_p in Example 1, $y_{p2} = \frac{10}{24} \sin 5x$

17. $u'' + u = 0$ by substitution of $y = ux^{-1/2}$. $y_1 = x^{-1/2} \cos x$, $y_2 = x^{-1/2} \sin x$, $y_p = -\frac{1}{2}x^{1/2} \cos x + \frac{1}{2}x^{-1/2} \sin x$ from (2) with the ODE in standard form.

Chapter 2 Review Questions and Problems, page 102

9. $c_1 e^{4x} + c_2 e^{-2x} - 1.1 \cos 6x - 0.3 \sin 6x$

11. $e^{-4x}(A \cos 3x + B \sin 3x) - \frac{3}{4} \cos 3x + \frac{1}{2} \sin 3x$

13. $y_1 = x^3,\ y_2 = x^{-4},\ r = x^{-5},\ W = -7x^{-2},\ y_p = -\frac{1}{42}x^{-3} - \frac{1}{7}x^{-3} = -\frac{1}{6}x^{-3}$
15. $y_1 = e^x,\ y_2 = xe^x,\ W = e^{2x},\ y_p = e^x/(2x)$
17. $y_1 = e^x \cos x,\ y_2 = e^x \sin x,\ W = e^{2x},\ y_p = -xe^x \cos x + e^x(\sin x) \ln |\sin x|$
19. $y = 4e^{2x} + 2e^{-7x}$
21. $y = 9x^{-4} + 6x^6$
23. $y = e^{-2x} - 2e^{-3x} + 18x^2 - 30x + 19$
25. $y = \frac{1}{5}x^3 + 4x^2 - 5x^{-2}$
27. $y = -16 \cos 2t + 12 \sin 2t + 16(\cos 0.5t - \sin 1.5t)$.
Resonance for $\omega/(2\pi) = 2/(2\pi) = 1/\pi$
29. $\omega = 3.1$ is close to $\omega_0 = \sqrt{k/m} = 3$, $y = 25(\cos 3t - \cos 3.1t)$.
31. $R = 9\ \Omega,\ L = 0.5\ \text{H},\ C = 0.025\ \text{F},\ E = 17 \sin 6t\ \text{V}$, hence $0.5I'' + 9I' + 40I$
$= 102 \cos 6t,\quad I = -8.16e^{-8t} + 7.5e^{-10t} + 0.66 \cos 6t + 1.62 \sin 6t$
33. $E' = 220 \cdot 314 \cos 314t,\ I = e^{-50t}(A \cos 150t + B \sin 150t) + 0.847001 \sin 314t$
$- 1.985219 \cos 314t$

Problem Set 3.1, page 111

7. Linearly independent
9. Linearly dependent
11. $x|x| = x^2$ if $x > 0$, linearly dependent
13. Linearly independent
15. Linearly independent
17. Linearly independent
19. Linearly dependent

Problem Set 3.2, page 115

1. $y''' - 6y'' + 11y' - 6y = 0$
3. $y^{iv} - y = 0$
5. $y^{iv} + 4y'' = 0$
7. $c_1 + c_2 \cos x + c_3 \sin x$
9. $c_1e^x + (c_2 + c_3x)e^{-x}$
11. $c_1e^x + c_2e^{(1+\sqrt{7})x} + c_3e^{(1-\sqrt{7})x}$
13. $e^{0.25x} + 4.3e^{-0.7x} + 12.1 \cos 0.1x - 0.6 \sin 0.1x$
15. $2.4 + e^{-1.6x}(\cos 1.5x - 2 \sin 1.5x)$
17. $y = \cosh 5x - \cos 4x$
19. $y = c_1x^{-2} + c_2x + c_3x^2$. $W = 12/x^2$

Problem Set 3.3, page 122

1. $(c_1 + c_2x)e^{2x} + c_3e^{-2x} - 0.04e^{-3x} + x^2 + x + 1$
3. $c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x + x(c_3 \cos \frac{1}{2}x + c_4 \sin \frac{1}{2}x) - \frac{1}{2}e^{-x} \sin \frac{1}{2}x$
5. $c_1x^{0.5} + c_2x + c_3x^{1.5} + 0.1x^{5.5}$
7. $c_1 \cos x + c_2 \sin x + c_3 \cos 3x + c_4 \sin 3x + 0.2 \cosh 2x$
9. $y = (4 - x^2)e^{3x} - 0.5 \cos 3x + 0.5 \sin 3x$
11. $x^{-2} - x^2 + 5x^4 + x(\ln x + 1)$
13. $3 + 9e^{-2x} \cos 9x - (1.6 - 1.5x)e^x$

Chapter 3 Review Questions and Problems, page 122

7. $c_1 + c_2x^{1/2} + c_3x^{-1/2}$
9. $c_1e^{-0.5x} + c_2e^{0.5x} + c_3e^{-1.5x}$
11. $c_1x^2(\frac{1}{2} \ln x - \frac{3}{4}) + c_2x^2 + c_3x + c_4 + \frac{1}{7}x^7$
13. $c_1e^{-x} + e^{x/2}(c_2 \cos (\frac{1}{2}\sqrt{3}x) + c_3 \sin (\frac{1}{2}\sqrt{3}x)) + 8e^{x/2}$
15. $(c_1 + c_2x)e^x + c_3e^{-x} + 0.25x^2e^x$
17. $-0.5x^{-1} + 1.5x^{-5}$
19. $\cos 7x + e^{3x} - 0.02 \cosh x$

Problem Set 4.1, page 135

1. Yes **5.** $y_1' = 0.02(-y_1 + y_2)$, $y_2' = 0.02(y_1 - 2y_2 + y_3)$, $y_3' = 0.02(y_2 - y_3)$
7. $c_1 = 1$, $c_2 = -5$
9. 3 and 0
11. $y_1' = y_2$, $y_2' = 4y_1$, $y_1 = c_1e^{-2t} + c_2e^{2t} = y$, $y_2 = y_1'$
13. $y_1' = y_2$, $y_2' = y_2$, eigenvalues 0, 1, $y_1 = c_1 + c_2e^t$, $y_2 = y_1' = y'$
15. $y_1' = y_2$, $y_2' = 0.109375y_1 + 0.75y_2$ (divide by 64), $y_1 = c_1e^{-0.125t} + c_2e^{0.875t}$

Problem Set 4.3, page 146

1. $y_1 = c_1e^{-6t} + c_2e^{6t}$, $y_2 = -2c_1e^{-6t} + 2c_2e^{6t}$
3. $y_1 = c_1e^{2t} + c_2$, $y_2 = c_1e^{2t} - c_2$
5. $y_1 = c_1e^{4it} + c_2e^{-4it} = (c_1 + c_2)\cos 4t + i(c_1 - c_2)\sin 4t$
$= A\cos 4t + B\sin 4t$, $\quad y_2 = ic_1e^{4it} - ic_2e^{-4it}$
$= (ic_1 - ic_2)\cos 4t + i(ic_1 + ic_2)\sin 4t = B\cos 4t - A\sin 4t$, $A = c_1 + c_2$,
$B = i(c_1 - c_2)$
7. $y_1 = 2c_1 + c_2e^{-6t}$, $y_2 = -c_1 + c_3e^{-6t}$, $y_3 = -c_1 + 2(c_2 + c_3)e^{-6t}$
9. $y_1 = c_1e^{1.8t} + 2c_2e^{-0.9t} + 2c_3e^{-1.8t}$, $y_2 = 2c_1e^{1.8t} + c_2e^{-0.9t} - 2c_3e^{-1.8t}$,
$y_3 = 2c_1e^{1.8t} - 2c_2e^{-0.9t} + c_3e^{-1.8t}$
11. $y_1 = 10 + 6e^{2t}$, $y_2 = -5 + 3e^{2t}$
13. $y_1 = 2.4e^{-t} - 2e^{2.5t}$, $y_2 = 1.8e^{-t} + 2e^{2.5t}$
15. $y_1 = 2e^{14.5t} + 10$, $y_2 = 5e^{14.5t} - 4$
17. $y_2 = y_1' + y_1$, $y_2' = y_1'' + y_1' = -y_1 - y_2 = -y_1 - (y_1' + y_1)$, $y_1'' + 2y_1' + 2y_1 = 0$,
$y_1 = e^{-t}(A\cos t + B\sin t)$, $y_2 = y_1' + y_1 = e^{-t}(B\cos t - A\sin t)$. Note that
$r^2 = y_1{}^2 + y_2{}^2 = e^{-2t}(A^2 + B^2)$.
19. $I_1 = 4c_1e^{-200t} + c_2e^{-50t}$, $I_2 = -c_1e^{-200t} - 4c_2e^{-50t}$

Problem Set 4.4, page 150

1. Saddle point, unstable, $y_1 = c_1e^{-4t} + c_2e^{4t}$, $y_2 = -2c_1e^{-4t} + 2c_2e^{4t}$
3. Unstable node, $y_1 = c_1e^t + c_2e^{3t}$, $y_2 = -c_1e^t + c_2e^{3t}$
5. Stable and attractive node, $y_1 = c_1e^{-3t} + c_2e^{-5t}$, $y_2 = c_1e^{-3t} - c_2e^{-5t}$
7. Center, stable, $y_1 = A\cos 4t + B\sin 4t$, $y_2 = -2B\cos 4t + 2A\sin 4t$
9. Saddle point, unstable, $y_1 = c_1e^{3t} + c_2e^{-t}$, $y_2 = c_1e^{3t} - c_2e^{-t}$
11. $y_1 = y = c_1e^{kt} + c_2e^{-kt}$, $y_2 = y'$, hyperbolas $k^2y_1{}^2 - y_2{}^2 = const$
13. $y = e^{-2t}(A\cos t + B\sin t)$, stable and attractive spirals
17. For instance, (a) -2, (b) -1, (c) $-\frac{1}{2}$, (d) 1, (e) 4.

Problem Set 4.5, page 158

1. (0, 0), $y_1' = y_2$, $y_2' = 3y_1$, saddle point; $(0, -1)$, $y_1 = \tilde{y}_1$, $y_2 = -1 + \tilde{y}_2$, $\tilde{y}_1' = -\tilde{y}_2$,
$\tilde{y}_2' = 3\tilde{y}_1$, center
3. (0, 0), $y_1' = 4y_2$, $y_2' = 2y_1$, saddle point; (2, 0), $y_1 = 2 + \tilde{y}_1$, $y_2 = \tilde{y}_2$, $\tilde{y}_1' = 4\tilde{y}_2$,
$\tilde{y}_2' = -2\tilde{y}_1$, center
5. (0, 0), $y_1' = -y_1 + y_2$, $y_2' = -y_1 - y_2$, stable and attractive spiral point; $(-2, 2)$,
$y_1 = -2 + \tilde{y}_1$, $y_2 = 2 + \tilde{y}_2$, $\tilde{y}_1' = -\tilde{y}_1 - 3\tilde{y}_2$, $\tilde{y}_2' = -\tilde{y}_1 - \tilde{y}_2$, saddle point

7. $y_1' = y_2$, $\tilde{y}_2' = -y_1(1 - 4y_1)$, $(0, 0)$, $y_1' = y_2$, $y_2' = -y_1$, center; $(\frac{1}{4}, 0)$, $y_1 = \frac{1}{4} + \tilde{y}_1$, $y_2 = \tilde{y}_2$, $\tilde{y}_1' = \tilde{y}_2$, $\tilde{y}_2' = (-\frac{1}{4} - \tilde{y}_1)(-4\tilde{y}_1)$, $\tilde{y}_2' = \tilde{y}_1$, saddle

9. $(\frac{1}{2}\pi \pm 2n\pi, 0)$ saddle points; $(-\frac{1}{2}\pi \pm 2n\pi, 0)$ centers. Use $-\cos(\pm\frac{1}{2}\pi + \tilde{y}_1) = \sin(\pm\tilde{y}_1) \approx \pm\tilde{y}_1$.

11. $y_1' = y_2$, $y_2' = -y_1(2 + y_1)(2 - y_1)$. $(0, 0)$, $y_2' = -4y_1$, center; $(-2, 0)$, $\tilde{y}_2' = 8\tilde{y}_1$, saddle point; $(2, 0)$, $\tilde{y}_2' = 8\tilde{y}_1$ saddle point

13. $y''/y' + 2y'/y = 0$, $\ln y' + 2 \ln y = c$, $y'y^2 = y_2 y_1{}^2 = const$

15. $y = A \cos t + B \sin t$, radius $\sqrt{A^2 + B^2}$

Problem Set 4.6, page 162

3. $y_1 = A \cos 4t + B \sin 4t + \frac{29}{16}$, $y_2 = B \cos 4t - A \sin 4t - \frac{9}{4}t$

5. $y_1 = c_1 e^{4t} + c_2 e^{-3t} + 4$, $y_2 = c_1 e^{4t} - 2.5 c_2 e^{-3t} - 10$

7. $y_1 = 2c_1 e^{-9t} + c_2 e^{-4t} - 90t + 28$, $y_2 = c_1 e^{-9t} + c_2 e^{-4t} - 126t + 14$

9. $y_1 = c_1 e^t + 4c_2 e^{2t} - 3t - 4 - 2e^{-t}$, $y_2 = -c_1 e^t - 5c_2 e^{2t} + 5t + 7.5 + e^{-t}$

11. $y_1 = 3 \cos 2t - \sin 2t + t + 1$, $y_2 = \cos 2t + 3 \sin 2t + 2t - \frac{1}{2}$

13. $y_1 = 4e^{-t} - 4e^t + e^{2t}$, $y_2 = -4e^{-t} + t$

15. $y_1 = 7 - 2e^{2t} + e^{3t} - 4e^{-3t}$, $y_2 = -e^{2t} + 3e^{-3t}$

17. $I_1' + 2.5(I_1 - I_2) = 845 \sin t$, $2.5(I_2' - I_1') + 25I_2 = 0$,
$I_1 = (95 + 162.5t)e^{-5t} - 95 \cos t + 312.5 \sin t$,
$I_2 = (-30 - 162.5t)e^{-5t} + 30 \cos t + 12.5 \sin t$

19. $I_1' + 2(I_1 - I_2) = 200$, $2(I_2 - I_1) + 8I_2 + 2 \int I_2\, dt = 0$.
$I_1 = 2c_1 e^{\lambda_1 t} + 2c_2 e^{\lambda_2 t} + 100$,
$I_2 = (1.1 + \sqrt{0.41})c_1 e^{\lambda_1 t} + (1.1 - \sqrt{0.41})c_2 e^{\lambda_2 t}$, $\lambda_1 = -0.9 + \sqrt{0.41}$,
$\lambda_2 = -0.9 - \sqrt{0.41}$

Chapter 4 Review Questions and Problems, page 163

11. $y_1 = c_1 e^{8t} + c_2 e^{-8t}$, $y_2 = 2c_1 e^{8t} - 2c_2 e^{-8t}$. Saddle point

13. $y_1 = c_1 e^t + c_2 e^{-6t}$, $y_2 = c_1 e^t - 6c_2 e^{-6t}$. Saddle point

15. $y_1 = c_1 e^{7.5t} + c_2 e^{-3t}$, $y_2 = -c_1 e^{7.5t} + 0.75 c_2 e^{-3t}$. Saddle point

17. $y_1 = c_1 e^{5t} + c_2 e^t$, $y_2 = c_1 e^{5t} - c_2 e^t$. Unstable node

19. $y_1 = e^{-t}(A \cos 2t + B \sin 2t)$, $y_2 = e^{-t}(B \cos 2t - A \sin 2t)$. Stable and attractive spiral point

21. $y_1 = c_1 e^t + c_2 e^{-t} + e^{2t} + e^{-2t}$, $y_2 = -c_2 e^{-t} - 1.5e^{-2t}$

23. $y_1 = c_1 e^t + c_2 e^{-2t} - 6e^{-t} - 5$, $y_2 = -c_1 e^t - 2c_2 e^{-2t} + 10e^{-t} + 6$

25. $y_1 = c_1 e^{3t} + c_2 e^{-t} + t^2 - 2t + 2$, $y_2 = c_1 e^{3t} - c_2 e^{-t} - t^2 + 2t - 2$

27. A saddle point at $(0, 0)$

29. $I_1 = 4e^{-40t} - e^{-10t}$, $I_2 = -e^{-40t} + 4e^{-10t}$

31. $(n\pi, 0)$ center for even n and saddle point for odd n

33. Saddle points at $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, centers at $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$

Problem Set 5.1, page 170

1. $a_0(1 + x + \frac{1}{2}x^2 + \cdots) = a_0 e^x$

3. $a_0(1 - 2x^2 + \frac{2}{3}x^4 - + \cdots) + a_1(x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - + \cdots)$
$= a_0 \cos 2x + \frac{1}{2}a_1 \sin 2x$

5. $a_0(1 + \frac{1}{2}x)$

7. $a_0 + a_0x + (\frac{1}{2}a_0 + \frac{1}{2})x^2 + \cdots = a_0e^x + e^x - x - 1 = ce^x - x - 1,\ c = a_0 + 1$

9. $a_0 + a_1x + \frac{1}{2}a_1x^2 + \cdots = a_0 - a_1 + a_1e^x$

11. $s = \frac{5}{4} - 4x + 8x^2 - \frac{32}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{15}x^5,\quad s(0.2) = 0.69900$

13. $s = \frac{1}{2} + \frac{1}{4}x - \frac{1}{48}x^3 + \frac{1}{480}x^5,\quad s(1) = 0.73125$

15. $s = 1 + x - x^2 - \frac{5}{6}x^3 + \frac{2}{3}x^4 + \frac{11}{24}x^5,\quad s(\frac{1}{2}) = \frac{923}{768}$

Problem Set 5.2, page 176

1. $|c|$

3. 2 (as function of $t = (x - 3)^2$). *Ans.* $\sqrt{2}$

5. 0

7. 2

9. 1

11. π

13. $\sum_{s=3}^{\infty} \frac{(-1)^{s-1}}{5(s-2)} x^s;\ R = 1$

15. $\sum_{s=5}^{\infty} \frac{(s-4)^2}{(s-3)!} x^s;\ R = \infty$

17. $a_0(1 - \frac{1}{12}x^4 - \frac{1}{60}x^5 - \cdots) + a_1(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{24}x^5 - \cdots)$

19. $a_0 + a_1(x - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{4}{21}x^7 + \frac{2}{27}x^9 - \frac{4}{165}x^{11} + - \cdots)$

21. $a_0(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{13}{720}x^6 + \cdots) + a_1(x - \frac{1}{6}x^3 - \frac{1}{24}x^5 + \frac{5}{1008}x^7 + \cdots)$

23. $a_0(1 + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) + a_1x$

Problem Set 5.3, page 180

3. $P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$,
$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$

7. Set $x = az$. $y = c_1P_n(x/a) + c_2Q_n(x/a)$

15. $P_1^{\,1} = \sqrt{1 - x^2},\ P_2^{\,1} = 3x\sqrt{1 - x^2},\ P_2^{\,2} = 3(1 - x^2)$,
$P_4^{\,2} = (1 - x^2)(105x^2 - 15)/2$

Problem Set 5.4, page 187

1. $y_1 = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots = \frac{\sinh x}{x},\quad y_2 = \frac{1}{x} + \frac{x}{2!} + \frac{x^3}{4!} + \cdots = \frac{\cosh x}{x}$

3. $y_1 = 1 - \frac{x^2}{12} + \frac{1}{384}x^4 - \cdots,\ y_2 = 9y_1 \ln x - \frac{144}{x^4} - \frac{36}{x^2} + \frac{x^2}{2} - \frac{25x^4}{1024} + \cdots$

5. $r(r - 1) + 4r + 2 = 0,\ r_1 = -1,\ r_2 = -2;\ y_1 = \frac{1}{x} - \frac{x}{6} + \frac{x^3}{120} - + \cdots$,
$y_2 = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + - \cdots$

7. Euler-Cauchy equation with $t = x + 3$, $y_1 = (x + 3)^5$, $y_2 = y_1 \ln(x + 3)$

9. $b_0 = 1,\ c_0 = 0,\ r^2 = 0,\ y_1 = e^{-x},\ y_2 = e^{-x}\ln x$

11. $y_1 = 1/(x + 1),\ y_2 = 1/x$

13. $b_0 = \frac{1}{2},\ c_0 = 0,\ r_1 = \frac{1}{2},\ r_2 = 0,\ y_1 = x^{1/2}(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots)$,
$y_2 = 1 + 2x + 2x^2 + \cdots$

15. $y_1 = (x - 4)^7,\ y_2 = (x - 4)^{-5}$ (Euler-Cauchy with $t = x - 4$)

17. $y_1 = x + x^3 - \frac{1}{12}x^4 + \frac{3}{10}x^5 - \frac{1}{20}x^6 + \cdots,\ y_2 = 1 + 3x^2 - \frac{1}{6}x^3 + \frac{3}{2}x^4 - \frac{1}{5}x^5$
$+ \frac{11}{36}x^6 - + \cdots$

19. $y = c_1F(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x) + c_2\sqrt{x}F(1, 1, \frac{3}{2}; x)$
21. $y = A(1 - 4x + \frac{8}{3}x^2) + B\sqrt{x}F(-\frac{3}{2}, \frac{3}{2}, \frac{3}{2}; x)$
23. $y = c_1F(2, -2, -\frac{1}{2}; t - 2) + c_2(t - 2)^{3/2}F(\frac{7}{2}, -\frac{1}{2}, \frac{5}{2}; t - 2)$

Problem Set 5.5, page 197

1. Use (7b) in Sec. 5.2.
3. 0.77958 (exact 0.76520), 0.19674 (0.22389), −0.27651 (−0.26005), −0.39788 (−0.39715), −0.17038 (−0.17760), 0.15680 (0.15065), 0.30086 (0.30008), 0.16833 (0.17165)
5. $y = c_1J_\nu(\lambda x) + c_2J_{-\nu}(\lambda x),\ \nu \neq 0, \pm 1, \cdots$
7. $y = c_1J_\nu(\sqrt{x}) + c_2J_{-\nu}(\sqrt{x}),\ \nu \neq 0, \pm 1, \cdots$
9. $y = c_1xJ_1(2x)$, J_1, J_{-1} linearly dependent
11. $y = x^{-\nu}[c_1J_\nu(x) + c_2J_{-\nu}(x)],\ \nu \neq 0, \pm 1, \cdots$
13. $y = c_1J_\nu(x^3) + c_2J_{-\nu}(x^3),\ \nu \neq 0, \pm 1, \cdots$
15. $y = c_1\sqrt{x}J_1(2\sqrt{x})$, J_1, J_{-1} linearly dependent
17. $y = x^{1/4}J_1(\frac{2}{3}x^{1/4})$, J_1, J_{-1} linearly dependent
19. $y = x^{2/5}(c_1J_{8/5}(4x^{1/4}) + c_2J_{-8/5}(4x^{1/4}))$
21. Use (24b) with $\nu = 0$, (24a) with $\nu = 1$, (24d) with $\nu = 2$, respectively.
23. $J_n(x_1) = J_n(x_2) = 0$ implies ${x_1}^{-n}J_n(x_1) = {x_2}^{-n}J_n(x_2) = 0$ and $[x^{-n}J_n(x)]' = 0$ somewhere between x_1 and x_2 by Rolle's theorem. Now use (24b) to get $J_{n+1}(x) = 0$ there. Conversely, $J_{n+1}(x_3) = J_{n+1}(x_4) = 0$, thus ${x_3}^{n+1}J_{n+1}(x_3) = {x_4}^{n+1}J_{n+1}(x_4) = 0$ implies $J_n(x) = 0$ in between by Rolle's theorem and (24a) with $\nu = n + 1$.
25. Integrate the formulas in (24).
27. Use (24a) with $\nu = 1$, partial integration, (24b) with $\nu = 0$, partial integration.
33. CAS Experiment. (b) $x_0 = 1$, $x_1 = 2.5$, $x_2 = 20$, approximately. It increases with n. (c) (14) is exact. (d) It oscillates. (e) Formula (24b) with $\nu = 0$

Problem Set 5.6, page 202

1. $y = c_1J_5(x) + c_2Y_5(x)$
3. $y = c_1J_0(\sqrt{x}) + c_2Y_0(\sqrt{x})$
5. $y = c_1J_2(x^2) + c_2Y_2(x^2)$
7. $y = x^{-5}(c_1J_5(x) + c_2Y_5(x))$
9. $y = x^3(c_1J_3(x^3) + c_2Y_3(x^3))$
11. Set $H^{(1)} = kH^{(2)}$, use (10).
13. Set $x = is$ in (1), Sec. 5.5, to get the present ODE (12) in terms of s. Use (20), Sec. 5.5.

Problem Set 5.7, page 209

3. Set $x = ct + k$.
5. $x = \cos\theta$, $dx = -\sin\theta\, d\theta$, etc.
7. $\lambda_m = (m\pi/5)^2,\ m = 1, 2, \cdots;\quad y_m = \sin(m\pi x/5)$
9. $\lambda_m = [(2m + 1)\pi/2L]^2,\ m = 0, 1, \cdots;\quad y_m(x) = \sin[(2m + 1)\pi x/2L]$
11. $\lambda_m = m^2,\ m = 0, 1, \cdots;\quad y_0 = 1,\ y_m = \cos mx, \sin mx,\ m = 1, 2, \cdots$
13. $k = k_m$ from $\tan k = -k$. $\quad \lambda_m = {k_m}^2,\ m = 1, 2, \cdots;\quad y_m = \sin k_mx$
15. $\lambda_m = m^2,\ m = 1, 2, \cdots;\quad y_m = x\sin(m\ln|x|)$
17. $p = e^{8x},\ q = 0,\ r = e^{8x},\ \lambda_m = m^2;\ y_m = e^{-4x}\sin mx,\ m = 1, 2, \cdots$
19. $\lambda_m = (m\pi)^2,\ y_m = x\cos m\pi x,\ x\sin m\pi x,\ m = 0, 1, \cdots$

Problem Set 5.8, page 216

1. $1.6P_4(x) - 0.6P_0(x)$ **3.** $\frac{2}{5}P_3(x) - \frac{2}{3}P_2(x) + \frac{8}{5}P_1(x) - \frac{4}{3}P_0(x)$

7. $-0.4775P_1(x) - 0.6908P_3(x) + 1.844P_5(x) - 0.8234P_7(x) + 0.1544P_9(x) + \cdots$, $m_0 = 9$. Rounding seems to have considerable influence in Probs. 6–15.

9. $0.3799P_2(x) + 1.673P_4(x) - 1.397P_6(x) + 0.3968P_8(x) + \cdots$, $m_0 = 8$

11. $1.175P_0(x) + 1.104P_1(x) + 0.3575P_2(x) + 0.0700P_3(x) - \cdots$, $m_0 = 3$ or 4

13. $0.7855P_0(x) - 0.3550P_2(x) + 0.0900P_4(x) - \cdots$, $m_0 = 4$

15. $0.1212P_0(x) - 0.7955P_2(x) + 0.9600P_4(x) - 0.3360P_6(x) + \cdots$, $m_0 = 8$

17. (c) $a_m = (2/J_1^2(\alpha_{0,m}))(J_1(\alpha_{0,m})/\alpha_{0,m}) = 2/(\alpha_{0,m}J_1(\alpha_{0,m}))$

Chapter 5 Review Questions and Problems, page 217

11. e^{3x}, e^{-3x}, or $\cosh 3x$, $\sinh 3x$ **13.** e^x, $1 + x$

15. e^{-x^2}, xe^{-x^2} **17.** e^{-x}, $e^{-x}\ln x$

19. $1/(1 - x^2)$, $x/(1 - x^2)$ or $1/(1 - x)$, $1/(1 + x)$

21. $y = c_1J_{\sqrt{2}}(6x) + c_2J_{-\sqrt{2}}(6x)$ **23.** $y = c_1J_1(x^2) + c_2Y_1(x^2)$

25. $y = \sqrt{x}[c_1J_{1/4}(\frac{1}{2}kx^2) + c_2J_{-1/4}(\frac{1}{2}kx^2)]$

27. $\lambda_m = (2m\pi)^2$, $y_0 = 1$, $y_m = \cos 2m\pi x$, $\sin 2m\pi x$, $m = 1, 2, \cdots$

29. $y = c_1J_1(kx) + c_2Y_1(kx)$, $c_2 = 0$, $y(1) = c_1J_1(k) = 0$, $k = k_m = \alpha_{1,m}$ (the positive zeros of J_1), $y_m = J_1(\alpha_{1,m}x)$

31. $1.813P_0(x) + 2.923P_1(x) + 1.759P_2(x) + 0.663P_3(x) + 0.185P_4(x) + \cdots$

33. $0.693P_0(x) - 0.285P_2(x) + 0.144P_4(x) - 0.091P_6(x) + \cdots$

35. $0.25P_0(x) + 0.5P_1(x) + 0.3125P_2(x) - 0.0938P_4(x) + 0.0508P_6(x) + \cdots$

Problem Set 6.1, page 226

1. $\dfrac{2}{s^3} - \dfrac{2}{s^2}$ **3.** $\dfrac{s}{s^2 + 4\pi^2}$ **5.** $\dfrac{s - 2}{(s - 2)^2 - 1}$

7. $\dfrac{s\cos\theta - \omega\sin\theta}{s^2 + \omega^2}$ **9.** $\dfrac{e^{3a}}{s + 2b}$ **11.** $\dfrac{1}{s^2 + 4}$

13. $\dfrac{k}{s}(1 - e^{-bs})$ **15.** $\dfrac{1 - (1 + 2s)e^{-2s}}{2s^2}$ **17.** $\dfrac{1 - e^{-bs}}{s^2} - \dfrac{be^{-bs}}{s}$

19. $\dfrac{(1 - e^{-s})^2}{s}$ **21.** $\mathcal{L}(f_1) = \mathcal{L}(1) - \mathcal{L}(f) = \dfrac{1}{s} - \dfrac{1}{s}(1 - e^{-2s}) = e^{-2s}/s$

23. Set $ct = p$. Then $\mathcal{L}(f(ct)) = \displaystyle\int_0^\infty e^{-st}f(ct)\,dt = \int_0^\infty e^{-(s/c)p}f(p)\,dp/c = F(s/c)/c$.

29. $4\cos\pi t - 3\sin\pi t$ **31.** $1 - \frac{3}{2}t^2 + \frac{1}{8}t^4$ **33.** $\sin\dfrac{n\pi t}{L}$

35. $2 - 2e^{-4t}$ **37.** $(e^{\sqrt{3}t} - e^{-\sqrt{5}t})/(\sqrt{3} + \sqrt{5})$

39. $\dfrac{1}{\sqrt{5}}\sin\sqrt{5}t - e^{-5t}$ **41.** $\dfrac{3.8}{(s - 2.4)^2}$ **43.** $\dfrac{5\omega}{(s + a)^2 + \omega^2}$

45. $\dfrac{a(s + k) + b}{(s + k)^2 + 1}$ **47.** $3.5t^2e^t$ **49.** $\sqrt{2}t^2e^{-t\sqrt{2}}$

51. $3e^{-2t}\sin 5t$ **53.** $e^{-5\pi t}\sinh\pi t$

Problem Set 6.2, page 232

1. $\dfrac{1}{(s-k)^2}$ **3.** $\dfrac{2\omega^2}{s(s^2+4\omega^2)}$ **5.** $\dfrac{2a^2}{s(s^2-4a^2)}$

7. $\dfrac{\pi s}{(s^2+\frac{1}{4}\pi^2)^2}$

9. Use shifting. Use $\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$; use $\cos^2\alpha + \sin^2\alpha = 1$. *Ans.* $(2s^2+1)/[2s(s^2+1)]$

11. $(s+\frac{1}{2})Y = -1 + 17 \cdot 2/(s^2+4)$, $y = 7e^{-t/2} + 2\sin 2t - 8\cos 2t$

13. $(s^2 - \frac{1}{4})Y = 4s$, $y = 4\cosh \frac{1}{2}t$

15. $(s^2+2s+2)Y = s - 3 + 2 \cdot 1$, $Y = (s+1-2)/[(s+1)^2+1]$, $y = e^{-t}(\cos t - 2\sin t)$

17. $(s^2+7s+12)Y = 3.5s - 10 + 24.5 + 21/(s-3)$, $y = \frac{1}{2}e^{3t} + \frac{5}{2}e^{-4t} + \frac{1}{2}e^{-3t}$

19. $(s+1.5)^2 Y = s + 31.5 + 3 + 54/s^4 + 64/s$, $Y = 1/(s+1.5) + 1/(s+1.5)^2 + 24/s^4 - 32/s^3 + 32/s^2$, $y = (1+t)e^{-1.5t} + 4t^3 - 16t^2 + 32t$

21. $t = \tilde{t} + 2$, $\tilde{Y} = 4/(s-6)$, $\tilde{y} = 4e^{6t}$, $y = 4e^{6(t-2)}$

23. $t = \tilde{t} + 1$, $(s-1)(s+4)\tilde{Y} = 4s + 17 + 6/(s-2)$, $y = 3e^{t-1} + e^{2(t-1)}$

25. (b) In the proof, integrate from 0 to a and then from a to ∞ and see what happens. (c) Find $\mathscr{L}(f)$ and $\mathscr{L}(f')$ by integration and substitute them into (1*).

27. $2 - 2e^{-t/2}$ **29.** $\dfrac{1}{k^2}(e^{kt}-1) - \dfrac{t}{k}$ **31.** $\cosh\sqrt{5}t - 1$

33. $\frac{1}{8}\sinh 2t - \frac{1}{4}t$

Problem Set 6.3, page 240

3. $(1 - e^{2-2s})/(s-1)$

5. $\left(\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s}\right)e^{-s} - \left(\dfrac{2}{s^3} + \dfrac{4}{s^2} + \dfrac{4}{s}\right)e^{-2s}$

7. $\dfrac{s}{s^2+\pi^2}(-e^{-s} - e^{-4s})$ **9.** $\left(\dfrac{1}{s^2} + \dfrac{5}{s}\right)e^{-5s} - \left(\dfrac{1}{s^2} + \dfrac{10}{s}\right)e^{-10s}$

11. $\dfrac{-20s}{s^2+\pi^2}(e^{-3s} + e^{-6s})$ **13.** $\dfrac{1}{s-\pi}(e^{-2s+2\pi} - e^{-4s+4\pi})$

15. 0 if $t < 4$, $t - 4$ if $t > 4$ **17.** $\sin t$ if $2\pi < t < 8\pi$, 0 elsewhere

19. 0 if $t < 2$, $(t-2)^4/24$ if $t > 2$ **21.** $u(t-3)\cosh(2t-6)$

23. $e^{-t}\sin t$ **25.** $e^{-2t}\cos 3t + 9\cos 2t + 8\sin 2t$

27. $\sin 3t + \sin t$ if $0 < t < \pi$ and $\frac{4}{3}\sin 3t$ if $t > \pi$

29. $t - \sin t$ if $0 < t < 1$, $\cos(t-1) + \sin(t-1) - \sin t$ if $t > 1$

31. $e^t - \sin t + u(t-2\pi)(\sin t - \frac{1}{2}\sin 2t)$

33. $t = 1 + \tilde{t}$, $\tilde{y}'' + 4\tilde{y} = 8(1+\tilde{t})^2(1 - u(\tilde{t}-4))$, $\cos 2t + 2t^2 - 1$ if $t < 5$, $\cos 2t + 49\cos(2t-10) + 10\sin(2t-10)$ if $t > 5$

35. $Rq' + q/C = 0$, $Q = \mathscr{L}(q)$, $q(0) = CV_0$, $i = q'(t)$, $R(sQ - CV_0) + Q/C = 0$, $q = CV_0 e^{-t/(RC)}$

37. $10I + \dfrac{100}{s}I = \dfrac{100}{s^2}e^{-2s}$, $I = e^{-2s}\left(\dfrac{1}{s} - \dfrac{1}{s+10}\right)$, $i = 0$ if $t < 2$ and $1 - e^{-10(t-2)}$ if $t > 2$

39. $i = e^{-20t} + 20t - 1 + u(t-2)[-20t + 1 + 39e^{-20(t-2)}]$

41. $0.1i' + 25i = 490e^{-5t}[1 - u(t-1)]$, $i = 20(e^{-5t} - e^{-250t}) + 20u(t-1)[-e^{-5t} + e^{-250t+245}]$

43. $i = (10 \sin 10t + 100 \sin t)(u(t-\pi) - u(t-3\pi))$

45. $i' + 2i + 2\int_0^t i(\tau)\, d\tau = 1 - u(t-2)$, $I = (1 - e^{-2s})/(s^2 + 2s + 2)$,
$i = e^{-t} \sin t - u(t-2)\, e^{-t+2} \sin (t-2)$

47. $i = 27 \cos t + 6 \sin t - e^{-t}(27 \cos 3t + 11 \sin 3t) + u(t-2\pi)\,[-27 \cos t - 6 \sin t + e^{-(t-2\pi)}(27 \cos 3t + 11 \sin 3t)]$

Problem Set 6.4, page 247

1. $y = 10 \cos t$ if $0 < t < 2\pi$ and $10 \cos t + \sin t$ if $t > 2\pi$

3. $y = 5.5e^t + 4.5e^{-t} + 5(e^{t-1/2} - e^{-t+1/2})u(t - \frac{1}{2}) - 50(e^{t-1} - e^{-t+1})u(t-1)$

5. $y = 0.1[e^t + e^{-2t}(-\cos t + 7 \sin t)] + 0.1u(t-10)[-e^t + e^{-2t+30}(\cos (t-10) - 7 \sin (t-10))]$

7. $y = 1 + \frac{1}{3}e^{-t} \sin 3t + u(t-4)[-1 + e^{-t+4}(\cos (3t-12) + \frac{1}{3} \sin (3t-12))] - \frac{10}{3}u(t-5)e^{-t+5} \sin (3t-15)$

9. $y = 5t - 2 - 50u(t-\pi)e^{-t+\pi} \sin 2t$. Straight line, sharply deformed between π and about 8

11. $y = (0.4t + 1.52)e^t + 0.48e^{-4t} + 1.6u(t-2)[-e^t + e^{-4t+10}]$

Problem Set 6.5, page 253

1. t

3. $e^t - t - 1$

5. $\dfrac{1}{\omega} \sin \omega t$

7. $\dfrac{1}{2k}(e^{kt} - e^{-kt}) = \dfrac{1}{k} \sinh kt$

9. $\frac{1}{8}(e^{3t} - e^{-5t})$

11. $\frac{1}{2}(\frac{1}{2} - \frac{1}{2} \cos 2t) = \frac{1}{2} \sin^2 t$

13. $t - \sin t$

15. $\frac{1}{9}(\cosh 3t - 1)$

19. $Y = 3/((s^2 + 4)(s^2 + 9))$, $y = 0.3 \sin 2t - 0.2 \sin 3t$

21. $(s^2 + 9)Y = 4 + 8(1 + e^{-\pi s})/(s^2 + 1)$, $y = \sin t + \sin 3t$ if $t < \pi$, $\frac{4}{3} \sin 3t$ if $t > \pi$

23. 0 if $0 < t < 1$, $\frac{5}{2}\int_1^t \sin (2(\tau - 1))\, d\tau = -\frac{5}{4} \cos (2t-2) + \frac{5}{4}$ if $t > 1$

25. $y = 2e^{-2t} - e^{-4t} + (e^{-2t+2} - e^{-4t+4})u(t-1) + (e^{-2t+4} - e^{-4t+8})u(t-2)$

27. $y - 1 * y = 1$, $y = e^t$

29. $y - y * \sin t = \cos t$, $Y = 1/s$, $y = 1$

31. $Y(1 + 1/s^2) = 1/s$, $y = \cos t$

33. $Y(1 + 2/(s-1)) = (s-1)^{-2}$, $y = \sinh t$

Problem Set 6.6, page 257

1. $\dfrac{4}{(s-1)^2}$

3. $\dfrac{2\omega s}{(s^2 + \omega^2)^2}$

5. $\dfrac{2s + 4}{(s^2 + 4s + 5)^2}$

7. $\dfrac{24s^2 + 128}{(s^2 - 16)^3}$

9. $\dfrac{2\omega(3s^2 - \omega^2)}{(s^2 + \omega^2)^3}$

11. $\dfrac{2s \cos k + (s^2 - 1) \sin k}{(s^2 + 1)^2}$

13. $6te^{-t}$ **15.** $te^{-2t} \sin t$
17. $t^2 e^{kt}$ **19.** $\ln s - \ln (s-1)$; $(e^t - 1)/t$

Problem Set 6.7, page 262

1. $y_1 = -e^{-t} \sin t, \quad y_2 = e^{-t} \cos t$ **3.** $y_1 = 2e^{-4t} - 4e^{2t}, \quad y_2 = e^{-4t} - 8e^{2t}$
5. $y_1 = 2e^{-t} + 4e^{-2t} + \frac{1}{2}t - \frac{1}{4}, \quad y_2 = -3e^{-t} - 4e^{-2t} - \frac{1}{2}t + \frac{1}{4}$
7. $y_1 = e^{-t}(2 \cos 2t + 6 \sin 2t) + t^2, \quad y_2 = 10e^{-t} \sin 2t - t^2$
9. $y_1 = 4 \cos 5t + 6 \sin 5t - 2 \cos t - 25 \sin t, \quad y_2 = 2 \cos 5t - 10 \sin 5t + 20 \sin t$
11. $y_1 = -\cos t + \sin t + 1 + u(t-1)[-1 + \cos (t-1) - \sin (t-1)]$
$y_2 = \cos t + \sin t - 1 + u(t-1)[1 - \cos (t-1) - \sin (t-1)]$
13. $y_1 = 2u(t-2)(e^{4t} - e^{t+6}), \quad y_2 = e^{2t} + u(t-2)(e^{4t} - 3e^{2t+4} + 2e^{t+6})$
15. $y_1 = -e^{-2t} + e^t + \frac{1}{3}u(t-1)(-e^{-2t+3} + e^t), \quad y_2 = -e^{-2t} + 4e^t$
$+ \frac{1}{3}u(t-1)(-e^{-2t+3} + e^t)$
17. $y_1 = 3 \sin 2t + 8e^{-3t}, \quad y_2 = -3 \sin 2t + 5e^{-3t}$
19. $y_1 = e^t - e^{-t}, \quad y_2 = e^t, \quad y_3 = e^{-t}$
25. $4i_1 + 8(i_1 - i_2) + 2i_1' = 390 \cos t, \quad 8i_2 + 8(i_2 - i_1) + 4i_2' = 0, \quad i_1 = -26e^{-2t}$
$- 16e^{-8t} + 42 \cos t + 15 \sin t, \quad i_2 = -26e^{-2t} + 8e^{-8t} + 18 \cos t + 12 \sin t$

Chapter 6 Review Questions and Problems, page 267

11. $\dfrac{1}{(s-3)^2}$ **13.** $\dfrac{2}{s(s^2+4)}$ **15.** $e^{-\pi s}\left(\dfrac{\pi}{s} + \dfrac{1}{s^2}\right)$

17. $\dfrac{s}{(s-1)(s^2+4)}$ **19.** $\dfrac{2s^2}{s^4 - 1}$ **21.** $\dfrac{a-b}{(s-a)(s-b)}$

23. $10 \cos t\sqrt{2}$ **25.** $3e^{-2t} \sin 4t$ **27.** $u(t-2)(5 + 4(t-2))$

29. $te^{-2t} \sin t$ **31.** $(t^2 - 1)u(t-1)$ **33.** $\dfrac{\pi}{\omega^3}(\omega t - \sin \omega t)$

35. $20 \sin t + u(t-1)[1 - \cos (t-1)]$
37. $10 \cos 2t - \frac{1}{2} \sin 2t + 4u(t-5) \sin (2t - 10)$ **39.** $e^{-t}(7 \cos 3t + 2 \sin 3t)$
41. $e^{-t} + u(t-\pi)[1.2 \cos t - 3.6 \sin t + 2e^{-t+\pi} - 0.8e^{2t-2\pi}]$
43. $u(t-1)(t-1)e^{2t-2} + 4u(t-2)(2-t)e^{2t-4}$
45. $y_1 = e^t + \frac{1}{2}e^{-t} - \frac{1}{2} \cos t - \frac{1}{2} \sin t, \quad y_2 = -e^t + \frac{1}{2}e^{-t} + \frac{1}{2} \cos t + \frac{1}{2} \sin t$
47. $y_1 = \frac{1}{2}e^{-t} \sin 2t, \quad y_2 = e^{-t}(\cos 2t - \frac{1}{2} \sin 2t)$
49. $y_1 = e^{2t}, \quad y_2 = e^{2t} + e^t$
51. $I = (1 - e^{-2s})/[s(s+10)], \quad i = 0.1(1 - e^{-10t}) + 0.1u(t-2)[-1 + e^{-10t+20}]$
53. $I = e^{-2t}(76 \cos 4t - 42 \sin 4t) - 76 \cos 20t + 16 \sin 20t$
55. $i_1' + 10(i_1 - i_2) = 100\, t^2, \quad 30i_2' + 10(i_2' - i_1') + 100i_2 = 0,$
$i_1 = (\frac{4}{5} + 4t)e^{-5t} + 10t^2 - \frac{4}{5}, \quad i_2 = (\frac{4}{5} + 2t)e^{-5t} + 2t - \frac{4}{5}$

Problem Set 7.1, page 277

1. $\begin{bmatrix} 6 & 3 \\ -2 & 11 \\ -7 & 6 \end{bmatrix}$, same, $\begin{bmatrix} 36 & -6 \\ -36 & 18 \\ -54 & 0 \end{bmatrix}$, $\begin{bmatrix} -36 & 6 \\ 36 & -18 \\ 54 & 0 \end{bmatrix}$

3. Undef., $\begin{bmatrix} -6 & 1 \\ 6 & -3 \\ 9 & 0 \end{bmatrix}$, $\begin{bmatrix} 6 & -1 \\ -6 & 3 \\ -9 & 0 \end{bmatrix}$, undef.

5. $\begin{bmatrix} -48 & -2 \\ 38 & -44 \\ 67 & -15 \end{bmatrix}$, $\begin{bmatrix} 36 & 0 & 48 \\ -12 & 24 & 24 \\ 72 & 60 & -48 \end{bmatrix}$, same, $\begin{bmatrix} -0.3 & -5.0 & -3.4 \\ -4.9 & 1.8 & 3.8 \\ -3.6 & 3.5 & 0.4 \end{bmatrix}$

7. $\begin{bmatrix} 66 \\ 0 \\ -33 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3.2 \\ -4.2 \end{bmatrix}$, same, $\begin{bmatrix} 6.5 \\ -0.8 \\ -2.2 \end{bmatrix}$

9.
$$\begin{aligned} -5x_2 &= -3 \\ -5x_1 + 2x_2 &= 4 \\ -3x_1 + 4x_2 &= 0 \end{aligned}$$

Problem Set 7.2, page 286

1. $\begin{bmatrix} 24 \\ 49 \\ -43 \end{bmatrix}$, undef., $\begin{bmatrix} 2 \\ 38 \\ -22 \end{bmatrix}$, $\begin{bmatrix} 54 & 10 & -46 \\ 74 & 19 & -29 \\ -74 & -5 & 51 \end{bmatrix}$

3. $\begin{bmatrix} 54 & 10 & -46 \\ 74 & 19 & -29 \\ -74 & -5 & 51 \end{bmatrix}$, $\begin{bmatrix} 134 & -50 & -18 \\ 94 & -29 & -1 \\ -134 & 63 & 19 \end{bmatrix}$, $\begin{bmatrix} 44 & 64 & -72 \\ 64 & 110 & -114 \\ -72 & -114 & 126 \end{bmatrix}$,
$\begin{bmatrix} 236 & -92 & -12 \\ -92 & 38 & 6 \\ -12 & 6 & 6 \end{bmatrix}$

5. $[20 \quad -3 \quad -7]$, $[-62 \quad 34 \quad 2]$, $\begin{bmatrix} 565 \\ 525 \\ 790 \end{bmatrix}$, same

7. $\begin{bmatrix} 15 & 0 & 40 \\ 3 & 0 & 8 \\ 6 & 0 & 16 \end{bmatrix}$, 31, $\begin{bmatrix} -310 & 170 & 10 \\ -62 & 34 & 2 \\ -124 & 68 & 4 \end{bmatrix}$, same

9. $\begin{bmatrix} 337 & 8 & -160 \\ 252 & 49 & -68 \\ -308 & 52 & 233 \end{bmatrix}$, same, $\begin{bmatrix} 257 & 68 & -188 \\ 232 & 97 & -96 \\ -248 & -16 & 265 \end{bmatrix}$

11. $\begin{bmatrix} 324 & 32 & -320 \\ 244 & 38 & -322 \\ -244 & -10 & 366 \end{bmatrix}$, $\begin{bmatrix} 216 & -104 & -104 \\ 280 & -132 & -68 \\ -280 & 140 & 76 \end{bmatrix}$, $\begin{bmatrix} 7060 & 960 & -5120 \\ 7548 & 1246 & -5434 \\ -8140 & -1090 & 6150 \end{bmatrix}$

$\begin{bmatrix} 4324 & 1520 & -4816 \\ 3636 & 1242 & -4518 \\ -3700 & -1046 & 5002 \end{bmatrix}$

13. 83, 166, 593, 0
19. (d) $\mathbf{AB} = (\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} = \mathbf{BA}$; etc. (e) *Ans.* If $\mathbf{AB} = -\mathbf{BA}$.
21. Triangular are $\mathbf{U}_1 + \mathbf{U}_2$, $\mathbf{U}_1\mathbf{U}_2$, $\mathbf{U}_1{}^2$, $\mathbf{L}_1 + \mathbf{L}_2$, $\mathbf{L}_1\mathbf{L}_2$, $\mathbf{L}_1{}^2$.
23. $[0.8 \quad 1.2]^\mathsf{T}$, $[0.76 \quad 1.24]^\mathsf{T}$, $[0.752 \quad 1.248]^\mathsf{T}$
27. $\mathbf{p} = [110 \quad 45 \quad 80]^\mathsf{T}$, $\mathbf{v} = [92000 \quad 86300]^\mathsf{T}$

Problem Set 7.3, page 295

1. $x = 2.5$, $y = -4.2$
3. $x = 0.2$, $y = 1.6$
5. $x = 0$, $y = -2$, $z = 9$
7. $x = 4$, $y = 0$, $z = -2$
9. $x = 3y + 2$, y arb., $z = -y + 6$
11. $y = 2x + 3z + 1$, x, z arb.
13. $w = 1$, $y = 2z - x$, x, z arb.
15. $w = 3$, $x = 0$, $y = -2$, $z = 8$
17. $I_1 = (R_1 + R_2)E_0/(R_1R_2)$, $I_2 = E_0/R_1$, $I_3 = E_0/R_2$ [Amps]
19. $I_1 - I_2 - I_3 = 0$, $(3 + 2 + 5)I_1 + 10I_2 = 95 + 35$, $10I_2 - 5I_3 = 35$, $I_1 = 8$, $I_2 = 5$, $I_3 = 3$ Amps
21. $x_1 + x_4 = 500$, $x_1 + x_2 = 800$, $x_2 + x_3 = 1100$, $x_3 + x_4 = 800$, $x_1 = 500 - x_4$, $x_2 = 300 + x_4$, $x_3 = 800 - x_4$, x_4 arbitrary

Problem Set 7.4, page 301

1. 1, $[1 \quad -2]$; $[1 \quad 0 \quad -3]^\mathsf{T}$
3. 3, $[1 \quad 4 \quad 0 \quad 7]$, $[0 \quad -2 \quad 1 \quad 3]$, $[0 \quad 0 \quad 5 \quad 105]$; $[-2 \quad 4 \quad 5]^\mathsf{T}$, $[0 \quad 1 \quad 5]^\mathsf{T}$, $[0 \quad 0 \quad 1]^\mathsf{T}$
5. 2, $[3 \quad 0 \quad 5]$, $[0 \quad 3 \quad 4]$; $[3 \quad 0 \quad 5]^\mathsf{T}$, $[0 \quad 3 \quad 4]^\mathsf{T}$
7. 2, $[8 \quad 0 \quad 4]$, $[0 \quad 2 \quad 0]$; $[8 \quad 0 \quad 4 \quad 0]^\mathsf{T}$, $[0 \quad 2 \quad 0 \quad 4]^\mathsf{T}$
9. 3, $[1 \quad 0 \quad 3 \quad 0]$, $[0 \quad 5 \quad 8 \quad -37]$, $[0 \quad 0 \quad -74 \quad 296]$; same transposed
11. 4, $[1 \quad 0 \quad 0 \quad 0]$, $[0 \quad 1 \quad 0 \quad 0]$, $[0 \quad 0 \quad 1 \quad 0]$, $[0 \quad 0 \quad 0 \quad 1]$; same transposed
13. No
15. No
17. Yes
19. Yes
21. (c) 1
27. 2, $[1 \quad -1 \quad 0]$, $[0 \quad 0 \quad 1]$
29. No
31. 1, $[-\frac{1}{4} \quad \frac{1}{3} \quad 1]$
33. No
35. 1, $[5 \quad \frac{5}{2} \quad \frac{5}{3} \quad \frac{5}{4} \quad 1]$

Problem Set 7.7, page 314

5. 107
7. $\cos(\alpha + \beta)$
9. -66.88
11. 0
13. $u^3 + v^3 + w^3 - 3uvw$
15. 4
19. $x = -1.2$, $y = 0.8$, $z = 3.1$
21. 1
23. 3

Problem Set 7.8, page 322

1. $\begin{bmatrix} 1.80 & -2.32 \\ -0.25 & 0.60 \end{bmatrix}$ **3.** $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$ **9.** $\mathbf{A}^{-1} = \mathbf{A}$ **11.** No inverse

15. $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2 = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$

19. $\mathbf{AA}^{-1} = \mathbf{I}$, $(\mathbf{AA}^{-1})^{-1} = (\mathbf{A}^{-1})^{-1}\mathbf{A}^{-1} = \mathbf{I}$. Multiply by $\mathbf{A}$ from the right.
21. det $\mathbf{A} = -1$. $C_{12} = C_{21} = C_{33} = -1$, the other C_{jk} are zero.
23. det $\mathbf{A} = 1$. $C_{11} = 1$, $C_{12} = -2$, $C_{22} = 1$, $C_{13} = 3$, $C_{23} = -4$, $C_{33} = 1$

Problem Set 7.9, page 329

1. Yes, 2, $[3 \quad 5 \quad 0]^T$, $[2 \quad 0 \quad -5]^T$ **3.** No
5. Yes, 2, $[0 \quad 0 \quad 0 \quad 1 \quad 0]^T$, $[0 \quad 0 \quad 0 \quad 0 \quad 1]^T$
7. Yes, 1, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **9.** No
11. Yes, 2, xe^{-x}, e^{-x}
13. $[1 \quad 0]^T$, $[0 \quad 1]^T$; $[1 \quad 1]^T$, $[-1 \quad 1]^T$; $[1 \quad 0]^T$, $[0 \quad -1]^T$
15. $x_1 = -0.6y_1 + 0.4y_2$
$x_2 = -0.8y_1 + 0.2y_2$
17. $x_1 = 2y_1 + y_2$
$x_2 = 5y_1 + 3y_2$
19. $x_1 = 5y_1 + 3y_2 - 3y_3$
$x_2 = 3y_1 + 2y_2 - 2y_3$
$x_3 = 2y_1 - y_2 + 2y_3$
21. $\sqrt{56}$ **23.** $16\sqrt{5}$
25. 2 **29.** $4v_1 - 3v_2 = 0$, $\mathbf{v} = \pm\left[\frac{3}{5} \quad \frac{4}{5}\right]^T$

Chapter 7 Review Questions and Problems, page 330

11. $x = 4$, $y = 7$ **13.** $x = y + 6$, $z = y$, y arbitrary
15. $x = \frac{1}{4}$, $y = -\frac{1}{2}$, $z = \frac{3}{2}$ **17.** $x = 7$, $y = -3$
19. $x = 2z$, $y = 4$, z arbitrary **21.** 0
23. 638, 0, 0 **25.** $\begin{bmatrix} 8.0 & -3.6 & 1.2 \\ -3.6 & 2.6 & 2.4 \\ 1.2 & 2.4 & 9.0 \end{bmatrix}$

27. 14, 14, $\begin{bmatrix} 12 & 0 & 6 \\ 28 & 0 & 14 \\ 4 & 0 & 2 \end{bmatrix}$ **29.** $[-20 \quad 9 \quad -3]$, $\begin{bmatrix} -20 \\ 9 \\ -3 \end{bmatrix}$

31. 2, 2

33. 2, 2

35. 2, 2

37. $\frac{1}{51}\begin{bmatrix} 4 & 3 \\ -5 & 9 \end{bmatrix}$

39. $\frac{1}{45}\begin{bmatrix} -5 & 10 & 5 \\ 11 & 5 & -2 \\ 23 & -10 & 4 \end{bmatrix}$

41. $\frac{1}{12}\begin{bmatrix} 72 & -72 & 132 \\ 31 & -32 & 59 \\ -19 & 20 & -35 \end{bmatrix}$

43. $I_1 = 33$ A, $I_2 = 11$ A, $I_3 = 22$ A

45. $I_1 = 12$ A, $I_2 = 18$ A, $I_3 = 6$ A

Problem Set 8.1, page 338

1. -2, $[1 \quad 0]^\mathsf{T}$; 0.4, $[0 \quad 1]^\mathsf{T}$

3. 4, $2x_1 + (-4 - 4)x_2 = 0$, say, $x_1 = 4$, $x_2 = 1$; -4, $[0 \quad 1]^\mathsf{T}$

5. -4, $[2 \quad 9]^\mathsf{T}$; 3, $[1 \quad 1]^\mathsf{T}$ **7.** $0.8 + 0.6i$, $[1 \quad -i]^\mathsf{T}$; $0.8 - 0.6i$, $[1 \quad i]^\mathsf{T}$

9. 5, $[1 \quad 2]^\mathsf{T}$; 0, $[-2 \quad 1]^\mathsf{T}$ **11.** 4, $[1 \quad 0 \quad 0]^\mathsf{T}$; 0, $[0 \quad 1 \quad 0]^\mathsf{T}$; -1, $[0 \quad 0 \quad 1]^\mathsf{T}$

13. $-(\lambda^3 - 18\lambda^2 + 99\lambda - 162)/(\lambda - 3) = -(\lambda^2 - 15\lambda + 54)$; 3, $[2 \quad -2 \quad 1]^\mathsf{T}$; 6, $[1 \quad 2 \quad 2]^\mathsf{T}$; 9, $[2 \quad 1 \quad -2]^\mathsf{T}$

15. 1, $[-3 \quad 2 \quad 10]^\mathsf{T}$; 4, $[0 \quad 1 \quad 2]^\mathsf{T}$; 2, $[0 \quad 0 \quad 1]^\mathsf{T}$

17. $-(\lambda^3 - 7\lambda^2 - 5\lambda + 75)/(\lambda + 3) = -(\lambda^2 - 10\lambda + 25)$; -3, $[1 \quad 2 \quad -1]^\mathsf{T}$; 5, $[3 \quad 0 \quad 1]^\mathsf{T}$, $[-2 \quad 1 \quad 0]^\mathsf{T}$

19. $-(\lambda - 9)^3$; 9, $[2 \quad -2 \quad 1]^\mathsf{T}$; defect 2

21. $\lambda(\lambda^3 - 8\lambda^2 - 16\lambda + 128)/(\lambda - 4) = \lambda(\lambda^2 - 4\lambda - 32)$; 4, $[-1 \quad 3 \quad 1 \quad 1]^\mathsf{T}$; -4, $[1 \quad 1 \quad -1 \quad -1]^\mathsf{T}$; 0, $[1 \quad 1 \quad 1 \quad 1]^\mathsf{T}$; 8, $[1 \quad -3 \quad 1 \quad -3]^\mathsf{T}$

23. 2, $[8 \quad 8 \quad -16 \quad 1]^\mathsf{T}$; 1, $[0 \quad 7 \quad 0 \quad 4]^\mathsf{T}$; 3, $[0 \quad 0 \quad 9 \quad 2]^\mathsf{T}$, -6, $[0 \quad 0 \quad 0 \quad 1]^\mathsf{T}$

25. $(\lambda + 1)^2(\lambda^2 + 2\lambda - 15)$; -1, $[1 \quad 0 \quad 0 \quad 0]^\mathsf{T}$, $[0 \quad 1 \quad 0 \quad 0]^\mathsf{T}$; -5, $[-3 \quad -3 \quad 1 \quad 1]^\mathsf{T}$, 3, $[3 \quad -3 \quad 1 \quad -1]^\mathsf{T}$

29. Use that real entries imply real coefficients of the characteristic polynomial.

Problem Set 8.2, page 343

1. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; -1, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; 1, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; any point $(x, 0)$ on the x-axis is mapped onto $(-x, 0)$, so that $[1 \quad 0]^\mathsf{T}$ is an eigenvector corresponding to $\lambda = -1$.

3. (x, y) maps onto $(x, 0)$. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; 1, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; 0, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A point on the x-axis maps onto itself, a point on the y-axis maps onto the origin.

5. (x, y) maps onto $(5x, 5y)$. 2×2 diagonal matrix with entries 5.

7. -2, $[1 \quad -1]^\mathsf{T}$, $-45°$; 8, $[1 \quad 1]^\mathsf{T}$, $45°$

9. 2, $[3 \quad -1]^\mathsf{T}$, $-18.4°$; 7, $[1 \quad 3]^\mathsf{T}$, $71.6°$

11. 1, $[-1/\sqrt{6} \quad 1]$, $112.2°$; 8, $[1 \quad 1/\sqrt{6}]$, $22.2°$

13. 1, $[1 \quad 1]^\mathsf{T}$, $45°$; -5, $[1 \quad -1]^\mathsf{T}$, $-45°$

15. $c[15 \quad 24 \quad 50]^\mathsf{T}$, $c > 0$

17. $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} = [0.73 \quad 0.59 \quad 1.04]^\mathsf{T}$ (rounded)

19. $[1 \quad 1 \quad 1]^\mathsf{T}$ **21.** 1.8 **23.** 2.1

Problem Set 8.3, page 348

3. No

5. $\mathbf{A}^{-1} = (-\mathbf{A}^{\mathsf{T}})^{-1} = -(\mathbf{A}^{-1})^{\mathsf{T}}$

7. No since $\det \mathbf{A} = \det(\mathbf{A}^{\mathsf{T}}) = \det(-\mathbf{A}) = (-1)^3 \det \mathbf{A} = -\det \mathbf{A} = 0$.

9. Orthogonal, $0.96 \pm 0.28i$

11. Neither, 2, 2, defect 1

13. Symmetric, 9, 18, 18

15. Orthogonal, 1, i, $-i$

17. Symmetric, $a + 2b$, $a - b$, $a - b$

Problem Set 8.4, page 355

1. $[1 \quad 2]^{\mathsf{T}}$, $[2 \quad -1]^{\mathsf{T}}$; $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$

3. $[1 \quad -1]^{\mathsf{T}}$, $[1 \quad 1]^{\mathsf{T}}$, $\mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$

5. $[2 \quad -1]^{\mathsf{T}}$, $[2 \quad 1]^{\mathsf{T}}$, diag $(-2, 4)$

7. $[1 \quad 0 \quad 0]^{\mathsf{T}}$, $[1 \quad -2 \quad 1]^{\mathsf{T}}$, $[0 \quad 1 \quad 0]^{\mathsf{T}}$, diag $(1, 2, 3)$

9. $[0 \quad 3 \quad 2]^{\mathsf{T}}$, $[5 \quad 3 \quad 0]^{\mathsf{T}}$, $[1 \quad 0 \quad 2]^{\mathsf{T}}$, diag $(45, 9, -27)$

13. $\begin{bmatrix} 16 & -7 \\ 42 & -19 \end{bmatrix}$; -5, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$; 2, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

15. $\begin{bmatrix} -30 & -72 \\ \frac{40}{3} & 32 \end{bmatrix}$; 2, $\begin{bmatrix} 9 \\ -4 \end{bmatrix}$; 0, $\begin{bmatrix} 12 \\ -5 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

17. $\begin{bmatrix} -95 & 18 & -144 \\ 24 & -2 & 36 \\ 66 & -12 & 100 \end{bmatrix}$; 4, $\begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$; -2, $\begin{bmatrix} 6 \\ -1 \\ -4 \end{bmatrix}$; 1, $\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$;

$$\mathbf{x} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

19. $\mathbf{C} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$, $10y_1^2 - 15y_2^2 = 5$, $\mathbf{x} = \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix} \mathbf{y}$, hyperbola

21. $\mathbf{C} = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$, $5y_1^2 - 5y_2^2 = 0$, $\mathbf{x} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \mathbf{y}$, straight lines

23. $\mathbf{C} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$, $y_1^2 + 5y_2^2 = 10$, $\mathbf{x} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \mathbf{y}$, ellipse

25. $\mathbf{C} = \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix}$, $7y_1^2 - 5y_2^2 = 35$, $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \mathbf{y}$, hyperbola

27. $\mathbf{C} = \begin{bmatrix} 12 & 16 \\ 16 & 12 \end{bmatrix}$, $28y_1^2 - 4y_2^2 = 112$, $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \mathbf{y}$, hyperbola

Problem Set 8.5, page 361

3. $(\overline{\mathbf{ABC}})^\mathsf{T} = \overline{\mathbf{C}}^\mathsf{T}\overline{\mathbf{B}}^\mathsf{T}\overline{\mathbf{A}}^\mathsf{T} = \mathbf{C}^{-1}(-\mathbf{B})\mathbf{A}$
5. Hermitian, $3 + \sqrt{2}$, $[-i \quad 1 - \sqrt{2}]^\mathsf{T}$; $3 - \sqrt{2}$, $[-i \quad 1 + \sqrt{2}]^\mathsf{T}$
7. Hermitian, unitary, 1, $[1 \quad i - i\sqrt{2}]^\mathsf{T}$; -1, $[1 \quad i + i\sqrt{2}]^\mathsf{T}$
9. Skew-Hermitian, $5i$, $[1 \quad 0 \quad 0]^\mathsf{T}$, $[0 \quad 1 \quad 1]^\mathsf{T}$; $-5i$, $[0 \quad 1 \quad -1]^\mathsf{T}$
11. Skew-Hermitian, unitary, i, $[1 \quad 0 \quad 1]^\mathsf{T}$, $[0 \quad 1 \quad 0]^\mathsf{T}$; $-i$, $[1 \quad 0 \quad -1]^\mathsf{T}$
13. Skew-Hermitian, $-66i$
15. Hermitian, 10

Chapter 8 Review Questions and Problems, page 362

9. $\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} \mathbf{A} \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -7 \end{bmatrix}$

11. $\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -9 \end{bmatrix}$

13. $\begin{bmatrix} \frac{2}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & 0 & \frac{2}{9} \\ 0 & -\frac{2}{9} & -\frac{1}{9} \end{bmatrix} \mathbf{A} \begin{bmatrix} 4 & 1 & 2 \\ 1 & -2 & -4 \\ -2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} -1.0 & 0 \\ 4.8 & 5.0 \end{bmatrix}$, 5, -1

17. $\begin{bmatrix} -2 & 0 & -4.50 \\ 1 & 0 & -2.75 \\ 0 & 0 & 4 \end{bmatrix}$, 4, 0, -2

19. $1.1y_1^2 + y_2^2 = 1$, ellipse

Problem Set 9.1, page 370

1. 2, -4, 0; $\sqrt{20}$; $[1/\sqrt{5}, -2/\sqrt{5}, 0]$
3. -1, 0, 5; $\sqrt{26}$; $[-1/\sqrt{26}, 0, 5/\sqrt{26}]$
5. -8, -6, 0; 10; $[-0.8, -0.6, 0]$
7. (7, 5, 0); $\sqrt{10}$
9. $(\frac{1}{4}, \frac{3}{2}, \frac{9}{4})$; $\sqrt{37/8}$
11. $(0, 1, \frac{1}{2})$; $\sqrt{37/2}$
13. $[4, -2, 0]$, $[-2, 1, 0]$, $[-1, \frac{1}{2}, 0]$
15. $[10, -5, -15]$
17. $[28, -14, -14]$
19. $[-2, 1, 8]$, $[6, -3, -24]$
23. (5.5, 5.5, 0), $(\frac{1}{8}, \frac{1}{2}, \frac{15}{8})$
25. [0, 0, 9]; 9
27. $[-8, -2, 4]$; $\sqrt{84}$
29. $\mathbf{v} = [0, 0, -9]$
31. $[-9, 0, 0]$, $[0, -2, 0]$, $[0, 0, -11]$. Yes.
33. $|\mathbf{p} + \mathbf{q} + \mathbf{u}| \leqq 6$. Nothing
35. $\left[\frac{25}{\sqrt{2}}, \frac{25}{\sqrt{2}}\right] - \left[-\frac{20}{\sqrt{2}}, \frac{20}{\sqrt{2}}\right] = \left[\frac{45}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right]$
37. $|\mathbf{w}|/(2 \sin \alpha)$

Problem Set 9.2, page 376

1. 4
3. $\sqrt{241}$
5. $[12, -8, 4]$, $[-18, -9, -36]$
7. 17
9. -4, 4
11. -24
15. Use (1) and $|\cos \gamma| \leqq 1$.
17. $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = \mathbf{a}\bullet\mathbf{a} + 2\mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{b} + (\mathbf{a}\bullet\mathbf{a} - 2\mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet\mathbf{b}) = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$
19. 0
21. 15
23. Orthogonality. Yes

25. 2, 2, 0, −2
27. 79.11°
29. 82.45°
31. 54.74°
33. 54.79°, 79.11°, 46.10°
35. 63.43°, 116.57°
37. 3
39. 1.4
41. If $|\mathbf{a}| = |\mathbf{b}|$ or if **a** and **b** are orthogonal

Problem Set 9.3, page 383

1. [0, 0, −10], [0, 0, 10]
3. [−4, −8, 26]
5. [0, 0, −60]
7. −20, −20
9. 240
11. [19, −21, 24], $\sqrt{1378}$
13. [10, −5, −1]
15. 2
17. 30, −30
19. −20, −20
25. $[-2, 2, 0] \times [4, 4, 0] = -16\mathbf{k}$, 16
27. $[1, -1, 2] \times [1, 2, 3] = [-7, -1, 3]$, $\sqrt{59}$
29. $[0, 10, 0] \times [4, 3, 0] = [0, 0, -40]$, speed 40
31. $|[7, 0, 0] \times [1, 1, 0]| = 7$
33. $\frac{1}{2}\sqrt{3}$
35. [18, 14, 26]; $9x + 7y + 13z = c$, $9 \cdot 4 + 7 \cdot 8 + 13 \cdot 0 = 92 = c$
37. 16
39. $c = 2.5$

Problem Set 9.4, page 389

1. Hyperbolas
3. Hyperbolas
5. Circles
7. Ellipses; 288, 100, 409; elliptic ring between the ellipses

$$\frac{x^2}{(2/3)^2} + \frac{y^2}{(1/2)^2} = 1 \quad \text{and} \quad \frac{x^2}{(4/3)^2} + y^2 = 1.$$

9. Ellipsoids
11. Cones
13. Planes
23. $[8x, 0, yz]$, $[0, 0, xz]$, $[0, 18z, xy]$; $[0, z, y]$, $[z, 0, x]$, $[y, x, 0]$

Problem Set 9.5, page 398

1. $[4 + 3\cos t,\ \ 6 + 3\sin t]$
3. $[2 - t,\ \ 0,\ \ 4 + t]$
5. $[3,\ \ -2 + 3\cos t,\ \ 3\sin t]$
7. $[a + 3t,\ \ b - 2t,\ \ c + 5t]$
9. $[\sqrt{2}\cos t,\ \ \sin t,\ \ \sin t]$
11. Helix on $(x - 2)^2 + (y - 6)^2 = r^2$
13. Circle $(x - 2)^2 + (y + 2)^2 = 1$, $z = 5$
15. $x^4 + y^4 = 1$
17. Hyperbola $xy = 1$
23. $\mathbf{r}' = [-5\sin t,\ \ 5\cos t,\ \ 0]$, $\mathbf{u} = [-\sin t,\ \ \cos t,\ \ 0]$, $\mathbf{q} = [4 - 3w,\ \ 3 + 4w,\ \ 0]$
25. $\mathbf{r}' = [\sinh t,\ \ \cosh t]$, $\mathbf{u} = (\cosh 2t)^{-1/2}\,[\sinh t,\ \ \cosh t]$, $\mathbf{q} = [\frac{5}{3} + 4w,\ \ \frac{4}{3} + 5w]$
27. $\sqrt{\mathbf{r}' \bullet \mathbf{r}'} = \cosh t$, $\ell = \sinh 1 = 1.175$
29. Start from $\mathbf{r}(t) = [t,\ \ f(t)]$.
33. $\mathbf{v} = \mathbf{r}' = [1,\ \ 2t,\ \ 0]$, $|\mathbf{v}| = \sqrt{1 + 4t^2}$, $\mathbf{a} = [0,\ \ 2,\ \ 0]$
35. $\mathbf{v}(0) = 2\omega R\mathbf{i}$, $\mathbf{a}(0) = -\omega^2 R\mathbf{j}$
37. 1 year $= 365 \cdot 86400$ sec, $R = 30 \cdot 365 \cdot 86400/2\pi = 151 \cdot 10^6$ [km]. $|\mathbf{a}| = \omega^2 R = |\mathbf{v}|^2/R = 5.98 \cdot 10^{-6}$ [km/sec^2]
39. $R = 3960 + 80$ mi $= 2.133 \cdot 10^7$ ft, $g = |\mathbf{a}| = \omega^2 R = |\mathbf{v}|^2/R$, $|\mathbf{v}| = \sqrt{gR} = \sqrt{6.61 \cdot 10^8} = 25700$ [ft/sec] $= 17500$ [mph]
43. $\mathbf{r}(t) = [t,\ \ y(t),\ \ 0]$, $\mathbf{r}' = [1,\ \ y',\ \ 0]$, $\mathbf{r}' \bullet \mathbf{r}' = 1 + y'^2$, $\mathbf{r}'' = [0,\ \ y'',\ \ 0]$, etc.
47. $3/(1 + 9t^2 + 9t^4)$

Problem Set 9.6, page 403

1. $w' = 2\sqrt{2}(\sinh 4t)/(\cosh 4t)^{1/2}$

3. $w' = (\cosh t)^{\sinh t - 1}((\cosh^2 t) \ln(\cosh t) + \sinh^2 t)$

5. $w' = 3(2t^4 + t^8)^2(8t^3 + 8t^7)$

7. $e^{4u} \sin^2 2v, \frac{1}{2}e^{4u} \sin 4v$

9. $-2(u^2 + v^2)^{-3}u, -2(u^2 + v^2)^{-3}v$

Problem Set 9.7, page 409

1. $[2x, \quad 2y]$
3. $[1/y, \quad -x/y^2]$
5. $[y + 2, \quad x - 2]$
7. $[6, \quad 4, \quad 4]$
9. $[-1.25, \quad 0]$
11. $[0, \quad -e]$
13. $[-4, \quad 2]$
15. $[-18, \quad 24]$
17. $[48, \quad -36]$
19. $[6, \quad 4]$
21. $[-6, \quad -12]$
23. $[-0.0015, \quad 0, \quad -0.0020]$
27. $[a, \quad b, \quad c]$
29. $[8, \quad 6, \quad 0]$
31. $[108, \quad 108, \quad 108]$
33. $\sqrt{2/3}$
35. 7/3
37. $2e^2/\sqrt{13}$
39. $\frac{3}{2}x^2 + \frac{5}{2}y^2 - 2z^2$
41. $x^4 + y^3 - 3z^2$

Problem Set 9.8, page 413

1. $3(x + y)^2$
3. $2(x + xz + z)$
5. $(y + x + 1) \cos xy$
7. $9x^2y^2z^2$

9. $[v_1, v_2, v_3] = \mathbf{r}' = [x', y', z'] = [y, 0, 0], z' = 0, z = c_3, y' = 0, y = c_2, x' = y = c_2, x = c_2t + c_1$. Hence as t increases from 0 to 1, this "shear flow" transforms the cube into a parallelepiped of volume 1.

11. div $(\mathbf{w} \times \mathbf{r}) = 0$ because v_1, v_2, v_3 do not depend on x, y, z, respectively.

13. (b) $(fv_1)_x + (fv_2)_y + (fv_3)_z = f[(v_1)_x + (v_2)_y + (v_3)_z] + f_xv_1 + f_yv_2 + f_zv_3$, etc. (c) Use (b) with $\mathbf{v} = \nabla g$.

15. $4(x + y)/(y - x)^3$
17. 0
19. $e^{xyz}(y^2z^2 + x^2z^2 + x^2y^2)$

Problem Set 9.9, page 416

1. $[0, \quad 0, \quad 4x - 1]$
3. $[0, \quad 0, \quad 2e^x \sin y]$
5. $[0, \quad 0, \quad -4y/(x^2 + y^2)]$

9. curl $\mathbf{v} = [-2z, \quad 0, \quad 0]$, incompressible, $\mathbf{v} = \mathbf{r}' = [x', \quad y', \quad z'] = [0, \quad z^2, \quad 0]$, $x = c_1, z = c_3, y' = z^2 = c_3{}^2, y = c_3{}^2t + c_2$

11. curl $\mathbf{v} = [0, \quad 0, \quad -2]$, incompressible, $x' = y, y' = -x, z' = 0, z = c_3$, $y\,dy + x\,dx = 0, x^2 + y^2 = c$

13. Irrotational, div $\mathbf{v} = 1$, compressible, $\mathbf{r} = [c_1e^t, \quad c_2e^{-t}, \quad c_3e^t]$

17. $\mathbf{0}, \mathbf{0}, [xy - zx, \quad yz - xy, \quad zx - yz]$

19. $\mathbf{0}, \mathbf{0}, 0, -2yz^2 - 2zx^2 - 2xy^2$

Chapter 9 Review Questions and Problems, page 416

11. $[-1, \quad 9, \quad 24]$
13. $\mathbf{0}, [-43, \quad 54, \quad 3], [43, \quad -54, \quad -3]$
15. $[0, \quad 0, \quad -740], [0, \quad 0, \quad -740]$
17. $[-24, \quad 3, \quad -398], [114, \quad 95, \quad -76]$
19. $-495, -495$
21. $90°, 95.4°$
23. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. Always
25. $[v_1, v_2, -3]$
27. 3.4
29. If $\gamma > \frac{1}{2}\pi, \frac{1}{2}\pi$

33. 45/6
35. No
37. **0,** $2y^2 + (z + x)^2$
39. $[-1, \ 1, \ -1], [-2z, \ -2x, \ -2y]$
41. $0, 2x^2 + 4y^2 + 2z^2 + 4xz$
43. $488/\sqrt{3323}$ **45. 0**

Problem Set 10.1, page 425

1. $\mathbf{F}(\mathbf{r}(t)) = [125t^6, \ t^3, \ 0]$, $16448/7 = 2350$ **3.** $0 + 160$
5. $\mathbf{F}(\mathbf{r}(t)) = [\cosh t \sinh^2 t, \ \cosh^2 t \sinh t]$, 93.09
7. $\mathbf{F}(\mathbf{r}(t)) = [t, \ \cos t, \ \sin t]$, 6π
9. $\mathbf{F}(\mathbf{r}(t)) = [\cosh \frac{1}{2}t, \ \sinh \frac{1}{4}t, \ e^{t/8}]$, 0.6857
11. $\mathbf{F}(\mathbf{r}(t)) = [e^t, \ e^{t^2}, \ e^{t^2}]$, $e^2 + 2e^4 - 3$
15. 17/3
17. $[36\pi, \ \frac{4}{3}(8\pi)^3, \ 36\pi]$

Problem Set 10.2, page 432

1. $\sin xy$, 1
3. $-\frac{1}{2}e^{-(x^2+y^2)}$, 0
5. $e^{xz} + y$, -2
7. $x^2y + \cosh z$, 392
11. $\sinh ac$
13. No
15. $ce^a - ae^b$
17. $\frac{1}{2}a^2bc^2$
19. No

Problem Set 10.3, page 438

3. $\int_0^1 [x - x^3 - (x^2 - x^5)]\,dx = \frac{1}{12}$
5. $\frac{1}{2}\cosh 6 - \cosh 3 + \frac{1}{2}$
7. $\int_0^4 \frac{1}{2}(e^{3x} - e^{-x})\,dx = \frac{1}{6}e^{12} + \frac{1}{2}e^{-4} - \frac{2}{3}$
9. $\int_0^{\pi/2} (e^{\sin y} \cos y - \cos y)\,dy = e - 2$
11. $\int_{-1}^1 (2x^2 + \frac{2}{3})\,dx = \frac{8}{3}$
13. $\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx = \frac{1}{3}$
15. $\bar{x} = \frac{2}{3}b$, $\bar{y} = \frac{1}{3}h$
17. $I_x = bh^3/12$, $I_y = b^3h/4$
19. $I_x = (a + b)h^3/24$, $I_y = h(a^4 - b^4)/(48(a - b))$

Problem Set 10.4, page 444

1. $2x^3y - 2xy^3$, $81 - 36 = 45$
3. $3x^2 + 3y^2$, $1875\pi/2 = 2945$
5. $e^{x-y} - e^{x+y}$, $-\frac{1}{3}e^3 + \frac{1}{2}e^2 + e^{-1} - \frac{1}{6}$
7. $2x - 2y$, $-56/15$
9. 0 (why?)
11. Integrand 4. *Ans.* 40π
13. y from 0 to $\frac{1}{2}x$, x from 0 to 2. *Ans.* $\cosh 2 - \frac{1}{2}\sinh 2$
15. y from 1 to $5 - x^2$. *Ans.* 56
19. $4e^4 - 4$

Problem Set 10.5, page 448

1. Straight lines, **k**
3. $x^2/a^2 + y^2/b^2 = 1$, ellipses, straight lines, $[-b \cos v, \ a \sin v, \ 0]$
5. $z = (c/a)\sqrt{x^2 + y^2}$, circles, straight lines, $[-acu \cos v, \ -acu \sin v, \ a^2u]$
7. $x^2/9 + y^2/16 = z$, ellipses, parabolas, $[-8u^2 \cos v, \ -6u^2 \sin v, \ 12u]$
9. $x^2/4 + y^2/9 + z^2/16 = 1$, ellipses, $[12 \cos^2 v \cos u, \ 8 \cos^2 v \sin u, \ 6 \cos v \sin v]$
13. $[10u, \ 10v, \ 1.6 - 4u + 2v]$, $[40, \ -20, \ 100]$
15. $[-2 + \cos v \cos u, \ \cos v \sin u, \ 2 + \sin v]$,
$[\cos^2 v \cos u, \ \cos^2 v \sin u, \ \cos v \sin v]$

17. $[u, \quad v, \quad 3v^2]$, $[0, \quad -6v, \quad 1]$
19. $[u \cos v, \quad 3u \sin v, \quad 3u]$, $[-9u \cos v, \quad -3u \sin v, \quad 3u]$
21. Because $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangent to the coordinate curves $v = const$ and $u = const$, respectively.
23. $[\tilde{u}, \quad \tilde{v}, \quad \tilde{u}^2 + \tilde{v}^2]$, $\tilde{\mathbf{N}} = [-2\tilde{u}, \quad -2\tilde{v}, \quad 1]$

Problem Set 10.6, page 456

1. -64
3. -18
5. -128π
7. 2π
9. $\frac{1}{3}a^3$
11. $17h/4$
15. $140\sqrt{6}/3$
17. $128\pi\sqrt{2}/3 = 189.6$
19. $\frac{1}{96}\pi^2(37^{3/2} - 5^{3/2}) = 22.00$
25. $2\pi h$
27. $\pi h^4/\sqrt{2}$
29. $\pi h + 2\pi h^3/3$

Problem Set 10.7, page 463

1. $8a^3b^3c^3/27$
3. 6
5. $42\frac{2}{3}\pi$
7. 234π
9. $2a^5/3$
11. $ha^4\pi/2$
13. $\pi h^5/10$
17. 108π
19. 216π
21. 0
23. 8
25. 384π

Problem Set 10.8, page 468

1. Integrals $4 \cdot 1 \cdot 1$ $(x = 1)$, $4 \cdot 1 \cdot 1$ $(y = 1)$, $-8 \cdot 1 \cdot 1$ $(z = 1)$, 0 $(x = y = z = 0)$
3. 2 (volume integral of $6y^2$), 2 (surface integral over $x = 1$). Others 0
5. Volume integral of $6y^2 - 6x^2$ is 0. 2 $(x = 1)$, -2 $(y = 1)$, others 0.
7. $\mathbf{F} = [x, \quad y, \quad z]$, div $\mathbf{F} = 3$, In (2), Sec. 10.7, $\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}||\mathbf{n}| \cos \phi$ $= \sqrt{x^2 + y^2 + z^2} \cos \phi = r \cos \phi$.
9. $\mathbf{F} = [x, \quad 0, \quad 0]$, div $\mathbf{F} = 1$, use (2*), Sec. 10.7, etc.

Problem Set 10.9, page 473

1. $[0, \quad 8z, \quad 16] \cdot [0, \quad -1, \quad 1]$, ± 12
3. $[-e^z, \quad -e^x, \quad e^y] \cdot [-1, \quad -1, \quad 1]$, $\pm(e^2 - 1)$
5. S: $[u, \quad v, \quad v^2]$, $(\text{curl } \mathbf{F}) \cdot \mathbf{N} = -4ve^{2v^2}$, $\pm(4 - 4e^2)$
7. $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = 3/2$, $\pm 3a^2/2$
9. The sides contribute a, $3a^2/2$, $-a$, 0.
11. curl $\mathbf{F} = [0, \quad 0, \quad 6]$, 24π
13. $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = 2x - 2y$, $1/3$
15. $-\pi/4$
17. $(\text{curl } \mathbf{F}) \cdot \mathbf{N} = \pi(\cos \pi x + \sin \pi y)$, 2
19. $\mathbf{F} \cdot \mathbf{r}' = [-\sin \theta, \quad \cos \theta] \cdot [-\sin \theta, \quad \cos \theta] = 1$, 2π, 0

Chapter 10 Review Questions and Problems, page 473

11. Exact, $-542/3$
13. Not exact, $e^4 - 7$
15. By Green, 1152π
17. By Stokes, $\pm 18\pi$
19. By Stokes, $\pm 12\pi$
21. $4/5$, $8/15$
23. 0, $4a/3\pi$
25. $8/7$, $118/49$
27. Direct, 5
29. By Gauss, 100π
31. By Gauss, $40abc$
33. Direct, πh
35. Direct, $5(e^2 - 1)$

Problem Set 11.1, page 485

3. $2\pi/n,\ 2\pi/n,\ k,\ k,\ k/n,\ k/n$

13. $\frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right)$

15. $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

17. $\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

$+ \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots$

19. $-\frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

$+ 2\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

21. $\frac{1}{3}\pi^2 - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - + \cdots\right)$

23. $\frac{1}{6}\pi^2 - \frac{4}{\pi}\cos x - \frac{1}{2}\cos 2x + \frac{4}{27\pi}\cos 3x + \frac{1}{8}\cos 4x - \cdots$

29. $f' = 2x,\ f'' = 2,\ j_1 = 0,\ j_1' = -4\pi,\ j_1'' = 0,\ a_n = \frac{1}{n\pi}\left(-\frac{1}{n}\right)(-4\pi)\cos n\pi$, etc.

Problem Set 11.2, page 490

1. $\frac{4}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \cdots\right)$

3. $\frac{1}{3} - \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right)$

5. Rectifier, $\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\pi x + \frac{1}{3\cdot 5}\cos 4\pi x + \frac{1}{5\cdot 7}\cos 6\pi x + \cdots\right)$

7. Rectifier, $\frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$

9. $\frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - \frac{1}{16}\cos 4\pi x + - \cdots\right)$

11. $\frac{3}{4} - \frac{4}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{2}\cos \pi x + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \frac{1}{18}\cos 3\pi x\right.$

$\left. + \cdots\right)$

13. $\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

15. Translate by $\frac{1}{2}$.

17. Set $x = 0$.

Problem Set 11.3, page 496

1. Even, odd, neither, even, neither, odd

3. Odd

5. Neither

7. Odd

9. Odd

11. $\dfrac{\pi}{2} + \dfrac{4}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right)$

13. $\dfrac{4}{\pi}\left(\sin x - \dfrac{1}{9}\sin 3x + \dfrac{1}{25}\sin 5x - + \cdots\right)$

15. $1 - \dfrac{4}{\pi}\left(\sin\dfrac{\pi x}{2} + \dfrac{1}{3}\sin\dfrac{3\pi x}{2} + \dfrac{1}{5}\sin\dfrac{5\pi x}{2} + \cdots\right)$

17. (a) 1, (b) $\dfrac{4}{\pi}\left(\sin\dfrac{\pi x}{2} + \dfrac{1}{3}\sin\dfrac{3\pi x}{2} + \dfrac{1}{5}\sin\dfrac{5\pi x}{2} + \cdots\right)$

19. (a) $1 + \dfrac{8}{\pi^2}\left(\cos\dfrac{\pi x}{2} + \dfrac{1}{9}\cos\dfrac{3\pi x}{2} + \dfrac{1}{25}\cos\dfrac{5\pi x}{2} + \cdots\right)$

(b) $\dfrac{4}{\pi}\left(\sin\dfrac{\pi x}{2} + \dfrac{1}{2}\sin\pi x + \dfrac{1}{3}\sin\dfrac{3\pi x}{2} + \dfrac{1}{4}\sin 2\pi x + \cdots\right)$

21. (a) $\dfrac{3}{2} - \dfrac{2}{\pi}\left(\cos\dfrac{\pi x}{2} - \dfrac{1}{3}\cos\dfrac{3\pi x}{2} + \dfrac{1}{5}\cos\dfrac{5\pi x}{2} - \dfrac{1}{7}\cos\dfrac{7\pi x}{2} + - \cdots\right)$

(b) $\dfrac{6}{\pi}\left(\sin\dfrac{\pi x}{2} - \dfrac{1}{3}\sin\pi x + \dfrac{1}{3}\sin\dfrac{3\pi x}{2} + \dfrac{1}{5}\sin\dfrac{5\pi x}{2} - \dfrac{1}{9}\sin 3\pi x + \cdots\right)$

23. (a) $\dfrac{L}{2} - \dfrac{4L}{\pi^2}\left(\cos\dfrac{\pi x}{L} + \dfrac{1}{9}\cos\dfrac{3\pi x}{L} + \dfrac{1}{25}\cos\dfrac{5\pi x}{L} + \cdots\right)$

(b) $\dfrac{2L}{\pi}\left(\sin\dfrac{\pi x}{L} - \dfrac{1}{2}\sin\dfrac{2\pi x}{L} + \dfrac{1}{3}\sin\dfrac{3\pi x}{L} - + \cdots\right)$

25. (a) $\dfrac{\pi}{2} + \dfrac{4}{\pi}\left(\cos x + \dfrac{1}{9}\cos 3x + \dfrac{1}{25}\cos 5x + \cdots\right)$

(b) $2\left(\sin x + \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x + \cdots\right)$

Problem Set 11.4, page 499

3. Use (5).

7. $-\dfrac{2i}{\pi}\sum_{n=-\infty}^{\infty}\dfrac{1}{2n+1}e^{(2n+1)ix}$

9. $i\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\dfrac{(-1)^n}{n}e^{inx}$

11. $\dfrac{\pi^2}{3} + 2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\dfrac{(-1)^n}{n^2}e^{inx}$

13. $\pi + i\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\dfrac{1}{n}e^{inx}$

Problem Set 11.5, page 501

3. $(0.05n)^2$ in D_n changes to $(0.02n)^2$, which gives $C_5 = 0.5100$, leaving the other coefficients almost unaffected.

5. $y = c_1 \cos \omega t + c_2 \sin \omega t + A(\omega) \cos t$, $A(\omega) = 1/(\omega^2 - 1) < 0$ if $\omega^2 < 1$ (phase shift!) and > 0 if $\omega^2 > 1$

7. $y = c_1 \cos \omega t + c_2 \sin \omega t + \sum_{n=1}^{N} \frac{a_n}{\omega^2 - n^2} \cos nt$

9. $y = c_1 \cos \omega t + c_2 \sin \omega t + \frac{\pi}{2\omega^2} + \frac{4}{\pi}\left(\frac{1}{\omega^2 - 1} \cos t + \frac{1/9}{\omega^2 - 9} \cos 3t + \cdots\right)$

11. $y = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{2\omega^2} - \frac{1}{1 \cdot 3(\omega^2 - 4)} \cos 2t - \frac{1}{3 \cdot 5(\omega^2 - 16)} \cos 4t - \cdots$

13. The situation is the same as in Fig. 53 in Sec. 2.8.

15. $y = -\frac{3c}{64 + 9c^2} \cos 3t - \frac{8}{64 + 9c^2} \sin 3t$

17. $y = \sum_{n=1}^{N} \left(-\frac{ncb_n}{D_n} \cos nt + \frac{(1 - n^2)b_n}{D_n} \sin nt\right)$, $D_n = (1 - n^2)^2 + n^2c^2$

19. $I(t) = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt)$, $A_n = (-1)^{n+1} \frac{240(10 - n^2)}{n^2 D_n}$, $B_n = (-1)^{n+1} \frac{2400}{nD_n}$, $D_n = (10 - n^2)^2 + 100n^2$

Problem Set 11.6, page 505

1. $F = 2\left(\sin x - \frac{1}{2} \sin 2x + \cdots + \frac{(-1)^{N+1}}{N} \sin Nx\right)$, $E^* = 8.1, 5.0, 3.6, 2.8, 2.3$

3. $F = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \cdots\right)$, $E^* = 0.0748, 0.0748, 0.0119, 0.0119, 0.0037$

5. $F = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1 \cdot 3} \cos 2x + \frac{1}{3 \cdot 5} \cos 4x + \frac{1}{5 \cdot 7} \cos 6x + \cdots\right)$, $E^* = 0.5951, 0.0292, 0.0292, 0.0066, 0.0066$

7. $F = \frac{4}{\pi}\left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots\right)$, $E^* = 1.1902, 1.1902, 0.6243, 0.6243, 0.4206$ $(0.1272$ when $N = 20)$

9. $\frac{8}{\pi}\left(\sin x + \frac{1}{27} \sin 3x + \frac{1}{125} \sin 5x + \cdots\right)$, $E^* = 0.0295, 0.0295, 0.0015, 0.0015, 0.00023$

Problem Set 11.7, page 512

1. $f(x) = \pi e^{-x}$ $(x > 0)$ gives $A = \int_0^\infty e^{-v} \cos wv \, dv = \dfrac{1}{1 + w^2}$, $B = \dfrac{w}{1 + w^2}$ (see Example 3), etc.

3. $f(x) = \frac{1}{2}\pi e^{-x}$ gives $A = 1/(1 + w^2)$.

5. Use $f = (\pi/2) \cos v$ and (11) in App. 3.1 to get $A = (\cos(\pi w/2))/(1 - w^2)$.

7. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin aw \cos xw}{w} \, dw$

9. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos w + w \sin w - 1}{w^2} \cos xw \, dw$

11. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos \pi w + 1}{1 - w^2} \cos xw \, dw$

15. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin \pi w}{1 - w^2} \sin xw \, dw$

17. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\pi w - \sin \pi w}{w^2} \sin xw \, dw$

19. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{wa - \sin wa}{w^2} \sin xw \, dw$

Problem Set 11.8, page 517

1. $\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin 2w - 2 \sin w}{w}\right)$

5. $\sqrt{\pi/2}\, e^{-x}$ $(x > 0)$

7. $\sqrt{\pi/2} \cos w$ if $0 < w < \pi/2$ and 0 if $w > \pi/2$

9. Yes, no

11. $\sqrt{(2/\pi)}\, w/(w^2 + \pi^2)$

13. $\mathcal{F}_s(xe^{-x^2/2}) = \mathcal{F}_s(-(e^{-x^2/2})') = w\mathcal{F}_c(e^{-x^2/2}) = we^{-w^2/2}$

17. $\mathcal{F}_c(f') = \mathcal{F}_c(-af) = -a\mathcal{F}_c(f) = -\sqrt{\dfrac{2}{\pi}}\dfrac{a^2}{a^2 + w^2} = w\mathcal{F}_s(f) - \sqrt{\dfrac{2}{\pi}} \cdot 1$,
$\mathcal{F}_s(f) = \sqrt{\dfrac{2}{\pi}}\dfrac{w}{a^2 + w^2}$

19. In (5) for $f(ax)$ set $ax = v$.

Problem Set 11.9, page 528

3. $ik(e^{-ibw} - 1)/(\sqrt{2\pi} w)$

5. $\sqrt{(2/\pi)}\, k (\sin w)/w$

7. $[(1 + iw)e^{-iw} - 1]/(\sqrt{2\pi} w^2)$

9. $\sqrt{(2/\pi)}\, i(\cos w - 1)/w$

11. $\frac{1}{2}e^{-w^2/2}$

13. $(e^{ibw} - e^{-ibw})/(iw\sqrt{2\pi}) = \sqrt{2/\pi}(\sin bw)/w$

Chapter 11 Review Questions and Problems, page 532

11. $\dfrac{4k}{\pi}\left(\sin \pi x + \dfrac{1}{3}\sin 3\pi x + \dfrac{1}{5}\sin 5\pi x + \cdots\right)$

13. $4\left(\sin\dfrac{x}{2} - \dfrac{1}{2}\sin x + \dfrac{1}{3}\sin\dfrac{3x}{2} - \dfrac{1}{4}\sin 4x + \dfrac{1}{5}\sin\dfrac{5x}{2} - + \cdots\right)$

15. $\dfrac{8}{\pi^2}\left(\sin\dfrac{\pi x}{2} - \dfrac{1}{9}\sin\dfrac{3\pi x}{2} + \dfrac{1}{25}\sin\dfrac{5\pi x}{2} - + \cdots\right)$

17. $\dfrac{2}{\pi} - \dfrac{4}{\pi}\left(\dfrac{1}{1 \cdot 3}\cos 16\pi x + \dfrac{1}{3 \cdot 5}\cos 32\pi x + \dfrac{1}{5 \cdot 7}\cos 48\pi x + \cdots\right)$

19. $\frac{\pi^2}{12} - \cos 2x + \frac{1}{4}\cos 4x - \frac{1}{9}\cos 6x + \frac{1}{16}\cos 8x - + \cdots$

21. $\pi/4$ by Prob. 11 **23.** $\pi^2/8$ by Prob. 15

25. $\frac{1}{2}[f(x) + f(-x)],\ \frac{1}{2}[f(x) - f(-x)]$

27. $\pi - \frac{8}{\pi}\left(\cos\frac{x}{2} + \frac{1}{9}\cos\frac{3x}{2} + \frac{1}{25}\cos\frac{5x}{2} + \cdots\right)$

29. 8.105, 4.963, 3.567, 2.781, 2.279, 1.929, 1.673, 1.477

31. $y = C_1 \cos \omega t + C_2 \sin \omega t + \frac{\pi^2}{3\omega^2} - 4\left(\frac{\cos t}{\omega^2 - 1} - \frac{1}{4}\cdot\frac{\cos 2t}{\omega^2 - 4} + \frac{1}{9}\cdot\frac{\cos 3t}{\omega^2 - 9} - \frac{1}{16}\cdot\frac{\cos 4t}{\omega^2 - 16} + - \cdots\right)$

33. $\frac{1}{\pi}\int_0^\infty \frac{(\cos w + w \sin w - 1)\cos wx + (\sin w - w\cos w)\sin wx}{w^2}\,dw$

35. $\frac{2}{\pi}\int_0^\infty \frac{w - \sin w \cos w}{w^2}\sin wx\,dw$

37. $\frac{4}{\pi}\int_0^\infty \frac{\sin 2w - 2w\cos 2w}{w^3}\cos wx\,dw$ **39.** $\sqrt{\frac{8}{\pi}}\cdot\frac{1}{w^2 + 4}$

Problem Set 12.1, page 537

1. $u = c_1(x)\cos 4y + c_2(x)\sin 4y$

3. $u = c_1(x) + c_2(x)y$

5. $u = c(x)e^{-y} + e^{xy}/(x + 1)$

7. $u = c(x)\exp\left(\frac{1}{2}y^2\cosh x\right)$

9. $u = c_1(x)y + c_2(x)y^{-2}$

11. $u = c(x)e^y + h(y)$

15. $c = 1/4$

17. Any c

19. $\pi/4$

21. Any c and ω

27. $u = 110 - (110/\ln 100)\ln(x^2 + y^2)$

29. $u = c_1 x + c_2(y)$

Problem Set 12.3, page 546

1. $k \cos 2\pi t \sin 2\pi x$

3. $\frac{8k}{\pi^3}\left(\cos \pi t \sin \pi x + \frac{1}{27}\cos 3\pi t \sin 3\pi x + \frac{1}{125}\cos 5\pi t \sin 5\pi x + \cdots\right)$

5. $\frac{4}{5\pi^2}\left(\cos \pi t \sin \pi x - \frac{1}{9}\cos 3\pi t \sin 3\pi x + \frac{1}{25}\cos 5\pi t \sin 5\pi x - + \cdots\right)$

7. $\frac{2}{\pi^2}\left((\sqrt{2} - 1)\cos \pi t \sin \pi x + \frac{1}{2}\cos 2\pi t \sin 2\pi x + \frac{1}{9}(\sqrt{2} + 1)\cos 3\pi t \sin 3\pi x - \cdots\right)$

9. $\frac{2}{\pi^2}\left((2-\sqrt{2})\cos \pi t \sin \pi x - \frac{1}{9}(2+\sqrt{2})\cos 3\pi t \sin 3\pi x + \right.$
$\left.\frac{1}{25}(2+\sqrt{2})\cos 5\pi t \sin 5\pi x + \cdots\right)$

17. $u = \frac{8L^2}{\pi^3}\left(\cos\left[c\left(\frac{\pi}{L}\right)^2 t\right]\sin\frac{\pi x}{L} + \frac{1}{3^3}\cos\left[c\left(\frac{3\pi}{L}\right)^2 t\right]\sin\frac{3\pi x}{L} + \cdots\right)$

19. (a) $u(0, t) = 0$, (b) $u(L, t) = 0$, (c) $u_x(0, t) = 0$, (d) $u_x(L, t) = 0$. $C = -A$, $D = -B$ from (a), (b). Insert this. The coefficient determinant resulting from (c), (d) must be zero to have a nontrivial solution. This gives (22).

Problem Set 12.4, page 552

3. $c^2 = 300/[0.9/(2 \cdot 9.80)] = 80.83^2$ [m²/sec²]
11. Hyperbolic, $u = f_1(x) + f_2(x + y)$
13. Elliptic, $u = f_1(y + 3ix) + f_2(y - 3ix)$
15. Parabolic, $u = xf_1(x - y) + f_2(x - y)$
17. Parabolic, $u = xf_1(2x + y) + f_2(2x + y)$
19. Hyperbolic, $u = (1/y)f_1(xy) + f_2(y)$

Problem Set 12.5, page 560

5. $u = \sin 0.4\pi x \, e^{-1.752 \cdot 16\pi^2 t/100}$

7. $u = \frac{2}{\pi}\left(\frac{2}{\pi}\sin 0.1\pi x \, e^{-0.01752\pi^2 t} + \frac{1}{2}\sin 0.2\pi x \, e^{-0.01752(2\pi)^2 t} - \cdots\right)$

9. $u = \frac{20\sqrt{2}}{\pi^2}\left(\sin 0.1\pi x \, e^{-0.01752\pi^2 t} + \frac{1}{9}\sin 0.3\pi x \, e^{-0.01752(3\pi)^2 t} - - \cdots\right)$

11. $u = u_{\text{I}} + u_{\text{II}}$, where $u_{\text{II}} = u - u_{\text{I}}$ satisfies the boundary conditions of the text, so that $u_{\text{II}} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-(cn\pi/L)^2 t}$, $B_n = \frac{2}{L}\int_0^L [f(x) - u_{\text{I}}(x)] \sin\frac{n\pi x}{L}\,dx$

13. $F = A\cos px + B\sin px$, $F'(0) = Bp = 0$, $B = 0$, $F'(L) = -Ap\sin pL = 0$, $p = n\pi/L$, etc.

15. $u = 1$

17. $u = \frac{2\pi^2}{3} + 4\left(\cos x \, e^{-t} - \frac{1}{4}\cos 2x \, e^{-4t} + \frac{1}{9}\cos 3x \, e^{-9t} - + \cdots\right)$

19. $u = \frac{\pi^2}{12} + \cos 2x \, e^{-4t} + \frac{1}{4}\cos 4x \, e^{-16t} + \frac{1}{9}\cos 6x \, e^{-36t} + \cdots$

23. $-\frac{K\pi}{L}\sum_{n=1}^{\infty} nB_n e^{-\lambda_n^2 t}$

25. $w = e^{-\beta t}$

27. $v_t - c^2 v_{xx} = 0$, $w'' = -Ne^{-\alpha x}/c^2$, $w = \frac{N}{c^2\alpha^2}\left[-e^{-\alpha x} - \frac{1}{L}(1 - e^{-\alpha L})x + 1\right]$,

so that $w(0) = w(L) = 0$.

29. $u = (\sin \frac{1}{2}\pi x \sinh \frac{1}{2}\pi y)/\sinh \pi$

31. $u = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh(2n-1)\pi}\sin\frac{(2n-1)\pi x}{24}\sinh\frac{(2n-1)\pi y}{24}$

33. $u = A_0 x + \sum_{n=1}^{\infty} A_n \dfrac{\sinh (n\pi x/24)}{\sinh n\pi} \cos \dfrac{n\pi y}{24}$,

$$A_0 = \frac{1}{24^2} \int_0^{24} f(y)\, dy, \quad A_n = \frac{1}{12} \int_0^{24} f(y) \cos \frac{n\pi y}{24}\, dy$$

35. $\sum_{n=1}^{\infty} A_n \sin \dfrac{n\pi x}{a} \sinh \dfrac{n\pi(b-y)}{a}$, $A_n = \dfrac{2}{a \sinh (n\pi b/a)} \int_0^a f(x) \sin \dfrac{n\pi x}{a}\, dx$

Problem Set 12.6, page 568

1. $A = \dfrac{2 \sin ap}{\pi p}$, $B = 0$, $u = \dfrac{2}{\pi} \int_0^{\infty} \dfrac{\sin ap}{p} \cos px\, e^{-c^2p^2t}\, dp$

3. $A = e^{-p}$, $B = 0$, $u = \int_0^{\infty} \cos px\, e^{-p-c^2p^2t}\, dp$

5. Set $\pi v = s$. $A = 1$ if $0 < p/\pi < 1$, $B = 0$, $u = \int_0^{\pi} \cos px\, e^{-c^2p^2t}\, dp$

7. $A = 2[\cos p + p \sin p - 1)/(\pi p^2)]$, $B = 0$, $u = \int_0^{\infty} A \cos px\, e^{-c^2p^2t}\, dp$

Problem Set 12.8, page 578

1. (a), (b) It is multiplied by $\sqrt{2}$. (c) Half

3. $B_{mn} = 16/(mn\pi^2)$ if m, n odd, 0 otherwise

5. $B_{mn} = (-1)^{n+1}8/(mn\pi^2)$ if m odd, 0 if m even

7. $B_{mn} = (-1)^{m+n}4/(mn\pi^2)$

11. $k \cos \sqrt{29}\, t \sin 2x \sin 5y$

13. $\dfrac{6.4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \dfrac{1}{m^3n^3} \cos (t\sqrt{m^2+n^2}) \sin mx \sin ny$ (m, n odd)

17. $c\pi\sqrt{260}$ (corresponding eigenfunctions $F_{4,16}$ and $F_{16,14}$), etc.

19. $B_{mn} = 0$ (m or n even), $B_{mn} = 16k/(mn\pi^2)$ (m, n odd)

21. $B_{mn} = (-1)^{m+n}144a^3b^3/(m^3n^3\pi^6)$

23. $\cos \left(\pi t \sqrt{\dfrac{9}{a^2} + \dfrac{16}{b^2}}\right) \sin \dfrac{3\pi x}{a} \sin \dfrac{4\pi y}{b}$

Problem Set 12.9, page 585

7. $30r \cos \theta + 10r^3 \cos 3\theta$

9. $55 + \dfrac{220}{\pi}\left(r \cos \theta - \dfrac{1}{3} r^3 \cos 3\theta + \dfrac{1}{5} r^5 \cos 5\theta - + \cdots\right)$

11. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(r \cos \theta + \dfrac{1}{9} r^3 \cos 3\theta + \dfrac{1}{25} r^5 \cos 5\theta + \cdots\right)$

15. Solve the problem in the disk $r < a$ subject to u_0 (given) on the upper semicircle and $-u_0$ on the lower semicircle.

$$u = \frac{4u_0}{\pi}\left(\frac{r}{a} \sin \theta + \frac{1}{3a^3} r^3 \sin 3\theta + \frac{1}{5a^5} r^5 \sin 5\theta + \cdots\right)$$

17. Increase by a factor $\sqrt{2}$

19. $T = 6.826\rho R^2 f_1^{\,2}$

21. No

23. Differentiation brings in a factor $1/\lambda_m = R/(c\alpha_m)$.

Problem Set 12.10, page 593

11. $v = F(r)G(t)$, $F'' + k^2F = 0$, $\dot{G} + c^2k^2G = 0$, $F_n = \sin(n\pi r/R)$,

$G_n = B_n \exp(-c^2n^2\pi^2t/R^2)$, $B_n = \dfrac{2}{R}\displaystyle\int_0^R rf(r)\sin\frac{n\pi r}{R}\,dr$

13. $u = 100$

15. $u = \frac{8}{5}r^3P_3(\cos\phi) - \frac{3}{5}rP_1(\cos\phi)$

17. $64r^4P_4(\cos\phi)$

21. Analog of Example 1 in the text with 55 replaced by 50

23. $v = r(\cos\theta)/r^2 = x/(x^2 + y^2)$, $v = xy/(x^2 + y^2)^2$

Problem Set 12.11, page 596

5. $W = \dfrac{c(s)}{x^s} + \dfrac{x}{s^2(s+1)}$, $W(0, s) = 0$, $c(s) = 0$, $w(x, t) = x(t - 1 + e^{-t})$

7. $w = f(x)g(t)$, $xf'g + f\dot{g} = xt$, take $f(x) = x$ to get $g = ce^{-t} + t - 1$ and $c = 1$ from $w(x, 0) = x(c - 1) = 0$.

9. Set $x^2/(4c^2\tau) = z^2$. Use z as a new variable of integration. Use $\text{erf}(\infty) = 1$.

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19. $u = c_1(y)e^x + c_2(y)e^{-2x}$

21. $u = g(x)(1 - e^{-y}) + f(x)$

23. $u = \cos t \sin x - \frac{1}{2}\cos 2t \sin 2x$

25. $u = \frac{3}{4}\cos t \sin x - \frac{1}{4}\cos 3t \sin 3x$

27. $u = \sin(0.02\pi x)\, e^{-0.004572t}$

29. $u = \dfrac{200}{\pi^2}\left(\sin\dfrac{\pi x}{50}\, e^{-0.004572t} - \dfrac{1}{9}\sin\dfrac{3\pi x}{50}\, e^{-0.04115t} + \cdots\right)$

31. $u = 100\cos 4x\, e^{-16t}$

33. $u = \dfrac{\pi}{2} - \dfrac{16}{\pi}\left(\dfrac{1}{4}\cos 2x\, e^{-4t} + \dfrac{1}{36}\cos 6x\, e^{-36t} + \dfrac{1}{100}\cos 10x\, e^{-100t} + \cdots\right)$

37. $u = f_1(y) + f_2(x + y)$ **39.** $u = f_1(y - 2ix) + f_2(y + 2ix)$

41. $u = xf_1(y - x) + f_2(y - x)$

49. $u = (u_1 - u_0)(\ln r)/\ln(r_1/r_0) + (u_0 \ln r_1 - u_1 \ln r_0)/\ln(r_1/r_0)$

Problem Set 13.1, page 606

5. $x - iy = -(x + iy)$, $x = 0$

7. 484

9. $-5/169$

11. $-7/13 - (22/13)i$

13. $-273 + 136i$

15. $-7/17 - (11/17)i$

17. $x/(x^2 + y^2)$

19. $(x^2 - y^2)/(x^2 + y^2)^2$

Problem Set 13.2, page 611

1. $3\sqrt{2}(\cos(-\frac{1}{4}\pi) + i\sin(-\frac{1}{4}\pi))$

3. $5(\cos\pi + i\sin\pi) = 5\cos\pi$

5. $\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi$

7. $\frac{1}{3}\sqrt{61}\,(\cos \arctan \frac{6}{5} + i \sin \arctan \frac{6}{5})$ **9.** $-3\pi/4$
11. $\arctan(\pm 3/4)$ **13.** $\pm\pi/4$ **15.** $3\pi/4$
17. $2.94020 + 0.59601i$ **19.** $0.54030 - 0.84147i$
21. $\cos(-\frac{1}{4}\pi) + i \sin(-\frac{1}{4}\pi)$, $\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi$
23. $\pm(1 \pm i)/\sqrt{2}$ **25.** -1, $\cos \frac{1}{5}\pi \pm i \sin \frac{1}{5}\pi$, $\cos \frac{3}{5}\pi \pm i \sin \frac{3}{5}\pi$
27. $4 + 3i$, $4 - 8i$ **29.** $\frac{5}{2} - i$, $2 + \frac{1}{4}i$
35. $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$. Multiply out and use Re $z_1\bar{z}_2 \leqq |z_1\bar{z}_2|$ (Prob. 32):
$z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 = |z_1|^2 + 2 \text{ Re } z_1\bar{z}_2 + |z_2|^2 \leqq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$.
Take the square root to get (6).

Problem Set 13.3, page 617

1. Circle of radius $\frac{4}{3}$, center $3 + 2i$
3. Set obtained from an open disk of radius 1 by omitting its center $z = 1$
5. Hyperbola $xy = 1$ **7.** y-axis
9. The region above $y = x$
13. $f = 1 - 1/(z + 1) = 1 - (x + 1 - iy)/[(x + 1)^2 + y^2]$; $0.9 - 0.1i$
15. $(x^2 - y^2 - 2ixy)/(x^2 + y^2)^2$, $-i/2$ **17.** Yes since $r^2(\sin 2\theta)/r \to 0$
19. Yes **21.** $6z^2(z^3 + i)$
23. $2i(1 - z)^{-3}$

Problem Set 13.4, page 623

1. Yes **3.** No **5.** Yes
7. No **9.** Yes for $z \neq 0$
11. $r_x = x/r = \cos\theta$, $r_y = \sin\theta$, $\theta_x = -(\sin\theta)/r$, $\theta_y = (\cos\theta)/r$,
(a) $0 = u_x - v_y = u_r \cos\theta + u_\theta(-\sin\theta)/r - v_r \sin\theta - v_\theta(\cos\theta)/r$.
(b) $0 = u_y + v_x = u_r \sin\theta + u_\theta(\cos\theta)/r + v_r \cos\theta + v_\theta(-\sin\theta)/r$.
Multiply (a) by $\cos\theta$, (b) by $\sin\theta$, and add. Etc.
13. $z^2/2$ **15.** $\ln|z| + i \text{ Arg } z$ **17.** z^3
19. No **21.** No **23.** $c = 1$, $\cos x \sinh y$
27. Use (4), (5), and (1).

Problem Set 13.5, page 626

3. $-1.13120 + 2.47173i$, $e = 2.71828$ **5.** $-i$, 1
7. $e^{0.8}(\cos 5 - i \sin 5)$, 2.22554 **9.** $e^{-2x}\cos 2y$, $-e^{-2x}\sin 2y$
11. $\exp(x^2 - y^2)\cos 2xy$, $\exp(x^2 - y^2)\sin 2xy$
13. $e^{i\pi/4}$, $e^{5\pi i/4}$
15. $\sqrt[n]{r}\,\exp[i(\theta + 2k\pi)/n]$, $k = 0, \cdots, n - 1$
17. $9e^{\pi i}$ **19.** $z = \ln 2 + \pi i + 2n\pi i$ $(n = 0, \pm 1, \cdots)$
21. $z = \ln 5 - \arctan \frac{3}{4} i \pm 2n\pi i$ $(n = 0, 1, \cdots)$

Problem Set 13.6, page 629

3. Use (11), then (5) for e^{iy}, and simplify. **5.** Use (11) and simplify.
7. $\cos 1 \cosh 1 - i \sin 1 \sinh 1 = 0.83373 - 0.98890i$

9. 74.203, 74.210
11. $-3.7245 - 0.51182i$
13. -1
15. $\cosh 4 = 27.308$
17. $z = \pm(2n + 1)\pi i/2$
19. $z = \frac{1}{2}(2n + 1)\pi - (-1)^n 1.4436i$
21. $z = \pm n\pi i$
25. Insert the definitions on the left, multiply out, simplify.

Problem Set 13.7, page 633

1. $\ln 10 + \pi i$
3. $\frac{1}{2} \ln 8 - \frac{1}{4}\pi i$
5. $\ln 5 + (\arctan \frac{4}{3} - \pi)i = 1.609 - 2.214i$
7. $0.9273i$
9. $\frac{1}{2} \ln 2 - \frac{1}{4}\pi i$
11. $\pm(2n + 1)\pi i, \quad n = 0, 1, \cdots$
13. $\ln 6 \pm (2n + 1)\pi i, \quad n = 0, 1, \cdots$
15. $(\pi - 1 \pm 2n\pi)i, \quad n = 0, 1, \cdots$
17. $\ln (i^2) = (\pm 2n + 1)\pi i$, $2 \ln i = \pm(4n + 1)\pi i, \quad n = 0, 1, \cdots$
19. $e^{0.3}(\cos 0.7 + i \sin 0.7) = 1.032 + 0.870i$
21. $e^2(1 + i)/\sqrt{2}$
23. $64(\cos (\ln 4) + i \sin (\ln 4))$
25. $2.8079 + 1.3179i$
27. $(1 + i)/\sqrt{2}$

Chapter 13 Review Questions and Problems, page 634

17. $-32 - 24i$
19. $-\frac{1}{2} - \frac{1}{2}i$
21. $5 - 3i$
23. $6\sqrt{2}e^{3\pi i/4}$
25. $12e^{-\pi i/2}$
27. $\pm(2 + 2i)$
29. $(\pm 1 \pm i)/\sqrt{2}$
31. $f(z) = 1/z$
33. $f(z) = (1 + i)z^2$
35. $f(z) = e^{z^2}$
37. $(-x^2 + y^2)/2$
39. No
41. 0
43. $0.6435i$
45. -1.5431

Problem Set 14.1, page 645

1. Straight segment from $1 + 3i$ to $4 + 12i$
3. Circle of radius 3, center $4 + i$
5. Semicircle, radius 1, center 0
7. Ellipse, half-axes 6 and 5
9. Parabola $y = \frac{1}{2}x^3$ from $-1 - \frac{1}{2}i$ to $2 + 4i$
11. e^{-it} $(0 \leqq t \leqq 2\pi)$
13. $t + i/t$ $(1 \leqq t \leqq 4)$
15. $t + (4 - 4t^2)i$ $(-1 \leqq t \leqq 1)$
17. $-a - ib + re^{-it}$ $(0 \leqq t \leqq 2\pi)$
19. $\frac{1}{2} + \frac{1}{2}i$
21. 0
23. $\pi i + \frac{1}{2}i \sinh 2\pi$
25. $i/2$
27. $-1 + i \tanh \frac{1}{4}\pi = -1 + 0.6558i$
29. $2 \sinh \frac{1}{2}$

Problem Set 14.2, page 653

1. πi, no
3. 0, yes
5. 0, yes
7. 0, yes
9. 0, no
11. 0, yes
15. Yes, by the deformation principle
19. πi by path deformation
21. πi
23. $2\pi i$
25. 0
27. (a) 0, (b) π
29. 0

Problem Set 14.3, page 657

1. -4π **3.** 4π **5.** $8\pi i$
7. 0 **9.** $-\pi i$ **11.** πi
13. π **15.** $2\pi i \text{ Ln } 4 = 8.710i$
17. $\pi i \cosh^2(1 + i) = \pi(-0.2828 + 1.6489i)$

Problem Set 14.4, page 661

1. $27\pi i/4$ **3.** $-2\pi i e^{\pi/2}$ **5.** $\pi i a^3/3$
7. $2\pi i$ if $|a| < 2$, 0 if $|a| > 2$ **9.** $\pi i(\cos \frac{1}{2} - \sin \frac{1}{2})$
11. $2\pi^2 i$ **13.** $\pi i e^{a/2}/24$ if $|a - 2 - i| < 3$, 0 if $|a - 2 - i| > 3$

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17. $-6\pi i$ **19.** 0 **21.** $-\pi i$
23. $\frac{1}{4}i \sin 8$ **25.** 0 **27.** π **29.** 0

Problem Set 15.1, page 672

1. Bounded, divergent, ± 1 **3.** Bounded, convergent, 0
5. Unbounded
7. Bounded, divergent, $\pm 1/\sqrt{2} \pm i$, 0, 1, -2
9. Convergent, 0
13. $|z_n - l| < \frac{1}{2}\epsilon$, $|z_n^* - l^*| < \frac{1}{2}\epsilon$ $(n > N(\epsilon))$, hence $|z_n + z_n^* - (l + l^*)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$
17. Convergent **19.** Divergent
21. Conditionally convergent **23.** Divergent by Theorem 3
27. $n = |100 + 75i| = 125$ (why?); $|100 + 75i|^{125}/125! = 125^{125}/[\sqrt{250\pi}\,(125/e)^{125}]$
$= e^{125}/\sqrt{250\pi} = 6.91 \cdot 10^{52}$

Problem Set 15.2, page 677

1. $\Sigma\, a_n z^{2n} = \Sigma\, a_n(z^2)^n$, $|z^2| < R = \lim |a_n/a_{n+1}|$, hence $|z| < \sqrt{R}$.
3. $-i$, 1 **5.** -1, e by (6) and $(1 + 1/n)^n \to e$.
7. 0, $|b/a|$ **9.** 0, 1 **11.** 0, 1
13. $3 - 2i$, 1 **15.** i, $1/\sqrt[4]{2}$ **17.** 0, $\sqrt{2}$

Problem Set 15.3, page 682

1. 3 **3.** $\sqrt{2}$ **5.** $\sqrt{5/3}$
7. $1/\sqrt{7}$ **9.** 1

Problem Set 15.4, page 690

1. $1 - 2z + 2z^2 - \frac{4}{3}z^3 + \frac{2}{3}z^4 - + \cdots, \quad R = \infty$
3. $e^{-2i}(1 + (z + 2i) + \frac{1}{2}(z + 2i)^2 + \frac{1}{6}(z + 2i)^3 + \frac{1}{24}(z + 2i)^4 + \cdots, \quad R = \infty$
5. $1 - \frac{1}{2}(z - \frac{1}{2}\pi)^2 + \frac{1}{24}(z - \frac{1}{2}\pi)^4 - \frac{1}{720}(z - \frac{1}{2}\pi)^6 + - \cdots, \quad R = \infty$
7. $\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}i(z - i) + (-\frac{1}{4} + \frac{1}{4}i)(z - i)^2 - \frac{1}{4}(z - i)^3 + - \cdots, \quad R = \sqrt{2}$

9. $1 - \frac{1}{2}z^2 + \frac{1}{8}z^4 - \frac{1}{48}z^6 + \frac{1}{384}z^8 - + \cdots, \quad R = \infty$

11. $4(z-1) + 10(z-1)^2 + 16(z-1)^3 + 14(z-1)^4 + 6(z-1)^5 + (z-1)^6$

13. $(2/\sqrt{\pi})(z - z^3/3 + z^5(2!5) - z^7/(3!7) + \cdots), \quad R = \infty$

15. $z^3/(1!3) - z^7/(3!7) + z^{11}/(5!11) - + \cdots, \quad R = \infty$

19. $z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots, \quad R = \frac{1}{2}\pi$

Problem Set 15.5, page 697

1. Use Theorem 1.

3. $R = 1/\sqrt{\pi} > 0.56$

5. $|z^n| \leqq 1$ and $\Sigma\ 1/n^2$ converges.

7. $|\tanh^n |z|| \leqq 1,\ 1/(n^2 + 1) < 1/n^2$

9. $|z + 1 - 2i| \leqq r < R = 4$

11. $|z| \leqq 2 - \delta\ (\delta > 0)$

13. Nowhere

15. $|z| \leqq \sqrt{5} - \delta\ (\delta > 0)$

Chapter 15 Review Questions and Problems, page 698

11. $\infty,\ e^{3z}$

13. $1,\ \frac{1}{2}\,\mathrm{Ln}[(1+z)/(1-z)]$

15. ∞

17. $1/\sqrt{\pi},\ [1 - \pi(z - 2i)^2]^{-1}$

19. $1/3$

21. $-1 - (z - \pi i) - (z - \pi i)^2/2! - \cdots, \quad R = \infty$

23. $\frac{1}{2} + \frac{1}{4}(z+1) + \frac{1}{8}(z+1)^2 + \frac{1}{16}(z+1)^3 + \cdots, \quad R = 2$

25. $1 + 3z + 6z^2 + 10z^3 + \cdots,\ R = 1$. Differentiate the geometric series.

27. $i + (z+i) - i(z+i)^2 - (z+i)^3 + \cdots, \quad R = 1$

29. $-(z - \frac{1}{2}\pi) + \dfrac{1}{3!}(z - \frac{1}{2}\pi)^3 - \dfrac{1}{5!}(z - \frac{1}{2}\pi)^5 + - \cdots,\ R = \infty$

Problem Set 16.1, page 707

1. $\dfrac{1}{z^4} + \dfrac{1}{z^3} + \dfrac{1}{z^2} + \dfrac{1}{z} + 1 + z + z^2 + \cdots,\ R = 1$

3. $\dfrac{1}{z^3} - \dfrac{1}{z^2} + \dfrac{1}{2z} - \dfrac{1}{6} + \dfrac{1}{24}z - \dfrac{1}{120}z^2 + - \cdots,\ R = \infty$

5. $\dfrac{1}{z^3} + \dfrac{1}{z^5} + \dfrac{1}{2z^7} + \dfrac{1}{6z^9} + \cdots,\ R = \infty$

7. $\dfrac{e^{z-1}e}{z-1} = e\sum_{n=0}^{\infty}\dfrac{(z-1)^{n-1}}{n!} = e\left[\dfrac{1}{z-1} + 1 + \dfrac{z-1}{2!} + \cdots\right],\ R = \infty$

9. $-\sum_{n=0}^{\infty}\left(\dfrac{i}{2}\right)^{n+1}(z-i)^{n-1} = -\dfrac{i/2}{z-i} + \dfrac{1}{4} + \dfrac{i}{8}(z-i) - \dfrac{1}{16}(z-i)^2 - \cdots,$
$R = 2$

11. $-\sum_{n=0}^{\infty}(z+i)^{n-1} = -\dfrac{1}{z+i} - 1 - (z+i) - \cdots,\ R = 1$

13. $-\dfrac{3}{z-1} + 2 + (z-1)$

15. $\sum_{n=0}^{\infty} z^{3n}$, $|z| < 1$, $-\sum_{n=0}^{\infty} \frac{1}{z^{3n+3}}$, $|z| > 1$

17. $\sum_{n=0}^{\infty} z^{4n+2}$, $|z| < 1$, $-\sum_{n=0}^{\infty} \frac{1}{z^{4n+2}}$, $|z| > 1$

19. $\frac{i}{(z-i)^2} + \frac{1}{z-i} + i + (z-i)$

21. $(1-4z)\sum_{n=0}^{\infty} z^{4n}$, $|z| < 1$, $\left(\frac{4}{z^3} - \frac{1}{z^4}\right)\sum_{n=0}^{\infty} \frac{1}{z^{4n}}$, $|z| > 1$

23. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(z+\frac{1}{2}\pi)^{2n-1}}{(2n)!}$, $|z + \frac{1}{2}\pi| > 0$

Problem Set 16.2, page 711

1. $\pm\frac{1}{2}, \pm\frac{3}{2}, \cdots$ (poles of 2nd order), ∞ (essential singularity)
3. $0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \cdots$ (simple poles), ∞ (essential singularity)
5. ∞ (essential singularity)
7. $\pm 1, \pm i$ (fourth-order poles), ∞ (essential singularity)
9. $\pm i$ (essential singularities)
13. $-16i$ (fourth order)
15. $\pm 1, \pm 2, \cdots$ (third order)
17. $\pm i/\sqrt{3}$ (simple)
19. $\pm 2i$ (simple), $0, \pm 2\pi i, \pm 4\pi i, \cdots$ (second order)
21. $0, \pm 2\pi, \pm 4\pi, \cdots$ (fourth order)
23. $f(z) = (z-z_0)^n g(z)$, $g(z_0) \neq 0$, hence $f^2(z) = (z-z_0)^{2n} g^2(z)$.

Problem Set 16.3, page 717

1. $i, 4i$
3. $-\frac{1}{4}i$ (at $z = 2i$), $\frac{1}{4}i$ (at $-2i$)
5. $1/5!$ (at $z = 0$)
7. 1 (at $\pm n\pi$)
9. $-\frac{1}{4}$ (at $z = 1$), $\frac{1}{4}$ (at $z = -1$)
11. -1 (at $z = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \cdots$)
15. $e^{1/z} = 1 + 1/z + \cdots$, *Ans.* $2\pi i$
17. Simple poles at $\pm\frac{1}{2}$. *Ans.* $-4i$
19. $-4\pi i \sinh \frac{1}{2}\pi$
21. $-4i \sinh \frac{1}{2}$
23. 0
25. $\frac{1}{2}$ (at $z = \frac{1}{2}$), 2 (at $z = \frac{1}{3}$). *Ans.* $5\pi i$

Problem Set 16.4, page 725

1. $2\pi/\sqrt{13}$
3. $2\pi/35$
5. $2\pi/3$
7. 0
9. π
11. $2\pi/3$
13. $\pi/16$
15. 0
17. $\pi/2$
19. 0
21. 0
23. π
25. 0
27. $-\pi/2$

Chapter 16 Review Questions and Problems, page 726

17. $2\pi i/3$
19. 5π
21. $\frac{1}{2}\pi \cos 10$
23. $\pi i/4$
25. 0 (n even), $(-1)^{(n-1)/2}\, 2\pi i/(n-1)!$ (n odd)
27. $6\pi i$
29. $2\pi/7$
31. $4\pi/\sqrt{3}$
33. 0
35. $\pi/2$

Problem Set 17.1, page 733

3. Only the size
5. $x = c$, $w = -y + ic$, $y = k$, $w = -k + ix$
7. $-3\pi/4 < \text{Arg } w < 3\pi/4$, $|w| < 1/8$
9. $|w| > 3$
11. $|w| \leqq 16$, $v \geqq 0$
13. Annulus $3 < |w| < 5$
15. $\ln 2 \leqq u \leqq \ln 3$, $\pi/4 \leqq v \leqq \pi/2$
17. ± 1, $\pm i$
19. $0, \pm 1, \pm 2, \cdots$
21. $-a/2$
23. a and 0, $\sqrt[3]{a}$
25. $M = e^x = 1$ when $x = 0$, $J = e^{2x}$
27. $M = 1/|z| = 1$ on the unit circle, $J = 1/|z|^2$

Problem Set 17.2, page 737

5. $z = \dfrac{-iw}{-2w + 3}$
7. $z = \dfrac{5i}{4w - 2}$
9. $z = 0$
11. $z = \frac{1}{2} + i \pm \sqrt{\frac{1}{4} + i}$
13. $z = \pm i$
15. $w = 4/z$, etc.
17. $a - d = 0$, $b/c = 1$ by (5)
19. $w = az/d$ $(a \neq 0, d \neq 0)$

Problem Set 17.3, page 741

5. Apply the inverse g of f on both sides of $z_1 = f(z_1)$ to get $g(z_1) = g(f(z_1)) = z_1$.
7. $w = (z + 2i)/(z - 2i)$
9. $w = z - 4$
11. $w = 1/z$
13. $w = (3iz + 1)/z$
15. $w = (z + 1)/(-3z + 1)$
17. $w = (2z - i)/(-iz - 2)$
19. $w = (z^4 - i)/(-iz^4 + 1)$

Problem Set 17.4, page 745

1. Annulus $1 \leqq |w| \leqq e^2$
3. $1/\sqrt{e} < |w| < \sqrt{e}$, $3\pi/4 < \arg w < 5\pi/4$
5. $1 < |w| < e$, $v > 0$
7. w-plane without 0
9. $u^2/\cosh^2 1 + v^2/\sinh^2 1 \leqq 1$, $u \geqq 0$
11. Elliptic annulus bounded by $u^2/\cosh^2 1 + v^2/\sinh^2 1 = 1$ and $u^2/\cosh^2 5 + v^2/\sinh^2 5 = 1$
13. $\pm(2n + 1)\pi/2$, $n = 0, 1, \cdots$
15. $0 < \text{Im } t < \pi$ is the image of R under $t = z^2$. *Ans.* $e^t = e^{z^2}$
17. $0, \pm i, \pm 2i, \cdots$
19. $u^2/\cosh^2 1 + v^2/\sinh^2 1 \leqq 1$, $v < 0$
21. $v < 0$
23. $-1 \leqq u \leqq 1$, $v = 0$ $(c = 0)$, $u^2/\cosh^2 c + v^2/\sinh^2 c = 1$ $(c \neq 0)$
25. $\ln 2 \leqq u \leqq \ln 3$, $\pi/4 \leqq v \leqq \pi/2$

Problem Set 17.5, page 747

1. w moves once around the unit circle.
5. $-5/3$, 2 sheets
7. $-i/2$, 3 sheets
9. 0, 2 sheets

Chapter 17 Review Questions and Problems, page 747

11. $u = \frac{1}{4}v^2 - 1, \frac{1}{4}v^2 - 1$ **13.** $|w| = 20.25$, $|\arg w| < \pi/2$
15. The domain between $u = \frac{1}{4} - v^2$ and $u = 1 - \frac{1}{4}v^2$
17. $|w + \frac{1}{2}| = \frac{1}{2}$ **19.** $u = 1$ **21.** $|\arg w| < \pi/4$
23. $0, (\pm 1 \pm i)/\sqrt{2}$ **25.** $\pi/8 \pm n\pi/2, n = 0, 1, \cdots$
27. $0, \pm i/\sqrt{2}$ **29.** $w = iz$ **31.** $w = 1/z$
33. $w = z/(z + 2)$ **35.** $\pm\sqrt{2}$ **37.** $2 \pm \sqrt{6}$
39. $1 + i \pm \sqrt{1 + 2i}$ **41.** $w = e^{3z}$ **43.** $z^2/2k$
45. $iz^3 + 1$

Problem Set 18.1, page 753

1. $20x + 200, 20z + 200$ **3.** $110 - 50xy, 110 + 25iz^2$
5. $F = (110/\ln 2)\text{Ln } z$ **7.** $F = 200 - (100/\ln 2)\text{ Ln } z$
13. Use Fig. 388 in Sec. 17.4 with the z- and w-planes interchanged, and $\cos z = \sin(z + \frac{1}{2}\pi)$.
15. $\Phi = 220 - 110xy$

Problem Set 18.2, page 757

1. $u^2 - v^2 = e^{2x}(\cos^2 y - \sin^2 y)$, $\Phi_{xx} = 4e^{2x}(\cos^2 y - \sin^2 y) = -\Phi_{yy}$, $\nabla^2\Phi = 0$
3. Straightforward calculation, involving the chain rule and the Cauchy-Riemann equations
5. See Fig. 389 in Sec. 17.4. $\Phi = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y$.
9. (i) $\Phi = U_1(1 - xy)$. (ii) $w = iz^2$ maps R onto $-2 \leqq u \leqq 0$, thus $\Phi^* = U_1(1 + \frac{1}{2}u) = U_1(1 + \frac{1}{2}(-2xy))$.
11. By Theorem 1 in Sec. 17.2
13. $\Phi = 10[1 - (1/\pi)\text{ Arg }(z - 4)]$, $F = 10[1 + (i/\pi)\text{ Ln }(z - 4)]$
15. Corresponding rays in the w-plane make equal angles, and the mapping is conformal.

Problem Set 18.3, page 760

3. $(100/d)y$. Rotate through $\pi/2$. **5.** $100 - 240\theta/\pi$
7. $\text{Re } F(z) = 100 + (200/\pi)\text{ Re }(\arcsin z)$ **9.** $(240/\pi)\text{ Arg } z$
11. $T_0 + (2/\pi)(T_1 - T_0)\text{ Arg } z$ **13.** $50 + (400/\pi)\text{ Arg } z$

Problem Set 18.4, page 766

1. $V = iV_2 = iK$, $\Psi = -Kx = const$, $\Phi = Ky = const$
3. $F(z) = Kz$ (K positive real)
5. $V = (1 + 2i)K$, $F = (1 - 2i)Kz$
7. $F(z) = z^3$
9. Hyperbolas $(x + 1)y = const$. Flow around a corner formed by $x = -1$ and the x-axis.
11. $y/(x^2 + y^2) = c$ or $x^2 + (y - k)^2 = k^2$
13. $F(z) = z/r_0 + r_0/z$

15. Use that $w = \arccos z$ is $w = \cos z$ with the roles of the z- and w-planes interchanged.

Problem Set 18.5, page 771

5. $1 - r^2 \cos 2\theta$

7. $2(r \sin \theta - \frac{1}{2}r^2 \sin 2\theta + \frac{1}{3}r^3 \sin 3\theta - + \cdots)$

9. $\frac{3}{4}r^2 \sin 2\theta - \frac{1}{4}r^6 \sin 6\theta$

11. $\frac{1}{3}\pi^2 - 4(r \cos \theta - \frac{1}{4}r^2 \cos 2\theta + \frac{1}{9}r^3 \cos 3\theta - + \cdots)$

13. $\dfrac{4}{\pi}\left(r \sin \theta - \dfrac{1}{9} r^3 \sin 3\theta + \dfrac{1}{25} r^5 \sin 5\theta - + \cdots\right)$

Problem Set 18.6, page 774

1. No; $|z|^2$ is not analytic. **3.** Use (2). $F(\frac{1}{3}) = \frac{25}{9}$ **5.** $\Phi(4, -4) = -12$

7. Use (3). $\Phi(1, 1) = -2$ **11.** $|F(e^{i\theta})|^2 = 2 - 2\cos 2\theta$, $\theta = \pi/2$, Max $= 2$

13. $|F(z)| = [\cos^2 2x + \sinh^2 2y]^{1/2}$, $z = \pm i$, Max $= [1 + \sinh^2 2]^{1/2} = \cosh 2 = 3.7622$

15. No

Chapter 18 Review Questions and Problems, page 775

11. $\Phi = 10(1 - x + y)$, $F = 10 - 10(1 + i)z$

13. $(20/\ln 10)\ \mathrm{Ln}\ z$

15. $(10/\ln 10)(\ln 100 - \ln r)$

17. $\mathrm{Arg}\ z = const$

19. $(-i/\pi)\ \mathrm{Ln}\ z$

23. $T(x, y) = x(2y + 1) = const$

25. Circles $(x - c)^2 + y^2 = c^2$

27. $F(z) = \dfrac{c}{2\pi}\ \mathrm{Ln}\ (z - 5)$, $\mathrm{Arg}\ (z - 5) = c$

29. $20 + \dfrac{80}{\pi}\left(r \sin \theta + \dfrac{1}{3} r^3 \sin 3\theta + \dfrac{1}{5} r^5 \sin 5\theta + \cdots\right)$

Problem Set 19.1, page 786

1. $0.9817 \cdot 10^2$, $-0.1010 \cdot 10^3$, $0.5787 \cdot 10^{-2}$, $-0.1360 \cdot 10^5$

3. $0.36443/(17.862 - 17.798) = 0.36443/0.064 = 5.6942$, $0.3644/(17.86 - 17.80) = 0.3644/0.06 = 6.073$, $0.364/(17.9 - 17.8) = 3.64$, impossible

5. $$\frac{0.36443(17.862 + 17.798)}{17.862^2 - 17.798^2} = \frac{0.36443 \cdot 35.660}{319.05 - 316.77} = \frac{12.996}{2.28} = 5.7000,$$
$$\frac{13.00}{2.28} = 5.702, \quad \frac{13.0}{2.28} = 5.70, \quad \frac{13}{2.3} = 5.7, \quad \frac{10}{2} = 5$$

7. 19.95, 0.049, 0.05013; 20, 0, 0.05

9. In the present calculation, (b) is more accurate than (a).

11. $-0.126 \cdot 10^{-2}$, $-0.402 \cdot 10^{-3}$; $-0.267 \cdot 10^{-6}$, $-0.847 \cdot 10^{-7}$

13. Add first, then round.

15. $$\frac{a_1}{a_2} = \frac{\tilde{a}_1 + \epsilon_1}{\tilde{a}_2 + \epsilon_2} = \frac{\tilde{a}_1 + \epsilon_1}{\tilde{a}_2}\left(1 - \frac{\epsilon_2}{\tilde{a}_2} + \frac{{\epsilon_2}^2}{{\tilde{a}_2}^2} - + \cdots\right) \approx \frac{\tilde{a}_1}{\tilde{a}_2} + \frac{\epsilon_1}{\tilde{a}_2} - \frac{\epsilon_2}{\tilde{a}_2} \cdot \frac{\tilde{a}_1}{\tilde{a}_2},$$

hence $\left|\left(\frac{a_1}{a_2} - \frac{\tilde{a}_1}{\tilde{a}_2}\right)\Big/\left|\frac{a_1}{a_2}\right|\right| \approx \left|\frac{\epsilon_1}{a_1} - \frac{\epsilon_2}{a_2}\right| \leqq |\epsilon_{r1}| + |\epsilon_{r2}| \leqq \beta_{r1} + \beta_{r2}$

19. (a) $19/21 = 0.904761905$, $\epsilon_{\text{chop}} = \epsilon_{\text{round}} = 0.1905 \cdot 10^{-5}$, $\epsilon_{r,\text{chop}} = \epsilon_{r,\text{round}} = 0.2106 \cdot 10^{-5}$, etc.

Problem Set 19.2, page 796

1. $g = 1.4 \sin x$, $1.37263\ (= x_5)$

5. $g = x^4 + 0.2$, $0.20165\ (= x_3)$

7. $2.403\ (= x_5$, exact to 3S)

9. $0.904557\ (= x_3)$

11. $1.834243\ (= x_4)$

13. $x_0 = 4.5$, $x_4 = 4.73004$ (6S exact)

15. (a) 0.5, 0.375, 0.377968, 0.377964; (b) $1/\sqrt{7} = 0.377964\,473$

17. $x_{n+1} = (2x_n + 7/x_n^2)/3$, $1.912931\ (= x_3)$

19. (a) **Algorithm Bisect** (f, a_0, b_0, N) **Bisection Method**

This algorithm computes an interval $[a_n, b_n]$ containing a solution of $f(x) = 0$ (f continuous) or it computes a solution c_n, given an initial interval $[a_0, b_0]$ such that $f(a_0)f(b_0) < 0$. Here N is determined by $(b - a)/2^N \leqq \beta$, β the required accuracy.

INPUT: Initial interval $[a_0, b_0]$, maximum number of iterations N.

OUTPUT: Interval $[a_N, b_N]$ containing a solution, or a solution c_n.

For $n = 0, 1, \cdots, N - 1$ do:

Compute $c_n = \frac{1}{2}(a_n + b_n)$.

If $f(c_n) = 0$ then OUTPUT c_n. Stop. [*Procedure completed*]

Else continue.

If $f(a_n)f(c_n) < 0$ then $a_{n+1} = a_n$ and $b_{n+1} = c_n$.

Else set $a_{n+1} = c_n$ and $b_{n+1} = b_n$.

End

OUTPUT $[a_N, b_N]$. Stop.

[*Procedure completed*]

End BISECT

Note that $[a_N, b_N]$ gives $(a_N + b_N)/2$ as an approximation of the zero and $(b_N - a_N)/2$ as a corresponding error bound.

(b) 0.739085; (c) 1.30980, 0.429494

21. 1.834243

23. 0.904557

Problem Set 19.3, page 808

1. $L_0(x) = -2x + 19$, $L_1(x) = 2x - 18$, $p_1(x) = 0.1082x + 1.2234$, $p_1(9.4) = 2.2405$

3. 0.9971, 0.9943, 0.9915 (0.9916 4D), 0.9861 (0.9862 4D), 0.9835, 0.9809

5. $p_2(x) = -0.44304x^2 + 1.30906x - 0.02322$, $p_2(0.75) = 0.70929$

7. $p_2(x) = -0.1434x^2 + 1.0895x$, $p_2(0.5) = 0.5089$, $p_2(1.5) = 1.3116$

9. $L_0 = -\frac{1}{6}(x - 1)(x - 2)(x - 3)$, $L_1 = \frac{1}{2}x(x - 2)(x - 3)$, $L_2 = -\frac{1}{2}x(x - 1)(x - 3)$, $L_3 = \frac{1}{6}x(x - 1)(x - 2)$; $p_3(x) = 1 + 0.039740x - 0.335187x^2 + 0.060645x^3$; $p_3(0.5) = 0.943654$ (6S-exact 0.938470), $p_3(1.5) = 0.510116$ (0.511828), $p_3(2.5) = -0.047993$ (-0.048384)

13. $p_2(x) = 0.9461x - 0.2868x(x - 1)/2 = -0.1434x^2 + 1.0895x$
15. 0.722, 0.786
17. $\delta f_{1/2} = 0.057\,839$, $\delta f_{3/2} = 0.069\,704$, etc.

Problem Set 19.4, page 815

9. $[-1.39(x - 5)^2 + 0.58(x - 5)^3]'' = 0.004$ at $x = 5.8$ (due to roundoff; should be 0).
11. $1 - \frac{5}{4}x^2 + \frac{1}{4}x^4$
13. $4 - 12x^2 - 8x^3$, $4 - 12x^2 + 8x^3$. Yes
15. $1 - x^2$, $-2(x - 1) - (x - 1)^2 + 2(x - 1)^3$,
$-1 + 2(x - 2) + 5(x - 2)^2 - 6(x - 2)^3$
17. Curvature $f''/(1 + f'^2)^{3/2} \approx f''$ if $|f'|$ is small
19. Use that the third derivative of a cubic polynomial is constant, so that g''' is piecewise constant, hence constant throughout under the present assumption. Now integrate three times.

Problem Set 19.5, page 828

1. 0.747131
3. 0.69377 (5S-exact 0.69315)
5. 1.566 (4S-exact 1.557)
7. 0.894 (3S-exact 0.908)
9. $J_{h/2} + \epsilon_{h/2} = 1.55963 - 0.00221 = 1.55742$ (6S-exact 1.55741)
11. $J_{h/2} + \epsilon_{h/2} = 0.90491 + 0.00349 = 0.90840$ (5S-exact 0.90842)
13. 0.94508, 0.94583 (5S-exact 0.94608)
15. 0.94614588, 0.94608693 (8S-exact 0.94608307)
17. 0.946083 (6S-exact)
19. 0.9774586 (7S-exact 0.9774377)
21. $x - 2 = t$, 1.098609 (7S-exact 1.098612)
23. $x = \frac{1}{2}(t + 1)$, 0.7468241268 (10S-exact 0.7468241330)
25. (a) $M_2 = 2$, $M_2^* = \frac{1}{4}$, $|KM_2| = 2/(12n^2)$, $n = 183$, (b) $f^{(iv)} = 24/x^5$, $2m = 14$
27. 0.08, 0.32, 0.176, 0.256 (exact)
29. $5(0.1040 - \frac{1}{2}\cdot 0.1760 + \frac{1}{3}\cdot 0.1344 - \frac{1}{4}\cdot 0.0384) = 0.256$

Chapter 19 Review Questions and Problems, page 830

17. 4.266, 4.38, 6.0, impossible
19. 49.980, 0.020; 49.980, 0.020008
21. $17.5565 \leqq s \leqq 17.5675$
23. The same as that of $\bar{a}$.
25. −0.2, −0.20032, −0.200323
27. 3, 2.822785, 2.801665, 2.801386, 2.801386
29. 2.95647, 2.96087
31. 0.26, $M_2 = 6$, $M_2^* = 0$, $-0.02 \leqq \epsilon \leqq 0$, $0.24 \leqq a \leqq 0.26$
33. 1.001005, $-0.001476 \leqq \epsilon \leqq 0$

Problem Set 20.1, page 839

1. $x_1 = -2.4$, $x_2 = 5.3$
3. No solution
5. $x_1 = 2$, $x_2 = 1$
7. $x_1 = 6.78$, $x_2 = -11.3$, $x_3 = 15.82$

9. $x_1 = 0$, $x_2 = t_1$ arbitrary, $x_3 = 5t_1 + 10$
11. $x_1 = t_1$, $x_2 = t_2$, both arbitrary, $x_3 = 1.25t_1 - 2.25t_2$
13. $x_1 = 1.5$, $x_2 = -3.5$, $x_3 = 4.5$, $x_4 = -2.5$

Problem Set 20.2, page 844

1. $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{matrix} x_1 = 4.2 \\ x_2 = 1.3 \end{matrix}$ **3.** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 0 & 5 \end{bmatrix}, \quad \begin{matrix} x_1 = -4 \\ x_2 = 3 \end{matrix}$

5. $\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{2} & 6 & 1 \end{bmatrix} \begin{bmatrix} 6 & 4 & 3 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad \begin{matrix} x_1 = 0.5 \\ x_2 = -2.5 \\ x_3 = 3.0 \end{matrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix} \begin{bmatrix} 3 & 9 & 6 \\ 0 & -6 & 3 \\ 0 & 0 & -3 \end{bmatrix}, \quad \begin{matrix} x_1 = -1/30 \\ x_2 = 2/15 \\ x_3 = 1/5 \end{matrix}$

9. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{matrix} x_1 = 3 \\ x_2 = 6 \\ x_3 = 2 \end{matrix}$

11. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 7 \end{bmatrix}, \quad \begin{matrix} x_1 = 4 \\ x_2 = 0 \\ x_3 = -2 \end{matrix}$

13. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{matrix} x_1 = 6 \\ x_2 = -2 \\ x_3 = 0 \\ x_4 = 14 \end{matrix}$

17. $\begin{bmatrix} -2 & 4 & -1 \\ -2 & 3 & 0 \\ 7 & -12 & 2 \end{bmatrix}$ **19.** $\begin{bmatrix} \frac{341}{6} & -10 & \frac{7}{6} \\ -\frac{15}{2} & \frac{4}{3} & -\frac{1}{6} \\ -17 & 3 & -\frac{1}{3} \end{bmatrix}$

Problem Set 20.3, page 850

3. Exact 21.5, 0, −13.8 **5.** Exact 2, 1, 4 **7.** Exact 0.5, 0.5, 0.5
9. (a) $\mathbf{x}^{(3)\mathsf{T}} = [0.49982 \quad 0.50001 \quad 0.50002]$, (b) $\mathbf{x}^{(3)\mathsf{T}} = [0.50333 \quad 0.49985 \quad 0.49968]$
11. 6, 15, 46, 96 steps; spectral radius 0.09, 0.35, 0.72, 0.85, approximately
13. $[1.99934 \quad 1.00043 \quad 3.99684]^{\mathsf{T}}$ (Jacobi, step 5); $[2.00004 \quad 0.998059 \quad 4.00072]^{\mathsf{T}}$ (Gauss-Seidel)
17. $\sqrt{306} = 17.49$, 12, 12 **19.** $\sqrt{18k^2} = 4.24|k|$, $4|k|$, $4|k|$

Problem Set 20.4, page 858

1. 12, $\sqrt{62} = 7.87$, 6, $[\frac{1}{6} \quad -1 \quad \frac{5}{6}]$
3. 14, $\sqrt{50} = 7.07$, 4, $[-1 \quad 1 \quad \frac{3}{4} \quad -\frac{3}{4}]$
5. 1.9, $\sqrt{1.35} = 1.16$, 1, $[0.3 \quad -0.1 \quad 0.5 \quad 1.0]$
7. 6, $\sqrt{6}$, 1, $[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$
11. $\| \mathbf{A} \|_1 = 17$, $\| \mathbf{A}^{-1} \|_1 = 17$, $\kappa = 289$
13. $\kappa = 100 \cdot 100$
15. $\kappa = 1.2 \cdot \frac{120}{98} = 1.469$
17. $46 \leqq 6 \cdot 17$ or $7 \cdot 17$
19. $[0 \quad 1]^\mathsf{T}$, $[1 \quad -0.4]^\mathsf{T}$, 289
21. $[-0.6 \quad 2.8]^\mathsf{T}$
23. 27, 748, 28375, 943656, 29070279

Problem Set 20.5, page 862

1. $-11.4 + 5.4x$
3. $8.95 - 0.388x$
5. $s = -675 + 90t$, $v_{av} = 90$ km/h
9. $4 - 0.75x - 0.125x^2$
11. $5.248 + 1.543x$, $3.900 + 0.5321x + 2.021x^2$
13. $-2.448 + 16.23x$, $-9.114 + 13.73x + 2.500x^2$,
$-2.270 + 1.466x - 1.778x^2 + 2.852x^3$

Problem Set 20.7, page 871

1. $5 \leqq \lambda \leqq 9$
3. 5, 0, 7; radii 4, 6, 6
5. $|\lambda - 4i| \leqq \sqrt{2} + 0.1$, $|\lambda| \leqq 0.1$, $|\lambda - 9i| \leqq \sqrt{2}$
7. $t_{11} = 100$, $t_{22} = t_{33} = 1$
9. They lie in the intervals with endpoints $a_{jj} \pm (n-1)10^{-6}$. (Why?)
11. 0 lies in no Gerschgorin disk, by (3) with $>$; hence $\det \mathbf{A} = \lambda_1 \cdots \lambda_n \neq 0$.
13. $\rho(\mathbf{A}) \leqq$ Row sum norm $\|\mathbf{A}\|_\infty = \max_j \sum_k |a_{jk}| = \max_j (|a_{jj}| + \text{Gerschgorin radius})$
15. $\sqrt{153} = 12.37$
17. $\sqrt{122} = 11.05$
19. $6 \leqq \lambda \leqq 10$, $8 \leqq \lambda \leqq 8$

Problem Set 20.8, page 875

1. $q = 4, 4.493, 4.4999$; $|\epsilon| \leqq 1.5, 0.1849, 0.0206$
3. $q = 8, 8.1846, 8.2252$; $|\epsilon| \leqq 1, 0.4769, 0.2200$
5. $q = 4, 4.786, 4.917$; $|\epsilon| \leqq 1.63, 0.619, 0.399$
7. $q = 5.5, 5.5738, 5.6018$; $|\epsilon| \leqq 0.5, 0.3115, 0.1899$; eigenvalues (4S) 1.697, 3.382, 5.303, 5.618
9. $\mathbf{y} = \mathbf{Ax} = \lambda\mathbf{x}$, $\mathbf{y}^\mathsf{T}\mathbf{x} = \lambda\mathbf{x}^\mathsf{T}\mathbf{x}$, $\mathbf{y}^\mathsf{T}\mathbf{y} = \lambda^2\mathbf{x}^\mathsf{T}\mathbf{x}$,
$\epsilon^2 \leqq \mathbf{y}^\mathsf{T}\mathbf{y}/\mathbf{x}^\mathsf{T}\mathbf{x} - (\mathbf{y}^\mathsf{T}\mathbf{x}/\mathbf{x}^\mathsf{T}\mathbf{x})^2 = \lambda^2 - \lambda^2 = 0$
11. $q = 1, \cdots, -2.8993$ approximates -3 (0 of the given matrix),
$|\epsilon| \leqq 1.633, \cdots, 0.7024$ (Step 8)

Problem Set 20.9, page 882

1. $\begin{bmatrix} 3.500000 & -1.802776 & 0 \\ -1.802776 & 6.730769 & 1.846154 \\ 0 & 1.846154 & 1.769230 \end{bmatrix}$
3. $\begin{bmatrix} 0.980000 & -0.441814 & 0 \\ -0.441814 & 0.870164 & 0.371803 \\ 0 & 0.371803 & 0.489836 \end{bmatrix}$

5. Eigenvalues 8, 3, 1

$$\begin{bmatrix} 5.64516 & -2.50867 & 0 \\ -2.50867 & 5.307219 & 0.374953 \\ 0 & 0.374953 & 1.04762 \end{bmatrix}, \begin{bmatrix} 7.45139 & -1.56325 & 0 \\ -1.56325 & 3.544142 & 0.0983071 \\ 0 & 0.0983071 & 1.00446 \end{bmatrix},$$

$$\begin{bmatrix} 7.91494 & -0.646602 & 0 \\ -0.646602 & 3.08458 & 0.0312469 \\ 0 & 0.0312469 & 1.000482 \end{bmatrix}$$

7. $$\begin{bmatrix} 18.3171 & 0.881767 & 0 \\ 0.881767 & 8.29042 & 0.360275 \\ 0 & 0.360275 & 1.39250 \end{bmatrix}, \begin{bmatrix} 18.3786 & 0.396511 & 0 \\ 0.396511 & 8.24727 & 0.0600924 \\ 0 & 0.0600924 & 1.37414 \end{bmatrix},$$

$$\begin{bmatrix} 18.3910 & 0.177669 & 0 \\ 0.177669 & 8.23540 & 0.0100214 \\ 0 & 0.0100214 & 1.37363 \end{bmatrix}$$

9. $$\begin{bmatrix} 7.00224 & 0.0571287 & 0 \\ 0.0571287 & 4.00088 & 0.0249333 \\ 0 & 0.0249333 & 0.996875 \end{bmatrix}, \begin{bmatrix} 7.00298 & 0.0326363 & 0 \\ 0.0326363 & 4.00034 & 0.00621221 \\ 0 & 0.00621221 & 0.996681 \end{bmatrix},$$

$$\begin{bmatrix} 7.00322 & 0.0186419 & 0 \\ 0.0186419 & 4.00011 & 0.00154782 \\ 0 & 0.00154782 & 0.996669 \end{bmatrix}$$

Chapter 20 Review Questions and Problems, page 883

17. $[4 \quad -1 \quad 2]^{\mathsf{T}}$ **19.** $[6 \quad -3 \quad 1]^{\mathsf{T}}$

21. All nonzero entries of the factors are 1.

23. $$\begin{bmatrix} 2.8193 & -1.5904 & -0.0482 \\ -1.5904 & 1.2048 & -0.0241 \\ -0.0482 & -0.0241 & 0.1205 \end{bmatrix}$$ (4D-values) **25.** Exact $[-2 \quad 1 \quad 2]^{\mathsf{T}}$

27. 15, $\sqrt{89}$, 8 **29.** 7, $\sqrt{21}$, 4 **31.** 14, $\sqrt{78}$, 7

33. 6 **35.** 9 **37.** $11.5 \cdot 4.4578 = 51.2651$

39. $y = 1.98 + 0.98x$

41. Centers 1, 1, 1, radii 2.5, 1, 2.5 ($\lambda = 2.944,\ 0.028 \pm 0.290i$, 3D)

43. Centers 5, 6, 8; radii 2, 1, 1, ($\lambda = 4.1864,\ 6.4707,\ 8.3429$, 5S)

45. $$\begin{bmatrix} 1.5 & -2.23607 & 0 \\ -2.23607 & 5.8 & -3.1 \\ 0 & -3.1 & 6.7 \end{bmatrix}, \text{ Step 3: } \begin{bmatrix} 9.44973 & -1.06216 & 0 \\ -1.06216 & 4.28682 & -0.00308 \\ 0 & -0.00308 & 0.26345 \end{bmatrix}$$

Problem Set 21.1, page 897

1. $y = e^x$, 0.0382, 0.1245 (error of x_5, x_{10})

3. $y = x - \tanh x$ (set $y - x = u$), 0.009292, 0.0188465 (error of x_5, x_{10})

5. $y = e^x$, 0.001275, 0.004200 (error of x_5, x_{10})

7. $y = 1/(1 - x^2/2)$, 0.00029, 0.01187 (error of x_5, x_{10})

9. $y = 1/(1 - x^2/2)$, 0.03547, 0.28715 (error of x_5, x_{10})

11. $y = 1/(1 - x^2/2)$; error -10^{-8}, $-4 \cdot 10^{-8}$, $\cdots$, $-6 \cdot 10^{-7}$, $+10^{-5}$; about $1.3 \cdot 10^{-5}$ by (10)

13. $y = xe^x$; error $\cdot\, 10^5$ (for $x = 1, \cdots, 3$) 19, 46, 85, 139, 213, 315, 454, 640, 889, 1219

15. $y = 3 \cos x - 2 \cos^2 x$; error $\cdot\, 10^7$: 0.18, 0.74, 1.73, 3.28, 5.59, 9.04, 14.33, 22.77, 36.80, 61.42

17. $y = 1/(x^5 + 1)$, 0.000307, -0.000259 (error of x_5, x_{10})

19. The errors are for E.-C. 0.02000, 0.06287, 0.05076, for Improved E.-C. -0.000455, 0.012086, 0.009601, for RK 0.0000012, 0.000016, 0.000536.

Problem Set 21.2, page 901

3. $y = e^{-0.1x^2}$; errors 10^{-6} to $6 \cdot 10^{-6}$

5. $y = \tan x$; $y_4, \cdots, y_{10}$ (error $\cdot\, 10^5$): 0.422798 (-0.48), 0.546315 (-1.2), 0.684161 (-2.4), 0.842332 (-4.4), 1.029714 (-7.5), 1.260288 (-13), 1.557626 (-22)

7. RK-error smaller, error $\cdot\, 10^5 = 0.4, 0.3, 0.2, 5.6$ (for $x = 0.4, 0.6, 0.8, 1.0$)

9. $y_4 = 4.229\,690$, $y_5 = 4.556\,859$, $y_6 = 5.360\,657$, $y_7 = 8.082\,563$

11. Errors between $-6 \cdot 10^{-7}$ and $+3 \cdot 10^{-7}$. Solution $e^x - x - 1$

13. Errors $\cdot\, 10^5$ from $x = 0.3$ to 0.7: $-5, -11, -19, -31, -47$

15. (a) 0, 0.02, 0.0884, 0.215 848, $y_4 = 0.417\,818$, $y_5 = 0.708\,887$ (poor).
(b) By 30–50%

Problem Set 21.3, page 908

3. $y_1 = e^x$, $y_2 = -e^x$, errors range from ± 0.02 to ± 0.23, monotone.

5. $y_1' = y_2$, $y_2' = -4y_1$, $y = y_1 = 1, 0.84, 0.52, 0.0656, -0.4720$; $y = \cos 2x$

7. $y_1 = 4e^{-x} \sin x$, $y_2 = 4e^{-x} \cos x$; errors from 0 to about ± 0.1

9. Errors smaller by about a factor 10^4

11. $y = 0.198669, 0.389494, 0.565220, 0.719632, 0.847790$;
$y' = 0.980132, 0.922062, 0.830020, 0.709991, 0.568572$

13. $y_1 = e^{-3x} - e^{-5x}$, $y_2 = e^{-3x} + e^{-5x}$; $y_1 = 0.1341, 0.1807, 0.1832, 0.1657, 0.1409$; $y_2 = 1.348, 0.9170, 0.6300, 0.4368, 0.3054$

17. You get the exact solution, except for a roundoff error [e.g., $y_1 = 2.761\,608$, $y(0.2) = 2.7616$ (exact), etc.]. Why?

19. $y = 0.198669, 0.389494, 0.565220, 0.719631, 0.847789$;
$y' = 0.980132, 0.922061, 0.830019, 0.709988, 0.568568$

Problem Set 21.4, page 916

3. 105, 155, 105, 115; Step 5: 104.94, 154.97, 104.97, 114.98

5. 0.108253, 0.108253, 0.324760, 0.324760; Step 10: 0.108538, 0.108396, 0.324902, 0.324831

7. 0, 0, 0, 0. All equipotential lines meet at the corners (why?). Step 5: 0.29298, 0.14649, 0.14649, 0.073245

9. $-3u_{11} + u_{12} = -200$, $u_{11} - 3u_{12} = -100$

11. $u_{12} = u_{32} = 31.25$, $u_{21} = u_{23} = 18.75$, $u_{jk} = 25$ at the others

13. $u_{21} = u_{23} = 0.25$, $u_{12} = u_{32} = -0.25$, $u_{jk} = 0$ else

15. (a) $u_{11} = -u_{12} = -66$. (b) Reduce to 4 equations by symmetry.
$u_{11} = u_{31} = -u_{15} = -u_{35} = -92.92$, $u_{21} = -u_{25} = -87.45$,
$u_{12} = u_{32} = -u_{14} = -u_{34} = -64.22$, $u_{22} = -u_{24} = -53.98$,
$u_{13} = u_{23} = u_{33} = 0$

17. $\sqrt{3}$, $u_{11} = u_{21} = 0.0849$, $u_{12} = u_{22} = 0.3170$. (0.1083, 0.3248 are 4S-values of the solution of the linear system of the problem.)

Problem Set 21.5, page 921

5. $u_{11} = 0.766$, $u_{21} = 1.109$, $u_{12} = 1.957$, $u_{22} = 3.293$

7. **A** as in Example 1, right sides $-2, -2, -2, -2$. Solution $u_{11} = u_{21} = 1.14286$, $u_{12} = u_{22} = 1.42857$

11. $-4u_{11} + u_{21} + u_{12} = -3$, $u_{11} - 4u_{21} + u_{22} = -12$, $u_{11} - 4u_{12} + u_{22} = 0$, $2u_{21} + 2u_{12} - 12u_{22} = -14$, $u_{11} = u_{22} = 2$, $u_{21} = 4$, $u_{12} = 1$. Here $-14/3 = -\frac{4}{3}(1 + 2.5)$ with 4/3 from the stencil.

13. $\mathbf{b} = [-380 \quad -190, \quad -190, \quad 0]^{\mathsf{T}}$; $u_{11} = 140$, $u_{21} = u_{12} = 90$, $u_{22} = 30$

Problem Set 21.6, page 927

5. 0.1636, 0.2545 ($t = 0.04$, $x = 0.2, 0.4$), 0.1074, 0.1752 ($t = 0.08$), 0.0735, 0.1187 ($t = 0.12$), 0.0498, 0.0807 ($t = 0.16$), 0.0339, 0.0548 ($t = 0.2$; exact 0.0331, 0.0535)

7. Substantially less accurate, 0.15, 0.25 ($t = 0.04$), 0.100, 0.163 ($t = 0.08$)

9. Step 5 gives 0, 0.06279, 0.09336, 0.08364, 0.04707, 0.

11. Step 2: 0 (exact 0), 0.0453 (0.0422), 0.0672 (0.0658), 0.0671 (0.0628), 0.0394 (0.0373), 0 (0)

13. 0.1018, 0.1673, 0.1673, 0.1018 ($t = 0.04$), 0.0219, 0.0355, $\cdots$ ($t = 0.20$)

15. 0.3301, 0.5706, 0.4522, 0.2380 ($t = 0.04$), 0.06538, 0.10604, 0.10565, 0.6543 ($t = 0.20$)

Problem Set 21.7, page 930

1. For $x = 0.2, 0.4$ we obtain 0.012, 0.02 ($t = 0.2$), 0.004, 0.008 ($t = 0.4$), -0.004, -0.008 ($t = 0.6$), etc.

3. $u(x, 1) = 0, -0.05, -0.10, -0.15, -0.075, 0$

5. 0.190, 0.308, 0.308, 0.190 (0.178, 0.288, 0.288, 0.178 exact to 3D)

7. 0, 0.354, 0.766, 1.271, 1.679, 1.834, $\cdots$ ($t = 0.1$); 0, 0.575, 0.935, 1.135, 1.296, 1.357, $\cdots$ ($t = 0.2$)

Chapter 21 Review Questions and Problems, page 930

17. $y = \tan x$; 0 (0), 0.10050 (-0.00017), 0.20304 (-0.00033), 0.30981 (-0.00047), 0.42341 (-0.00062), 0.54702 (-0.00072)

19. 0.1003349 $(0.8 \cdot 10^{-7})$ 0.2027099 $(1.6 \cdot 10^{-7})$, 0.3093360 $(2.1 \cdot 10^{-7})$, 0.4227930 $(2.3 \cdot 10^{-7})$, 0.5463023 $(1.8 \cdot 10^{-7})$

25. $y(0.4) = 1.822798$, $y(0.5) = 2.046315$, $y(0.6) = 2.284161$, $y(0.7) = 2.542332$, $y(0.8) = 2.829714$, $y(0.9) = 3.160288$, $y(1.0) = 3.557626$

27. $y_1 = 3e^{-9x}$, $y_2 = -5e^{-9x}$, $[1.23251 \quad -2.05419]$, $[0.506362 \quad -0.843937]$, $\cdots$, $[0.035113 \quad -0.058522]$

29. 1.96, 7.86, 29.46

31. $u(P_{11}) = u(P_{31}) = 270$, $u(P_{21}) = u(P_{13}) = u(P_{23}) = u(P_{33}) = 30$, $u(P_{12}) = u(P_{32}) = 90$, $u(P_{22}) = 60$

35. 0.06279, 0.09336, 0.08364, 0.04707

37. 0, −0.352, −0.153, 0.153, 0.352, 0 if $t = 0.12$ and 0, 0.344, 0.166, −0.166, −0.344, 0 if $t = 0.24$

39. 0.010956, 0.017720, 0.017747, 0.010964 if $t = 0.2$

Problem Set 22.1, page 939

3. $f = 3(x_1 - 2)^2 + 2(x_2 + 4)^2 - 44$. Step 3: $[2.0055 \quad -3.9975]^\mathsf{T}$

5. $f = 0.5(x_1 - 1)^2 + 0.7(x_2 + 3)^2 - 5.8$, Step 3: $[0.99406 \quad -3.0015]^\mathsf{T}$

7. $f = 0.2(x_1 - 0.2)^2 + {x_2}^2 - 0.008$. Step 3: $[0.493 \quad -0.011]^\mathsf{T}$, Step 6: $[0.203 \quad 0.004]^\mathsf{T}$

Problem Set 22.2, page 943

1. x_3, x_4 unused time on M_1, M_2, respectively

3. No

11. $f_{\max} = f(0, 5) = 10$

13. $f_{\max} = f(9, 6) = 36$

15. $f_{\min} = f(3.5, 2.5) = -30$

17. $x_1/3 + x_2/2 \leqq 100$, $x_1/3 + x_2/6 \leqq 80$, $f = 150x_1 + 100x_2$, $f_{\max} = f(210, 60) = 37500$

19. $0.5x_1 + 0.75x_2 \leqq 45$ (copper), $0.5x_1 + 0.25x_2 \leqq 30$, $f = 120x_1 + 100x_2$, $f_{\max} = f(45, 30) = 8400$

Problem Set 22.3, page 946

1. $f(120/11, 60/11) = 480/11$

3. $f\left(\dfrac{2100}{3}, \dfrac{200}{2/3}\right) = 78000$

5. Matrices with Rows 2 and 3 and Columns 4 and 5 interchanged

7. $f(0, \frac{5}{10}) = -10$

9. $f(5, 4, 6) = 478$

Problem Set 22.4, page 952

1. $f(4, 4) = 72$

3. $f(20, 30) = 50$

5. $f(10, 5) = 5500$

7. $f(1, 1, 0) = 12$

9. $f(\frac{1}{2}, 0, \frac{1}{2}) = 3$

Chapter 22 Review Questions and Problems, page 952

11. Step 5: $[0.353 \quad -0.028]^\mathsf{T}$. Slower

13. Of course! Step 5: $[-1.003 \quad 1.897]^\mathsf{T}$

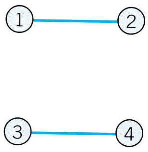
21. $f(2, 4) = 100$

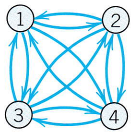
23. $f(3, 6) = -54$

25. $f(50, 100) = 150$

Problem Set 23.1, page 958

9. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **13.** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

15. 

15. 

21.

Vertex	Edge e_1	e_2	e_3
1	1	1	0
2	0	0	1
3	1	0	0
4	0	1	1

23.

Vertex					
1	−1	0	0	1	−1
2	1	−1	1	0	0
3	0	1	−1	−1	1

25.

Vertex	Incident Edges
1	$-e_1, -e_2, e_3, -e_4$
2	e_1
3	$e_2, -e_3$
4	e_4

Problem Set 23.2, page 962

1. 4 **3.** 5 **5.** 4

9. The idea is to go backward. There is a v_{k-1} adjacent to v_k and labeled $k - 1$, etc. Now the only vertex labeled 0 is s. Hence $\lambda(v_0) = 0$ implies $v_0 = s$, so that $v_0 - v_1 - \cdots - v_{k-1} - v_k$ is a path $s \to v_k$ that has length k.

15. No; there is no way of traveling along (3, 4) only once.

21. From m to $100m$, $10m$, $2.5m$, $m + 4.6$

Problem Set 23.3, page 966

1. (1, 2), (2, 4), (4, 3); $L_2 = 6$, $L_3 = 18$, $L_4 = 14$

3. (1, 2), (1, 4), (2, 3); $L_2 = 2$, $L_3 = 5$, $L_4 = 5$

5. (1, 4), (2, 4), (3, 4), (3, 5); $L_2 = 4$, $L_3 = 3$, $L_4 = 2$, $L_5 = 8$

7. (1, 5), (2, 3), (2, 6), (3, 4), (3, 5); $L_2 = 9$, $L_3 = 7$, $L_4 = 8$, $L_5 = 4$, $L_6 = 14$

Problem Set 23.4, page 969

1.
```
  2
 /
1   3   L = 12
 \ /
  4
   \
    5
```

3.
```
   1
  /
4 —2        L = 10
  \
   3 — 5
```

5.
```
          8
         /
1 — 2       5         L = 28
         \
          3 — 6 — 4
           \
            7
```

9. $1 - 3 - 4 \langle^{2}_{5\,-\,6,}$ $L = 38$ **11.** Yes

15. G is connected. If G were not a tree, it would have a cycle, but this cycle would provide two paths between any pair of its vertices, contradicting the uniqueness.

19. If we add an edge (u, v) to T, then since T is connected, there is a path $u \to v$ in T which, together with (u, v), forms a cycle.

Problem Set 23.5, page 972

1. (1, 2), (1, 4), (3, 4), (4, 5), $L = 12$

3. (1, 2), (2, 8), (8, 7), (8, 6), (6, 5), (2, 4), (4, 3), $L = 40$

5. (1, 4), (3, 4), (2, 4), (3, 5), $L = 20$

7. (1, 2), (1, 3), (1, 4), (2, 6), (3, 5), $L = 32$

11. If G is a tree

13. A shortest spanning tree of the largest connected graph that contains vertex 1

Problem Set 23.6, page 978

1. $1 - 2 - 5$, $\Delta f = 2$; $1 - 4 - 2 - 5$, $\Delta f = 2$, etc.

3. $1 - 2 - 4 - 6$, $\Delta f = 2$; $1 - 2 - 3 - 5 - 6$, $\Delta f = 1$, etc.

5. $f_{12} = 4$, $f_{13} = 1$, $f_{14} = 4$, $f_{42} = 4$, $f_{43} = 0$, $f_{25} = 8$, $f_{35} = 1$, $f = 9$

7. $f_{12} = 4$, $f_{13} = 3$, $f_{24} = 4$, $f_{35} = 3$, $f_{54} = 2$, $f_{46} = 6$, $f_{56} = 1$, $f = 7$

9. {4, 5, 6}, 28 **11.** {2, 4, 6}, 50

13. $1 - 2 - 3 - 7$, $\Delta f = 2$; $1 - 4 - 5 - 6 - 7$, $\Delta f = 1$; $1 - 2 - 3 - 6 - 7$, $\Delta f = 1$; $f_{max} = 14$

15. {3, 5, 7}, 22 **17.** $S = \{1, 4\}$, cap $(S, T) = 6 + 8 = 14$

19. If $f_{ij} < c_{ij}$ as well as $f_{ij} > 0$

Problem Set 23.7, page 982

3. (2, 3) and (5, 6)

5. $1 - 2 - 5$, $\Delta_t = 2$; $1 - 4 - 2 - 5$, $\Delta_t = 1$; $f = 6 + 2 + 1 = 9$

7. $1 - 2 - 4 - 6$, $\Delta_t = 2$; $1 - 3 - 5 - 6$, $\Delta_t = 1$; $f = 4 + 2 + 1 = 7$

9. By considering only edges with one labeled end and one unlabeled end

17. $S = \{1, 2, 4, 5\}$, $T = \{3, 6\}$, cap $(S, T) = 14$

Problem Set 23.8, page 986

1. No **3.** No **5.** Yes, $S = \{1, 4, 5, 8\}$

7. Yes; a graph is not bipartite if it has a nonbipartite subgraph.

9. 1 − 2 **—** 3 − 5

11. (1, 5), (2, 3) by inspection. The augmenting path 1 − 2 **—** 3 − 5 gives 1 **—** 2 − 3 **—** 5, that is, (1, 2), (3, 5).

13. (1, 4), (2, 3), (5, 7) by inspection. Or (1, 2), (3, 4), (5, 7) by the use of the path 1 − 2 **—** 3 − 4.

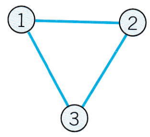
15. 3 **19.** 3 **23.** No; K_5 is not planar.

25. K_3

Chapter 23 Review Questions and Problems, page 987

13. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **15.** $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

19. 

21.

Vertex	Incident Edges
1	$e_2, -e_3$
2	$-e_1, e_3$
3	$e_1, -e_2$

23. 4

25. 4

27. $L_2 = 10, L_3 = 15, L_4 = 13$

29. $1 - 4 - 3 - 2, L = 16$

33. $f = 7$

Problem Set 24.1, page 996

1. $q_L = 19, q_M = 20, q_U = 20.5$

3. $q_L = 38, q_M = 44, q_U = 54$

5. $q_L = 69.7, q_M = 70.5, q_U = 71.2$

7. $q_L = 2.3, q_M = 2.4, q_U = 2.45$

9. $q_L = 399, q_M = 401, q_U = 401$

11. $\bar{x} = 19.875, s = 0.835$, IQR $= 1.5$

13. $\bar{x} = 70.49, s = 1.047$, IQR $= 1.5$

15. $\bar{x} = 400.4, s = 1.618$, IQR $= 2$

17. 0 0 300

19. 3.54, 1.29

Problem Set 24.2, page 999

1. 4 outcomes: HH, HT, TH, TT (H = Head, T = Tail)

3. $6^2 = 36$ outcomes (1, 1), (1, 2), $\cdots$, (6, 6)

5. Infinitely many outcomes $S, S^cS, S^cS^cS, \cdots$ (S = "Six")

7. The space of ordered triples of nonnegative numbers

9. The space of ordered pairs of numbers

11. Yes

13. $E = \{S, S^cS, S^cS^cS\}, E^c = \{S^cS^cS^cS, S^cS^cS^cS^cS, \cdots\}$ (S = "Six")

Problem Set 24.3, page 1005

1. (a) $0.9^3 = 72.9\%$, (b) $\frac{90}{100} \cdot \frac{89}{99} \cdot \frac{88}{98} = 72.65\%$

3. $\frac{490}{500} \cdot \frac{489}{499} \cdot \frac{488}{498} \cdot \frac{487}{497} \cdot \frac{486}{496} = 90.35\%$

5. $1 - \frac{1}{25} = 0.96$

7. $1 - 0.75^2 = 0.4375 < 0.5$

9. $P(MMM) + P(MMFM) + P(MFMM) + P(FMMM) = \frac{1}{8} + 3 \cdot \frac{1}{16} = \frac{5}{16}$

11. $\frac{6}{36} + \frac{27}{36} - \frac{3}{36} = \frac{30}{36}$ by Theorem 3, or by counting outcomes
13. $0.08 + 0.04 - 0.08 \cdot 0.04 = 11.68\%$
15. $0.95^4 = 81.5\%$
17. $1 - 0.97^4 = 11.5\%$

Problem Set 24.4, page 1010

3. In 40 320 ways
5. $\binom{20}{3} = 1140$
7. 210, 70, 112, 28
9. $9!/(2!3!4!) = 1260$. *Ans.* 1/1260
11. $\binom{52}{13} = 635\,013\,559\,600$
13. 1/84, 5/21
15. 676 000

Problem Set 24.5, page 1015

1. $k = 1/55$ by (6)
3. $k = 1/8$ by (10)
5. No because of (6)
7. $1 - P(X \leqq 3) = 0.5$
9. $P(X > 1200) = \int_{1.2}^{2} 6[0.25 - (x - 1.5)^2]\,dx = 0.896$. *Ans.* $0.896^3 = 72\%$
11. $k = 2.5$; 50%
13. $k = 1.1565$; 26.9%
17. $X > b$, $X \geqq b$, $X < c$, $X \leqq c$, etc.

Problem Set 24.6, page 1019

1. 2/3, 1/18
3. 3.5, 2.917
5. 4, 16/3
7. \$643.50
9. $\mu = 1/\theta = 25$; $P = 20.2\%$
11. $\frac{1}{2}, \frac{1}{20}, (X - \frac{1}{2})\sqrt{20}$
13. 750, 1, 0.002
15. $15c - 500c^3 = 0.97$, $c = 0.0855$

Problem Set 24.7, page 1025

1. 0.0625, 0.25, 0.9375, 0.9375
3. 64%
5. 0.265
7. $f(x) = 0.5^x e^{-0.5}/x!$, $f(0) + f(1) = e^{-0.5}(1.0 + 0.5) = 0.91$. *Ans.* 9%
9. $1 - e^{-0.2} = 18\%$
11. $0.99^{100} = 36.6\%$
13. $\frac{120}{286}, \frac{135}{286}, \frac{30}{286}, \frac{1}{286}$

Problem Set 24.8, page 1031

1. 0.1587, 0.6306, 0.5, 0.4950
3. 17.29, 10.71, 19.152
5. 16%
7. 31.1%, 95.5%
9. About 23
11. About 58%
13. $t = 1084$ hours

Problem Set 24.9, page 1040

1. 1/8, 3/16, 3/8
3. 2/9, 2/9, 1/2
5. $f_2(y) = 1/(\beta_2 - \alpha_2)$ if $\alpha_2 < y < \beta_2$ and 0 elsewhere
7. 27.45 mm, 0.38 mm
9. 25.26 cm, 0.0078 cm

13. Independent, $f_1(x) = 0.1e^{-0.1x}$ if $x > 0$, $f_2(y) = 0.1e^{-0.1y}$ if $y > 0$, 36.8%

15. 50%

17. No

Chapter 24 Review Questions and Problems, page 1041

21. $Q_L = 22.3$, $Q_M = 23.3$, $Q_U = 23.5$

23. $\bar{x} = 22.89$, $s = 1.028$, $s^2 = 1.056$

25. H, TH, TTH, etc.

27. $f(0) = 0.80816$, $f(1) = 0.18367$, $f(2) = 0.00816$

29. Always $B \subseteq A \cup B$. If also $A \subseteq B$, then $B = A \cup B$, etc.

31. 7/3, 8/9

33. 118.019, 1.98, 1.65%

35. 0, 2

37. $\mu = 100/30$

39. 16%, 2.3% (see Fig. 520 in Sec. 24.8)

Problem Set 25.2, page 1048

3. $l = p^k(1 - p)^{n-k}$, $\hat{p} = k/n$, k = number of successes in n trials

5. 11/20

7. $l = f(x)$, $\partial(\ln l)/\partial p = 1/p - (x - 1)/(1 - p) = 0$, $\hat{p} = 1/x$

9. $\hat{\mu} = \bar{x}$

11. $\hat{\theta} = n/\Sigma x_j = 1/\bar{x}$

13. $\hat{\theta} = 1$

15. Variability larger than perhaps expected

Problem Set 25.3, page 1057

1. $\text{CONF}_{0.95}\,\{37.47 \leqq \mu \leqq 43.87\}$

3. Shorter by a factor $\sqrt{2}$

5. 4, 16

7. Cf. Example 2. $n = 166$

9. $\text{CONF}_{0.99}\{20.07 \leqq \mu \leqq 20.33\}$

11. $\text{CONF}_{0.99}\{63.71 \leqq \mu \leqq 66.29\}$

13. $c = 1.96$, $\bar{x} = 87$, $s^2 = 87 \cdot 413/500 = 71.86$, $k \approx cs/\sqrt{n} = 0.743$, $\text{CONF}_{0.95}\,\{86 \leqq \mu \leqq 88\}$, $\text{CONF}_{0.95}\{0.17 \leqq p \leqq 0.18\}$

15. $\text{CONF}_{0.95}\{0.00045 \leqq \sigma^2 \leqq 0.00131\}$

17. $\text{CONF}_{0.95}\{0.73 \leqq \sigma^2 \leqq 5.19\}$

19. $\text{CONF}_{0.95}\{23 \leqq \sigma^2 \leqq 548\}$. Hence a larger sample would be desirable.

21. Normal distributions, means −27, 81, 133, variances 16, 144, 400

23. $Z = X + Y$ is normal with mean 105 and variance 1.25.
Ans. $P(104 \leqq Z \leqq 106) = 63\%$

Problem Set 25.4, page 1067

1. $t = \sqrt{7}(0.286 - 0)/4.31 = 0.18 < c = 1.94$; do not reject the hypothesis.

3. $c = 6090 > 6019$; do not reject the hypothesis.

5. $\sigma^2/n = 1$, $c = 28.36$; do not reject the hypothesis.

7. $\mu < 28.76$ or $\mu > 31.24$

9. Alternative $\mu \neq 1000$, $t = \sqrt{20}(996 - 1000)/5 = -3.58 < c = -2.09$ (Table A9, 19 degrees of freedom). Reject the hypothesis $\mu = 1000$ g.

11. Test $\mu = 0$ against $\mu \neq 0$. $t = 2.11 < c = 2.36$ (7 degrees of freedom). Do not reject the hypothesis.

13. $\alpha = 5\%$, $c = 16.92 > 9 \cdot 0.5^2/0.4^2 = 14.06$; do not reject hypothesis.

15. $t_0 = \sqrt{10 \cdot 9 \cdot 17/19}\,(21.8 - 20.2)/\sqrt{9 \cdot 0.6^2 + 8 \cdot 0.5^2} = 6.3 > c = 1.74$ (17 degrees of freedom). Reject the hypothesis and assert that B is better.

17. $v_0 = 50/30 = 1.67 < c = 2.59$ [(9, 15) degrees of freedom]; do not reject the hypothesis.

Problem Set 25.5, page 1071

1. $\text{LCL} = 1 - 2.58 \cdot 0.03/\sqrt{6} = 0.968$, $\text{UCL} = 1.032$
3. $n = 10$
5. Choose 4 times the original sample size (why?).
7. $2.58\sqrt{0.024}/\sqrt{2} = 0.283$, $\text{UCL} = 27.783$, $\text{LCL} = 27.217$
11. In 30% (5%) of the cases, approximately
13. $\text{UCL} = np + 3\sqrt{np(1-p)}$, $\text{CL} = np$, $\text{LCL} = np - 3\sqrt{np(1-p)}$
15. $\text{CL} = \mu = 2.5$, $\text{UCL} = \mu + 3\sqrt{\mu} = 7.2$, $\text{LCL} = \mu - 3\sqrt{\mu}$ is negative in (b) and we set $\text{LCL} = 0$.

Problem Set 25.6, page 1076

1. 0.9825, 0.9384, 0.4060
3. 0.8187, 0.6703, 0.1353
5. $P(A;\ \theta) \approx e^{-30\theta}(1 + 30\theta)$
7. $P(A;\ \theta) \approx e^{-50\theta}$
9. 19.5%, 14.7%
11. $(1-\theta)^5$, $(1-\theta)^5 + 5\theta(1-\theta)^4$
13. Because n is finite
15. $\Phi((9 - 12 + \frac{1}{2})/\sqrt{12(1 - 0.12)}) = 0.22$ (if $c = 9$)
17. $(1 - \frac{1}{2})^3 + 3 \cdot \frac{1}{2}(1 - \frac{1}{2})^2 = \frac{1}{2}$

Problem Set 25.7, page 1079

1. $\chi_0^2 = (30-50)^2/50 + (70-50)^2/50 = 16 > c = 3.84$; no
3. 41
5. $\chi_0^2 = 2.33 < c = 11.07$. Yes
7. $e_j = np_j = 370/5 = 74$, $\chi_0^2 = 984/74 = 13.3$, $c = 9.49$. Reject the hypothesis.
9. $\chi_0^2 = 1 < 3.84$; yes
13. Combining the results for $x = 10, 11, 12$, we have $K - r - 1 = 9$ ($r = 1$ since we estimated the mean, $\frac{10094}{2608} \approx 3.87$). $\chi_0^2 = 12.98 < c = 16.92$. Do not reject.
15. $\chi_0^2 = 49/20 + 49/60 = 3.27 < c = 3.84$ (1 degree of freedom, $\alpha = 5\%$), which supports the claim.
17. 42 even digits, accept.

Problem Set 25.8, page 1082

3. $(\frac{1}{2})^{18}(1 + 18 + 153 + 816) = 0.0038$
5. Hypothesis: A and B are equally good. Then the probability of at least 7 trials favorable to A is $\frac{1}{2}^{18} + 8 \cdot \frac{1}{2}^{18} = 3.5\%$. Reject the hypothesis.
7. Hypothesis $\mu = 0$. Alternative $\mu > 0$, $\bar{x} = 1.58$, $t = \sqrt{10} \cdot 1.58/1.23 = 4.06 > c = 1.83$ ($\alpha = 5\%$). Hypothesis rejected.
9. $\bar{x} = 9.67$, $s = 11.87$, $t_0 = 9.67/(11.87/\sqrt{15}) = 3.15 > c = 1.76$ ($\alpha = 5\%$). Hypothesis rejected.
11. Consider $y_j = x_j - \tilde{\mu}_0$.
13. $P(T \leqq 2) = 0.1\%$ from Table A12. Reject.

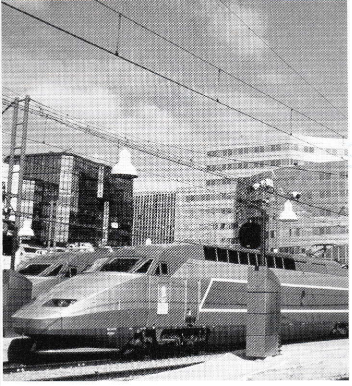
15. $P(T \leqq 15) = 10.8\%$. Do not reject.
17. $P(T \leqq 2) = 2.8\%$. Reject.

Problem Set 25.9, page 1091

1. $y = 1.9 + x$ **3.** $y = 6.7407 + 3.068x$ **5.** $y = 4 + 4.8x$, 172 ft
7. $y = -1146 + 4.32x$ **9.** $y = 0.5932 + 0.1138x$, $R = 1/0.1138$
11. $q_0 = 76$, $K = 2.36\sqrt{76/(7 \cdot 944)} = 0.253$, $\text{CONF}_{0.95}\{-1.58 \leqq \kappa_1 \leqq -1.06\}$
13. $3s_x^{\ 2} = 500$, $3s_{xy} = 33.5$, $k_1 = 0.067$, $3s_y^{\ 2} = 2.268$, $q_0 = 0.023$, $K = 0.021$
$\text{CONF}_{0.95}\{0.046 \leqq \kappa_1 \leqq 0.088\}$

Chapter 25 Review Questions and Problems, page 1092

21. $\hat{\mu} = 5.33$, $\tilde{\sigma}^2 = 1.722$ **23.** It will double.
25. $\text{CONF}_{0.99}\{19.1 \leqq \mu \leqq 33.7\}$ **27.** $\text{CONF}_{0.95}\{0.726 \leqq \mu \leqq 0.751\}$
29. $\text{CONF}_{0.95}\{1.373 \leqq \mu \leqq 1.451\}$ **31.** $\text{CONF}_{0.99}\{0.05 \leqq \sigma^2 \leqq 10\}$
33. $c = 14.74 > 14.5$; reject μ_0. **35.** $\Phi\left(\dfrac{14.74 - 14.40}{\sqrt{0.025}}\right) = 0.9842$
37. $30.14/3.8 = 7.93 < 8.25$. Reject.
39. $v_0 = 2.5 < 6.0$ [(9, 4) degrees of freedom]; accept the hypothesis.
41. Decrease by a factor $\sqrt{2}$. By a factor $2.58/1.96 = 1.32$.
43. 0.9953, 0.9825, 0.9384, etc. **45.** $y = 1.70 + 0.55x$



APPENDIX 3
Auxiliary Material

A3.1 Formulas for Special Functions

For tables of numeric values, see Appendix 5.

Exponential function e^x (Fig. 544)

$$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71353$$

(1) $$e^x e^y = e^{x+y}, \qquad e^x/e^y = e^{x-y}, \qquad (e^x)^y = e^{xy}$$

Natural logarithm (Fig. 545)

(2) $$\ln(xy) = \ln x + \ln y, \qquad \ln(x/y) = \ln x - \ln y, \qquad \ln(x^a) = a \ln x$$

$\ln x$ is the inverse of e^x, and $e^{\ln x} = x$, $e^{-\ln x} = e^{\ln(1/x)} = 1/x$.

Logarithm of base ten $\log_{10} x$ or simply $\log x$

(3) $$\log x = M \ln x, \qquad M = \log e = 0.43429\ 44819\ 03251\ 82765\ 11289\ 18917$$

(4) $$\ln x = \frac{1}{M} \log x, \qquad \frac{1}{M} = \ln 10 = 2.30258\ 50929\ 94045\ 68401\ 79914\ 54684$$

$\log x$ is the inverse of 10^x, and $10^{\log x} = x$, $10^{-\log x} = 1/x$.

Sine and cosine functions (Figs. 546, 547). In calculus, angles are measured in radians, so that $\sin x$ and $\cos x$ have period 2π.

$\sin x$ is odd, $\sin(-x) = -\sin x$, and $\cos x$ is even, $\cos(-x) = \cos x$.

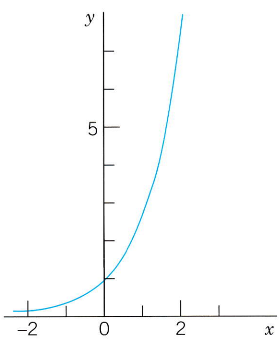


Fig. 544. Exponential function e^x

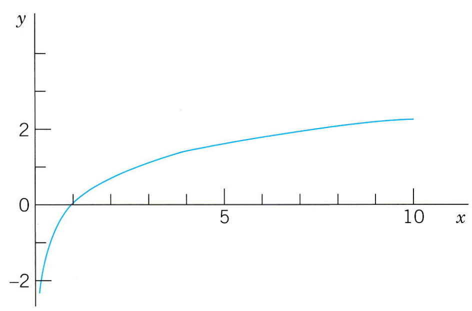


Fig. 545. Natural logarithm $\ln x$

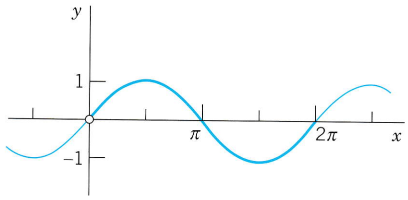


Fig. 546. sin *x*

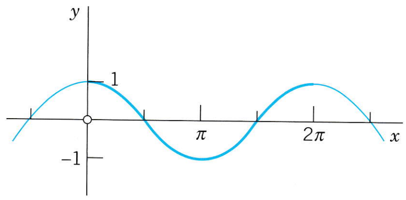


Fig. 547. cos *x*

$$1° = 0.01745\ 32925\ 19943 \text{ radian}$$

$$1 \text{ radian} = 57°\ 17'\ 44.80625''$$

$$= 57.29577\ 95131°$$

(5) $$\sin^2 x + \cos^2 x = 1$$

(6) $$\begin{cases} \sin(x+y) = \sin x \cos y + \cos x \sin y \\ \sin(x-y) = \sin x \cos y - \cos x \sin y \\ \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \cos(x-y) = \cos x \cos y + \sin x \sin y \end{cases}$$

(7) $$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x$$

(8) $$\begin{cases} \sin x = \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right) \\ \cos x = \sin\left(x + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - x\right) \end{cases}$$

(9) $$\sin(\pi - x) = \sin x, \qquad \cos(\pi - x) = -\cos x$$

(10) $$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

(11) $$\begin{cases} \sin x \sin y = \frac{1}{2}[-\cos(x+y) + \cos(x-y)] \\ \cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)] \\ \sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)] \end{cases}$$

(12) $$\begin{cases} \sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2} \\ \cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2} \\ \cos v - \cos u = 2 \sin \frac{u+v}{2} \sin \frac{u-v}{2} \end{cases}$$

(13) $$A \cos x + B \sin x = \sqrt{A^2 + B^2} \cos(x \pm \delta), \qquad \tan \delta = \frac{\sin \delta}{\cos \delta} = \mp \frac{B}{A}$$

(14) $$A \cos x + B \sin x = \sqrt{A^2 + B^2} \sin(x \pm \delta), \qquad \tan \delta = \frac{\sin \delta}{\cos \delta} = \pm \frac{A}{B}$$

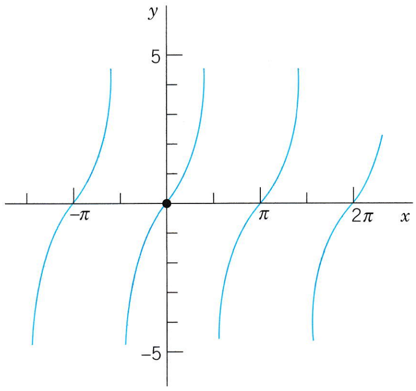


Fig. 548. tan x

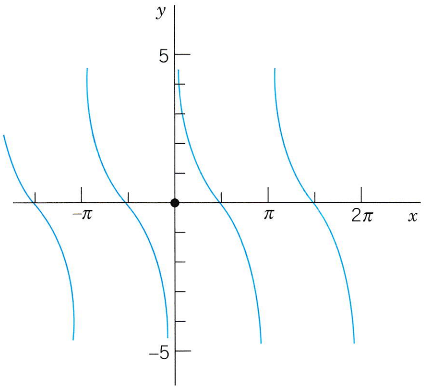


Fig. 549. cot x

Tangent, cotangent, secant, cosecant (Figs. 548, 549)

$$(15)\quad \tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}$$

$$(16)\qquad \tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \qquad \tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Hyperbolic functions (hyperbolic sine sinh x, etc.; Figs. 550, 551)

$$(17)\qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

$$(18)\qquad \tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$(19)\qquad \cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x}$$

$$(20)\qquad \cosh^2 x - \sinh^2 x = 1$$

$$(21)\qquad \sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1), \qquad \cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1)$$

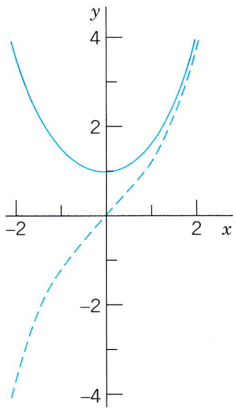


Fig. 550. sinh x (dashed) and cosh x

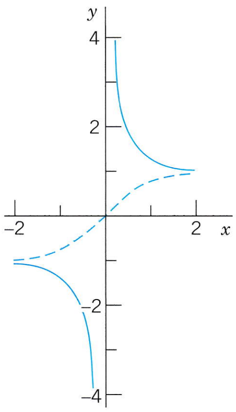


Fig. 551. tanh x (dashed) and coth x

(22) $$\begin{cases} \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \\ \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \end{cases}$$

(23) $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Gamma function (Fig. 552 and Table A2 in App. 5). The gamma function $\Gamma(\alpha)$ is defined by the integral

(24) $$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt \qquad (\alpha > 0)$$

which is meaningful only if $\alpha > 0$ (or, if we consider complex α, for those α whose real part is positive). Integration by parts gives the important *functional relation of the gamma function,*

(25) $$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

From (24) we readily have $\Gamma(1) = 1$; hence if α is a positive integer, say k, then by repeated application of (25) we obtain

(26) $$\Gamma(k + 1) = k! \qquad (k = 0, 1, \cdots).$$

This shows that *the gamma function can be regarded as a generalization of the elementary factorial function.* [Sometimes the notation $(\alpha - 1)!$ is used for $\Gamma(\alpha)$, even for noninteger values of α, and the gamma function is also known as the **factorial function.**]

By repeated application of (25) we obtain

$$\Gamma(\alpha) = \frac{\Gamma(\alpha + 1)}{\alpha} = \frac{\Gamma(\alpha + 2)}{\alpha(\alpha + 1)} = \cdots = \frac{\Gamma(\alpha + k + 1)}{\alpha(\alpha + 1)(\alpha + 2)\cdots(\alpha + k)}$$

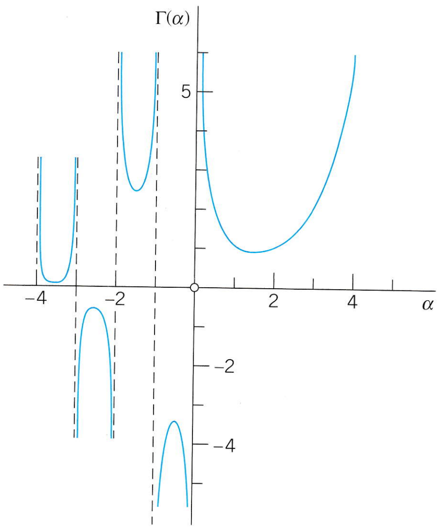


Fig. 552. Gamma function

and we may use this relation

$$\Gamma(\alpha) = \frac{\Gamma(\alpha + k + 1)}{\alpha(\alpha + 1) \cdots (\alpha + k)} \qquad (\alpha \neq 0, -1, -2, \cdots) \tag{27}$$

for defining the gamma function for negative α ($\neq -1, -2, \cdots$), choosing for k the smallest integer such that $\alpha + k + 1 > 0$. *Together with* (24), *this then gives a definition of* $\Gamma(\alpha)$ *for all* α *not equal to zero or a negative integer* (Fig. 552).

It can be shown that the gamma function may also be represented as the limit of a product, namely, by the formula

$$\Gamma(\alpha) = \lim_{n \to \infty} \frac{n!\, n^{\alpha}}{\alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)} \qquad (\alpha \neq 0, -1, \cdots). \tag{28}$$

From (27) or (28) we see that, for complex α, the gamma function $\Gamma(\alpha)$ is a meromorphic function with simple poles at $\alpha = 0, -1, -2, \cdots$.

An approximation of the gamma function for large positive α is given by the **Stirling formula**

$$\Gamma(\alpha + 1) \approx \sqrt{2\pi\alpha} \left(\frac{\alpha}{e}\right)^{\alpha} \tag{29}$$

where e is the base of the natural logarithm. We finally mention the special value

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{30}$$

Incomplete gamma functions

$$P(\alpha, x) = \int_0^x e^{-t} t^{\alpha - 1}\, dt, \qquad Q(\alpha, x) = \int_x^{\infty} e^{-t} t^{\alpha - 1}\, dt \qquad (\alpha > 0) \tag{31}$$

$$\Gamma(\alpha) = P(\alpha, x) + Q(\alpha, x) \tag{32}$$

Beta function

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\, dt \qquad (x > 0, y > 0) \tag{33}$$

Representation in terms of gamma functions:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \tag{34}$$

Error function (Fig. 553 and Table A4 in App. 5)

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{35}$$

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + - \cdots \right) \tag{36}$$

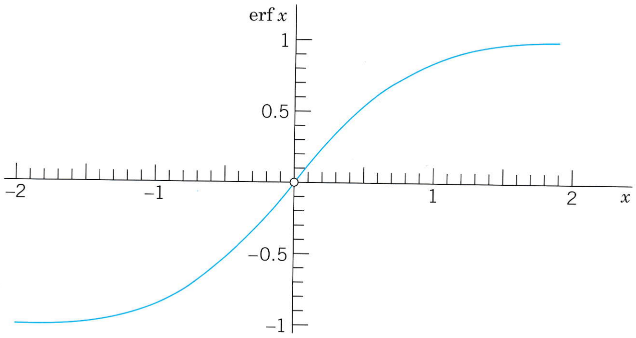


Fig. 553. Error function

erf (∞) = 1, *complementary error function*

(37) $$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Fresnel integrals[1] (Fig. 554)

(38) $$C(x) = \int_0^x \cos(t^2)\, dt, \qquad S(x) = \int_0^x \sin(t^2)\, dt$$

$C(\infty) = \sqrt{\pi/8}$, $S(\infty) = \sqrt{\pi/8}$, *complementary functions*

(39) $$c(x) = \sqrt{\frac{\pi}{8}} - C(x) = \int_x^\infty \cos(t^2)\, dt$$
$$s(x) = \sqrt{\frac{\pi}{8}} - S(x) = \int_x^\infty \sin(t^2)\, dt$$

Sine integral (Fig. 555 and Table A4 in App. 5)

(40) $$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

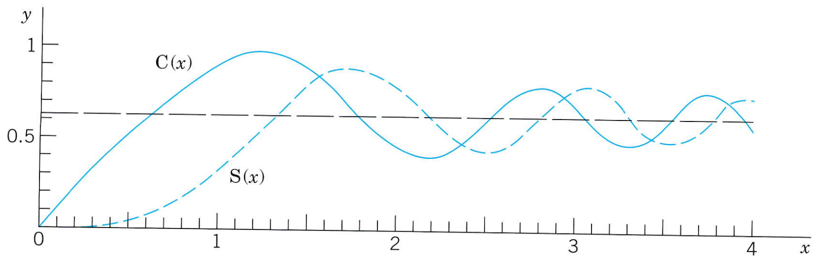


Fig. 554. Fresnel integrals

[1]AUGUSTIN FRESNEL (1788–1827), French physicist and mathematician. For tables see Ref. [GR1].

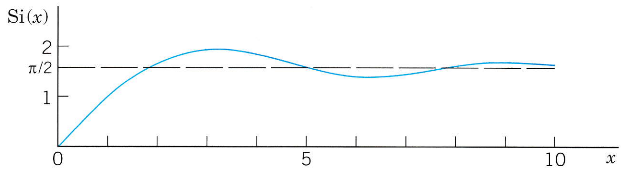


Fig. 555. Sine integral

$\mathrm{Si}(\infty) = \pi/2$, *complementary function*

$$\text{(41)} \qquad \mathrm{si}(x) = \frac{\pi}{2} - \mathrm{Si}(x) = \int_x^\infty \frac{\sin t}{t}\, dt$$

Cosine integral (Table A4 in App. 5)

$$\text{(42)} \qquad \mathrm{ci}(x) = \int_x^\infty \frac{\cos t}{t}\, dt \qquad (x > 0)$$

Exponential integral

$$\text{(43)} \qquad \mathrm{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt \qquad (x > 0)$$

Logarithmic integral

$$\text{(44)} \qquad \mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}$$

A3.2 Partial Derivatives

For differentiation formulas, see inside of front cover.

Let $z = f(x, y)$ be a real function of two independent real variables, x and y. If we keep y constant, say, $y = y_1$, and think of x as a variable, then $f(x, y_1)$ depends on x alone. If the derivative of $f(x, y_1)$ with respect to x for a value $x = x_1$ exists, then the value of this derivative is called the **partial derivative** of $f(x, y)$ *with respect to x at the point* (x_1, y_1) and is denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_1, y_1)} \qquad \text{or by} \qquad \left.\frac{\partial z}{\partial x}\right|_{(x_1, y_1)}$$

Other notations are

$$f_x(x_1, y_1) \qquad \text{and} \qquad z_x(x_1, y_1);$$

these may be used when subscripts are not used for another purpose and there is no danger of confusion.

We thus have, by the definition of the derivative,

$$\text{(1)} \qquad \left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1)} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x, y_1) - f(x_1, y_1)}{\Delta x} .$$

The partial derivative of $z = f(x, y)$ with respect to y is defined similarly; we now keep x constant, say, equal to x_1, and differentiate $f(x_1, y)$ with respect to y. Thus

$$\text{(2)} \qquad \left.\frac{\partial f}{\partial y}\right|_{(x_1,y_1)} = \left.\frac{\partial z}{\partial y}\right|_{(x_1,y_1)} = \lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y} .$$

Other notations are $f_y(x_1, y_1)$ and $z_y(x_1, y_1)$.

It is clear that the values of those two partial derivatives will in general depend on the point (x_1, y_1). Hence the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ at a variable point (x, y) are functions of x and y. The function $\partial z/\partial x$ is obtained as in ordinary calculus by differentiating $z = f(x, y)$ with respect to x, ***treating y as a constant,*** and $\partial z/\partial y$ is obtained by differentiating z with respect to y, ***treating x as a constant.***

EXAMPLE 1 Let $z = f(x, y) = x^2y + x \sin y$. Then

$$\frac{\partial f}{\partial x} = 2xy + \sin y, \qquad \frac{\partial f}{\partial y} = x^2 + x \cos y.$$

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ of a function $z = f(x, y)$ have a very simple **geometric interpretation.** The function $z = f(x, y)$ can be represented by a surface in space. The equation $y = y_1$ then represents a vertical plane intersecting the surface in a curve, and the partial derivative $\partial z/\partial x$ at a point (x_1, y_1) is the slope of the tangent (that is, $\tan \alpha$ where α is the angle shown in Fig. 556) to the curve. Similarly, the partial derivative $\partial z/\partial y$ at (x_1, y_1) is the slope of the tangent to the curve $x = x_1$ on the surface $z = f(x, y)$ at (x_1, y_1).

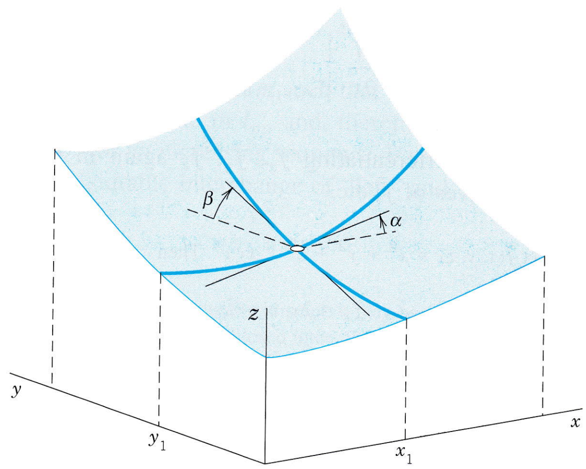


Fig. 556. Geometrical interpretation of first partial derivatives

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are called *first partial derivatives* or *partial derivatives of first order.* By differentiating these derivatives once more, we obtain the four *second partial derivatives* (or *partial derivatives of second order*)[2]

(3)
$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}\\ \frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}\\ \frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}\\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}.\end{aligned}$$

It can be shown that if all the derivatives concerned are continuous, then the two mixed partial derivatives are equal, so that the order of differentiation does not matter (see Ref. [GR4] in App. 1), that is,

(4)
$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}\,.$$

EXAMPLE 2 For the function in Example 1.

$$f_{xx} = 2y, \qquad f_{xy} = 2x + \cos y = f_{yx}, \qquad f_{yy} = -x \sin y.$$

By differentiating the second partial derivatives again with respect to x and y, respectively, we obtain the *third partial derivatives* or *partial derivatives of the third order* of f, etc.

If we consider a function $f(x, y, z)$ of ***three independent variables,*** then we have the three first partial derivatives $f_x(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$. Here f_x is obtained by differentiating f with respect to x, ***treating both y and z as constants.*** Thus, analogous to (1), we now have

$$\left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1,z_1)} = \lim_{\Delta x\to 0} \frac{f(x_1+\Delta x, y_1, z_1) - f(x_1, y_1, z_1)}{\Delta x},$$

etc. By differentiating f_x, f_y, f_z again in this fashion we obtain the second partial derivatives of f, etc.

EXAMPLE 3 Let $f(x, y, z) = x^2 + y^2 + z^2 + xy\,e^z$. Then

$$\begin{aligned}&f_x = 2x + y\,e^z, && f_y = 2y + x\,e^z, && f_z = 2z + xy\,e^z,\\ &f_{xx} = 2, && f_{xy} = f_{yx} = e^z, && f_{xz} = f_{zx} = y\,e^z,\\ &f_{yy} = 2, && f_{yz} = f_{zy} = x\,e^z, && f_{zz} = 2 + xy\,e^z.\end{aligned}$$

[2] CAUTION! In the subscript notation the subscripts are written in the order in which we differentiate, whereas in the "∂" notation the order is opposite.

A3.3 Sequences and Series

See also Chap. 15.

Monotone Real Sequences

We call a real sequence $x_1, x_2, \cdots, x_n, \cdots$ a **monotone sequence** if it is either **monotone increasing,** that is,

$$x_1 \leqq x_2 \leqq x_3 \leqq \cdots$$

or **monotone decreasing,** that is,

$$x_1 \geqq x_2 \geqq x_3 \geqq \cdots.$$

We call $x_1, x_2, \cdots$ a **bounded sequence** if there is a positive constant K such that $|x_n| < K$ for all n.

THEOREM 1 *If a real sequence is bounded and monotone, it converges.*

PROOF Let $x_1, x_2, \cdots$ be a bounded monotone increasing sequence. Then its terms are smaller than some number B and, since $x_1 \leqq x_n$ for all n, they lie in the interval $x_1 \leqq x_n \leqq B$, which will be denoted by I_0. We bisect I_0; that is, we subdivide it into two parts of equal length. If the right half (together with its endpoints) contains terms of the sequence, we denote it by I_1. If it does not contain terms of the sequence, then the left half of I_0 (together with its endpoints) is called I_1. This is the first step.

In the second step we bisect I_1, select one half by the same rule, and call it I_2, and so on (see Fig. 557 on p. A70).

In this way we obtain shorter and shorter intervals $I_0, I_1, I_2, \cdots$ with the following properties. Each I_m contains all I_n for $n > m$. No term of the sequence lies to the right of I_m, and, since the sequence is monotone increasing, all x_n with n greater than some number N lie in I_m; of course, N will depend on m, in general. The lengths of the I_m approach zero as m approaches infinity. Hence there is precisely one number, call it L, that lies in all those intervals,[3] and we may now easily prove that the sequence is convergent with the limit L.

In fact, given an $\epsilon > 0$, we choose an m such that the length of I_m is less than ϵ. Then L and all the x_n with $n > N(m)$ lie in I_m, and, therefore, $|x_n - L| < \epsilon$ for all those n. This completes the proof for an increasing sequence. For a decreasing sequence the proof is the same, except for a suitable interchange of "left" and "right" in the construction of those intervals. ■

[3]This statement seems to be obvious, but actually it is not; it may be regarded as an axiom of the real number system in the following form. Let $J_1, J_2, \cdots$ be closed intervals such that each J_m contains all J_n with $n > m$, and the lengths of the J_m approach zero as m approaches infinity. Then there is precisely one real number that is contained in all those intervals. This is the so-called **Cantor–Dedekind axiom,** named after the German mathematicians GEORG CANTOR (1845–1918), the creator of set theory, and RICHARD DEDEKIND (1831–1916), known for his fundamental work in number theory. For further details see Ref. [GR2] in App. 1. (An interval I is said to be **closed** if its two endpoints are regarded as points belonging to I. It is said to be **open** if the endpoints are not regarded as points of I.)

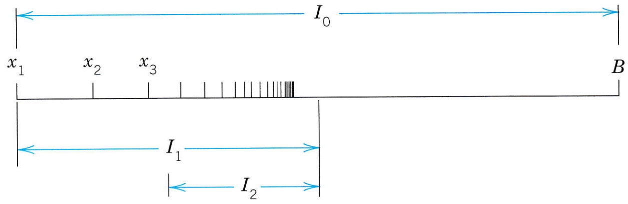


Fig. 557. Proof of Theorem 1

Real Series

THEOREM 2

Leibniz Test for Real Series

Let $x_1, x_2, \cdots$ be real and monotone decreasing to zero, that is,

$$\text{(1)} \qquad \text{(a)} \quad x_1 \geqq x_2 \geqq x_3 \geqq \cdots, \qquad \text{(b)} \quad \lim_{m\to\infty} x_m = 0.$$

Then the series with terms of alternating signs

$$x_1 - x_2 + x_3 - x_4 + - \cdots$$

converges, and for the remainder R_n after the nth term we have the estimate

$$\text{(2)} \qquad |R_n| \leqq x_{n+1}.$$

PROOF Let s_n be the nth partial sum of the series. Then, because of (1a),

$$s_1 = x_1, \qquad s_2 = x_1 - x_2 \leqq s_1,$$

$$s_3 = s_2 + x_3 \geqq s_2, \qquad s_3 = s_1 - (x_2 - x_3) \leqq s_1,$$

so that $s_2 \leqq s_3 \leqq s_1$. Proceeding in this fashion, we conclude that (Fig. 558)

$$\text{(3)} \qquad s_1 \geqq s_3 \geqq s_5 \geqq \cdots \geqq s_6 \geqq s_4 \geqq s_2$$

which shows that the odd partial sums form a bounded monotone sequence, and so do the even partial sums. Hence, by Theorem 1, both sequences converge, say,

$$\lim_{n\to\infty} s_{2n+1} = s, \qquad \lim_{n\to\infty} s_{2n} = s^*.$$

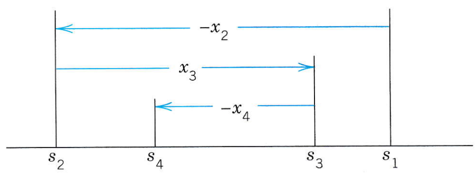


Fig. 558. Proof of the Leibniz test

Now, since $s_{2n+1} - s_{2n} = x_{2n+1}$, we readily see that (lb) implies

$$s - s^* = \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} x_{2n+1} = 0.$$

Hence $s^* = s$, and the series converges with the sum s.

We prove the estimate (2) for the remainder. Since $s_n \to s$, it follows from (3) that

$$s_{2n+1} \geqq s \geqq s_{2n} \qquad \text{and also} \qquad s_{2n-1} \geqq s \geqq s_{2n}.$$

By subtracting s_{2n} and s_{2n-1}, respectively, we obtain

$$s_{2n+1} - s_{2n} \geqq s - s_{2n} \geqq 0, \qquad 0 \geqq s - s_{2n-1} \geqq s_{2n} - s_{2n-1}.$$

In these inequalities, the first expression is equal to x_{2n+1}, the last is equal to $-x_{2n}$, and the expressions between the inequality signs are the remainders R_{2n} and R_{2n-1}. Thus the inequalities may be written

$$x_{2n+1} \geqq R_{2n} \geqq 0, \qquad 0 \geqq R_{2n-1} \geqq -x_{2n}$$

and we see that they imply (2). This completes the proof. ■

A3.4 Grad, Div, Curl, ∇^2 in Curvilinear Coordinates

To simplify formulas we write Cartesian coordinates $x = x_1$, $y = x_2$, $z = x_3$. We denote curvilinear coordinates by q_1, q_2, q_3. Through each point P there pass three coordinate surfaces $q_1 = const$, $q_2 = const$, $q_3 = const$. They intersect along coordinate curves. We assume the three coordinate curves through P to be **orthogonal** (perpendicular to each other). We write coordinate transformations as

$$(1) \qquad x_1 = x_1(q_1, q_2, q_3), \qquad x_2 = x_2(q_1, q_2, q_3), \qquad x_3 = x_3(q_1, q_2, q_3).$$

Corresponding transformations of grad, div, curl, and ∇^2 can all be written by using

$$(2) \qquad h_j^2 = \sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial q_j}\right)^2.$$

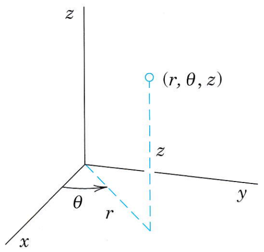
Next to Cartesian coordinates, most important are **cylindrical coordinates** $q_1 = r$, $q_2 = \theta$, $q_3 = z$ (Fig. 559a on p. A72) defined by

$$(3) \quad x_1 = q_1 \cos q_2 = r \cos\theta, \qquad x_2 = q_1 \sin q_2 = r \sin\theta, \qquad x_3 = q_3 = z$$

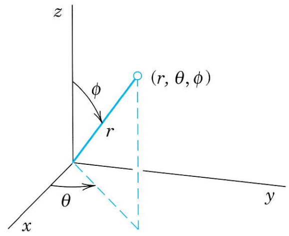
and **spherical coordinates** $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$ (Fig. 559b) defined by[4]

$$(4) \quad \begin{aligned} &x_1 = q_1 \cos q_2 \sin q_3 = r \cos\theta \sin\phi, \qquad x_2 = q_1 \sin q_2 \sin q_3 = r \sin\theta \sin\phi \\ &x_3 = q_1 \cos q_3 = r \cos\phi. \end{aligned}$$

[4]This is the notation used in calculus and in many other books. It is logical since in it, θ plays the same role as in polar coordinates. **CAUTION!** Some books interchange the roles of θ and ϕ.



(a) Cylindrical coordinates



(b) Spherical coordinates

Fig. 559. Special curvilinear coordinates

In addition to the general formulas for any orthogonal coordinates q_1, q_2, q_3, we shall give additional formulas for these important special cases.

Linear Element *ds*. In Cartesian coordinates,

$$ds^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2 \qquad \text{(Sec. 9.5).}$$

For the q-coordinates,

$$ds^2 = h_1{}^2\, dq_1{}^2 + h_2{}^2\, dq_2{}^2 + h_3{}^2\, dq_3{}^2. \tag{5}$$

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2 \qquad \text{(Cylindrical coordinates).} \tag{5$'$}$$

For polar coordinates set $dz^2 = 0$.

$$ds^2 = dr^2 + r^2 \sin^2\phi\, d\theta^2 + r^2\, d\phi^2 \qquad \text{(Spherical coordinates).} \tag{5$''$}$$

Gradient. grad $f = \nabla f = [f_{x_1}, \ f_{x_2}, \ f_{x_3}]$ (partial derivatives; Sec. 9.7). In the q-system, with **u, v, w** denoting unit vectors in the positive directions of the q_1, q_2, q_3 coordinate curves, respectively,

$$\text{grad } f = \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{u} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{v} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{w} \tag{6}$$

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\mathbf{u} + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{v} + \frac{\partial f}{\partial z}\mathbf{w} \qquad \text{(Cylindrical coordinates)} \tag{6$'$}$$

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\mathbf{u} + \frac{1}{r \sin\phi}\frac{\partial f}{\partial \theta}\mathbf{v} + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{w} \qquad \text{(Spherical coordinates).} \tag{6$''$}$$

Divergence div $\mathbf{F} = \nabla \cdot \mathbf{F} = (F_1)_{x_1} + (F_2)_{x_2} + (F_3)_{x_3}$ ($\mathbf{F} = [F_1, F_2, F_3]$, Sec. 9.8);

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 F_1) + \frac{\partial}{\partial q_2}(h_3 h_1 F_2) + \frac{\partial}{\partial q_3}(h_1 h_2 F_3)\right] \tag{7}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{\partial \theta} + \frac{\partial F_3}{\partial z} \qquad \text{(Cylindrical coordinates)} \tag{7$'$}$$

(7″) $$\operatorname{div}\mathbf{F} = \nabla\bullet\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_1) + \frac{1}{r\sin\phi}\frac{\partial F_2}{\partial\theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, F_3)$$ (Spherical coordinates).

Laplacian $\nabla^2 f = \nabla\bullet\nabla f = \operatorname{div}(\operatorname{grad} f) = f_{x_1x_1} + f_{x_2x_2} + f_{x_3x_3}$ (Sec. 9.8):

(8) $$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

(8′) $$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$ (Cylindrical coordinates)

(8″) $$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\cot\phi}{r^2}\frac{\partial f}{\partial\phi}$$ (Spherical coordinates).

Curl (Sec. 9.9):

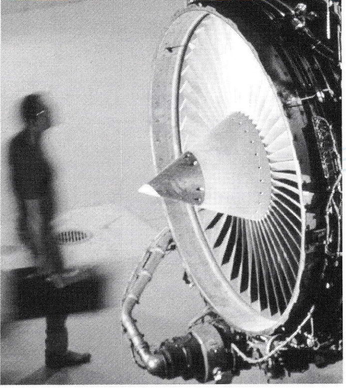
(9) $$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{u} & h_2\mathbf{v} & h_3\mathbf{w} \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}.$$

For cylindrical coordinates we have in (9) (as in the previous formulas)

$$h_1 = h_r = 1, \qquad h_2 = h_\theta = q_1 = r, \qquad h_3 = h_z = 1$$

and for spherical coordinates we have

$$h_1 = h_r = 1, \qquad h_2 = h_\theta = q_1\sin q_3 = r\sin\phi, \qquad h_3 = h_\phi = q_1 = r.$$



APPENDIX 4

Additional Proofs

Section 2.6, page 73

PROOF OF THEOREM 1 Uniqueness[1]

Assuming that the problem consisting of the ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

and the two initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1 \tag{3}$$

has two solutions $y_1(x)$ and $y_2(x)$ on the interval I in the theorem, we show that their difference

$$y(x) = y_1(x) - y_2(x)$$

is identically zero on I; then $y_1 \equiv y_2$ on I, which implies uniqueness.

Since (1) is homogeneous and linear, y is a solution of that ODE on I, and since y_1 and y_2 satisfy the same initial conditions, y satisfies the conditions

$$y(x_0) = 0, \qquad y'(x_0) = 0. \tag{10}$$

We consider the function

$$z(x) = y(x)^2 + y'(x)^2$$

and its derivative

$$z' = 2yy' + 2y'y''.$$

From the ODE we have

$$y'' = -py' - qy.$$

By substituting this in the expression for z' we obtain

$$z' = 2yy' - 2py'^2 - 2qyy'. \tag{11}$$

Now, since y and y' are real,

$$(y \pm y')^2 = y^2 \pm 2yy' + y'^2 \geqq 0.$$

[1]This proof was suggested by my colleague, Prof. A. D. Ziebur. In this proof we use formula numbers that have not yet been used in Sec. 2.6.

From this and the definition of z we obtain the two inequalities

(12) (a) $2yy' \leqq y^2 + y'^2 = z,$ (b) $-2yy' \leqq y^2 + y'^2 = z.$

From (12b) we have $2yy' \geqq -z$. Together, $|2yy'] \leqq z$. For the last term in (11) we now obtain

$$-2qyy' \leqq |-2qyy'| = |q||2yy'| \leqq |q|z.$$

Using this result as well as $-p \leqq |p|$ and applying (12a) to the term $2yy'$ in (11), we find

$$z' \leqq z + 2|p|y'^2 + |q|z.$$

Since $y'^2 \leqq y^2 + y'^2 = z$, from this we obtain

$$z' \leqq (1 + 2|p| + |q|)z$$

or, denoting the function in parentheses by h,

(13a) $$z' \leqq hz \qquad \text{for all } x \text{ on } I.$$

Similarly, from (11) and (12) it follows that

(13b) $$\begin{aligned} -z' &= -2yy' + 2py'^2 + 2qyy' \\ &\leqq z + 2|p|z + |q|z = hz. \end{aligned}$$

The inequalities (13a) and (13b) are equivalent to the inequalities

(14) $$z' - hz \leqq 0, \qquad z' + hz \geqq 0.$$

Integrating factors for the two expressions on the left are

$$F_1 = e^{-\int h(x)\,dx} \qquad \text{and} \qquad F_2 = e^{\int h(x)\,dx}.$$

The integrals in the exponents exist because h is continuous. Since F_1 and F_2 are positive, we thus have from (14)

$$F_1(z' - hz) = (F_1 z)' \leqq 0 \qquad \text{and} \qquad F_2(z' + hz) = (F_2 z)' \geqq 0.$$

This means that $F_1 z$ is nonincreasing and $F_2 z$ is nondecreasing on I. Since $z(x_0) = 0$ by (10), when $x \leqq x_0$ we thus obtain

$$F_1 z \geqq (F_1 z)_{x_0} = 0, \qquad F_2 z \leqq (F_2 z)_{x_0} = 0$$

and similarly, when $x \geqq x_0$,

$$F_1 z \leqq 0, \qquad F_2 z \geqq 0.$$

Dividing by F_1 and F_2 and noting that these functions are positive, we altogether have

$$z \leqq 0, \qquad z \geqq 0 \qquad \text{for all } x \text{ on } I.$$

This implies that $z = y^2 + y'^2 \equiv 0$ on I. Hence $y \equiv 0$ or $y_1 \equiv y_2$ on I. ■

Section 5.4, pages 184

PROOF OF THEOREM 2 Frobenius Method. Basis of Solutions. Three Cases

The formula numbers in this proof are the same as in the text of Sec. 5.4. An additional formula not appearing in Sec. 5.4 will be called (A) (see below).

The ODE in Theorem 2 is

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0, \tag{1}$$

where $b(x)$ and $c(x)$ are analytic functions. We can write it

$$x^2 y'' + x b(x) y' + c(x) y = 0. \tag{1'}$$

The indicial equation of (1) is

$$r(r-1) + b_0 r + c_0 = 0. \tag{4}$$

The roots r_1, r_2 of this quadratic equation determine the general form of a basis of solutions of (1), and there are three possible cases as follows.

Case 1. Distinct Roots not Differing by an Integer. A first solution of (1) is of the form

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5}$$

and can be determined as in the power series method. For a proof that in this case, the ODE (1) has a second independent solution of the form

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots), \tag{6}$$

see Ref. [A11] listed in App. 1.

Case 2. Double Root. The indicial equation (4) has a double root r if and only if $(b_0 - 1)^2 - 4c_0 = 0$, and then $r = \frac{1}{2}(1 - b_0)$. A first solution

$$y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots), \qquad r = \tfrac{1}{2}(1 - b_0), \tag{7}$$

can be determined as in Case 1. We show that a second independent solution is of the form

$$y_2(x) = y_1(x) \ln x + x^r(A_1 x + A_2 x^2 + \cdots) \qquad (x > 0). \tag{8}$$

We use the method of reduction of order (see Sec. 2.1), that is, we determine $u(x)$ such that $y_2(x) = u(x) y_1(x)$ is a solution of (1). By inserting this and the derivatives

$$y_2' = u' y_1 + u y_1', \qquad y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

into the ODE (1′) we obtain

$$x^2(u'' y_1 + 2u' y_1' + u y_1'') + x b(u' y_1 + u y_1') + c u y_1 = 0.$$

Since y_1 is a solution of (1′), the sum of the terms involving u is zero, and this equation reduces to

$$x^2 y_1 u'' + 2x^2 y_1' u' + xby_1 u' = 0.$$

By dividing by $x^2 y_1$ and inserting the power series for b we obtain

$$u'' + \left(2\frac{y_1'}{y_1} + \frac{b_0}{x} + \cdots\right)u' = 0.$$

Here and in the following the dots designate terms that are constant or involve positive powers of x. Now from (7) if follows that

$$\frac{y_1'}{y_1} = \frac{x^{r-1}[ra_0 + (r+1)a_1 x + \cdots]}{x^r[a_0 + a_1 x + \cdots]}$$

$$= \frac{1}{x}\left(\frac{ra_0 + (r+1)a_1 x + \cdots}{a_0 + a_1 x + \cdots}\right) = \frac{r}{x} + \cdots.$$

Hence the previous equation can be written

$$\text{(A)} \qquad u'' + \left(\frac{2r + b_0}{x} + \cdots\right)u' = 0.$$

Since $r = (1 - b_0)/2$, the term $(2r + b_0)/x$ equals $1/x$, and by dividing by u' we thus have

$$\frac{u''}{u'} = -\frac{1}{x} + \cdots.$$

By integration we obtain $\ln u' = -\ln x + \cdots$, hence $u' = (1/x)e^{(\cdots)}$. Expanding the exponential function in powers of x and integrating once more, we see that u is of the form

$$u = \ln x + k_1 x + k_2 x^2 + \cdots.$$

Inserting this into $y_2 = uy_1$, we obtain for y_2 a representation of the form (8).

Case 3. Roots Differing by an Integer. We write $r_1 = r$ and $r_2 = r - p$ where p is a ***positive*** integer. A first solution

$$\text{(9)} \qquad y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

can be determined as in Cases 1 and 2. We show that a second independent solution is of the form

$$\text{(10)} \qquad y_2(x) = ky_1(x)\ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots)$$

where we may have $k \neq 0$ or $k = 0$. As in Case 2 we set $y_2 = uy_1$. The first steps are literally as in Case 2 and give Eq. (A),

$$u'' + \left(\frac{2r + b_0}{x} + \cdots\right)u' = 0.$$

Now by elementary algebra, the coefficient $b_0 - 1$ of r in (4) equals minus the sum of the roots,

$$b_0 - 1 = -(r_1 + r_2) = -(r + r - p) = -2r + p.$$

Hence $2r + b_0 = p + 1$, and division by u' gives

$$\frac{u''}{u'} = -\left(\frac{p+1}{x} + \cdots\right).$$

The further steps are as in Case 2. Integrating, we find

$$\ln u' = -(p+1)\ln x + \cdots, \qquad \text{thus} \qquad u' = x^{-(p+1)}e^{(\cdots)}$$

where dots stand for some series of nonnegative integer powers of x. By expanding the exponential function as before we obtain a series of the form

$$u' = \frac{1}{x^{p+1}} + \frac{k_1}{x^p} + \cdots + \frac{k_{p-1}}{x^2} + \frac{k_p}{x} + k_{p+1} + k_{p+2}x + \cdots.$$

We integrate once more. Writing the resulting logarithmic term first, we get

$$u = k_p \ln x + \left(-\frac{1}{px^p} - \cdots - \frac{k_{p-1}}{x} + k_{p+1}x + \cdots\right).$$

Hence, by (9) we get for $y_2 = uy_1$ the formula

$$y_2 = k_p y_1 \ln x + x^{r_1-p}\left(-\frac{1}{p} - \cdots - k_{p-1}x^{p-1} + \cdots\right)(a_0 + a_1x + \cdots).$$

But this is of the form (10) with $k = k_p$ since $r_1 - p = r_2$ and the product of the two series involves nonnegative integer powers of x only. ■

Section 5.7, page 205

THEOREM

> **Reality of Eigenvalues**
>
> *If p, q, r, and p' in the Sturm–Liouville equation* (1) *of Sec.* 5.7 *are real-valued and continuous on the interval $a \leqq x \leqq b$ and $r(x) > 0$ throughout that interval (or $r(x) < 0$ throughout that interval), then all the eigenvalues of the Sturm–Liouville problem* (1), (2), *Sec.* 5.7, *are real.*

PROOF Let $\lambda = \alpha + i\beta$ be an eigenvalue of the problem and let

$$y(x) = u(x) + iv(x)$$

be a corresponding eigenfunction; here α, β, u, and v are real. Substituting this into (1), Sec. 5.7, we have

$$(pu' + ipv')' + (q + \alpha r + i\beta r)(u + iv) = 0.$$

This complex equation is equivalent to the following pair of equations for the real and the imaginary parts:

$$(pu')' + (q + \alpha r)u - \beta rv = 0$$
$$(pv')' + (q + \alpha r)v + \beta ru = 0.$$

Multiplying the first equation by v, the second by $-u$ and adding, we get

$$-\beta(u^2 + v^2)r = u(pv')' - v(pu')'$$
$$= [(pv')u - (pu')v]'.$$

The expression in brackets is continuous on $a \leqq x \leqq b$, for reasons similar to those in the proof of Theorem 1, Sec. 5.7. Integrating over x from a to b, we thus obtain

$$-\beta \int_a^b (u^2 + v^2)r\,dx = \Big[p(uv' - u'v)\Big]_a^b .$$

Because of the boundary conditions the right side is zero; this is as in that proof. Since y is an eigenfunction, $u^2 + v^2 \not\equiv 0$. Since y and r are continuous and $r > 0$ (or $r < 0$) on the interval $a \leqq x \leqq b$, the integral on the left is not zero. Hence, $\beta = 0$, which means that $\lambda = \alpha$ is real. This completes the proof. ■

Section 7.7, page 308

THEOREM

Determinants

The definition of a determinant

$$(7) \qquad D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

as given in Sec. 7.7 is unambiguous, that is, it yields the same value of D no matter which rows or columns we choose in developings.

PROOF In this proof we shall use formula numbers not yet used in Sec. 7.7.

We shall prove first that *the same value is obtained no matter which* **row** *is chosen.*

The proof is by induction. The statement is true for a second-order determinant, for which the developments by the first row $a_{11}a_{22} + a_{12}(-a_{21})$ and by the second row $a_{21}(-a_{12}) + a_{22}a_{11}$ give the same value $a_{11}a_{22} - a_{12}a_{21}$. Assuming the statement to be true for an $(n - 1)$st-order determinant, we prove that it is true for an nth-order determinant.

For this purpose we expand D in terms of each of two arbitrary rows, say, the ith and the jth, and compare the results. Without loss of generality let us assume $i < j$.

First Expansion. We expand D by the ith row. A typical term in this expansion is

$$a_{ik}C_{ik} = a_{ik} \cdot (-1)^{i+k}M_{ik}. \tag{19}$$

The minor M_{ik} of a_{ik} in D is an $(n - 1)$st-order determinant. By the induction hypothesis we may expand it by any row. We expand it by the row corresponding to the jth row of D. This row contains the entries a_{jl} $(l \neq k)$. It is the $(j - 1)$st row of M_{ik}, because M_{ik} does not contain entries of the ith row of D, and $i < j$. We have to distinguish between two cases as follows.

Case I. If $l < k$, then the entry a_{jl} belongs to the lth column of M_{ik} (see Fig. 560). Hence the term involving a_{jl} in this expansion is

$$a_{jl} \cdot (\text{cofactor of } a_{jl} \text{ in } M_{ik}) = a_{jl} \cdot (-1)^{(j-1)+l}M_{ikjl} \tag{20}$$

where M_{ikjl} is the minor of a_{jl} in M_{ik}. Since this minor is obtained from M_{ik} by deleting the row and column of a_{jl}, it is obtained from D by deleting the ith and jth rows and the kth and lth columns of D. We insert the expansions of the M_{ik} into that of D. Then it follows from (19) and (20) that the terms of the resulting representation of D are of the form

$$a_{ik}a_{jl} \cdot (-1)^{b}M_{ikjl} \qquad (l < k) \tag{21a}$$

where

$$b = i + k + j + l - 1.$$

Case II. If $l > k$, the only difference is that then a_{jl} belongs to the $(l - 1)$st column of M_{ik}, because M_{ik} does not contain entries of the kth column of D, and $k < l$. This causes an additional minus sign in (20), and, instead of (21a), we therefore obtain

$$-a_{ik}a_{jl} \cdot (-1)^{b}M_{ikjl} \qquad (l > k) \tag{21b}$$

where b is the same as before.

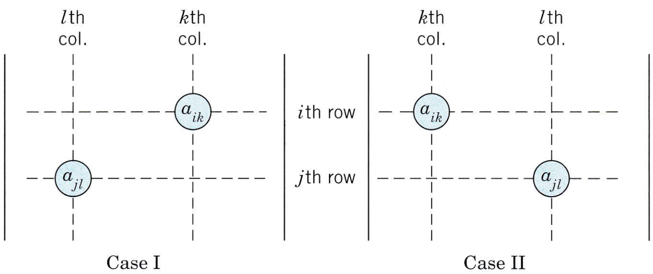


Fig. 560. Cases I and II of the two expansions of D

Second Expansion. We now expand D at first by the jth row. A typical term in this expansion is

$$a_{jl}C_{jl} = a_{jl} \cdot (-1)^{j+l}M_{jl}. \tag{22}$$

By the induction hypothesis we may expand the minor M_{jl} of a_{jl} in D by its ith row, which corresponds to the ith row of D, since $j > i$.

Case I. If $k > l$, the entry a_{ik} in that row belongs to the $(k-1)$st column of M_{jl}, because M_{jl} does not contain entries of the lth column of D, and $l < k$ (see Fig. 560). Hence the term involving a_{ik} in this expansion is

$$a_{ik} \cdot (\text{cofactor of } a_{ik} \text{ in } M_{jl}) = a_{ik} \cdot (-1)^{i+(k-1)}M_{ikjl}, \tag{23}$$

where the minor M_{ikjl} of a_{ik} in M_{jl} is obtained by deleting the ith and jth rows and the kth and lth columns of D [and is, therefore, identical with M_{ikjl} in (20), so that our notation is consistent]. We insert the expansions of the M_{jl} into that of D. It follows from (22) and (23) that this yields a representation whose terms are identical with those given by (21a) when $l < k$.

Case II. If $k < l$, then a_{ik} belongs to the kth column of M_{jl}, we obtain an additional minus sign, and the result agrees with that characterized by (21b).

We have shown that the two expansions of D consist of the same terms, and this proves our statement concerning rows.

The proof of the statement concerning ***columns*** is quite similar; if we expand D in terms of two arbitrary columns, say, the kth and the lth, we find that the general term involving $a_{jl}a_{ik}$ is exactly the same as before. This proves that not only all column expansions of D yield the same value, but also that their common value is equal to the common value of the row expansions of D.

This completes the proof and shows that *our definition of an nth-order determinant is unambiguous.* ■

Section 9.3, page 377

PROOF OF FORMULA (2)

We prove that in right-handed Cartesian coordinates, the vector product

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = [a_1, \quad a_2, \quad a_3] \times [b_1, \quad b_2, \quad b_3]$$

has the components

$$v_1 = a_2b_3 - a_3b_2, \qquad v_2 = a_3b_1 - a_1b_3, \qquad v_3 = a_1b_2 - a_2b_1. \tag{2}$$

We need only consider the case $\mathbf{v} \neq \mathbf{0}$. Since $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, Theorem 1 in Sec. 9.2 gives $\mathbf{a} \cdot \mathbf{v} = 0$ and $\mathbf{b} \cdot \mathbf{v} = 0$; in components [see (2), Sec. 9.2],

$$\begin{aligned} a_1v_1 + a_2v_2 + a_3v_3 &= 0 \\ b_1v_1 + b_2v_2 + b_3v_3 &= 0. \end{aligned} \tag{3}$$

Multiplying the first equation by b_3, the last by a_3, and subtracting, we obtain

$$(a_3b_1 - a_1b_3)v_1 = (a_2b_3 - a_3b_2)v_2.$$

Multiplying the first equation by b_1, the last by a_1, and subtracting, we obtain

$$(a_1b_2 - a_2b_1)v_2 = (a_3b_1 - a_1b_3)v_3.$$

We can easily verify that these two equations are satisfied by

$$\text{(4)} \qquad v_1 = c(a_2b_3 - a_3b_2), \qquad v_2 = c(a_3b_1 - a_1b_3), \qquad v_3 = c(a_1b_2 - a_2b_1)$$

where c is a constant. The reader may verify by inserting that (4) also satisfies (3). Now each of the equations in (3) represents a plane through the origin in $v_1v_2v_3$-space. The vectors **a** and **b** are normal vectors of these planes (see Example 6 in Sec. 9.2). Since $\mathbf{v} \neq \mathbf{0}$, these vectors are not parallel and the two planes do not coincide. Hence their intersection is a straight line L through the origin. Since (4) is a solution of (3) and, for varying c, represents a straight line, we conclude that (4) represents L, and every solution of (3) must be of the form (4). In particular, the components of **v** must be of this form, where c is to be determined. From (4) we obtain

$$|\mathbf{v}|^2 = v_1{}^2 + v_2{}^2 + v_3{}^2 = c^2[(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2].$$

This can be written

$$|\mathbf{v}|^2 = c^2[(a_1{}^2 + a_2{}^2 + a_3{}^2)(b_1{}^2 + b_2{}^2 + b_3{}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2],$$

as can be verified by performing the indicated multiplications in both formulas and comparing. Using (2) in Sec. 9.2, we thus have

$$|\mathbf{v}|^2 = c^2[(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2].$$

By comparing this with formula (12) in Team Project 24 of Problem Set 9.3 we conclude that $c = \pm 1$.

We show that $c = +1$. This can be done as follows.

If we change the lengths and directions of **a** and **b** continuously and so that at the end $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$ (Fig. 186a in Sec. 9.3), then **v** will change its length and direction continuously, and at the end, $\mathbf{v} = \mathbf{i} \times \mathbf{j} = \mathbf{k}$. Obviously we may effect the change so that both **a** and **b** remain different from the zero vector and are not parallel at any instant. Then **v** is never equal to the zero vector, and since the change is continuous and c can only assume the values $+1$ or -1, it follows that at the end c must have the same value as before. Now at the end $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$, $\mathbf{v} = \mathbf{k}$ and, therefore, $a_1 = 1$, $b_2 = 1$, $v_3 = 1$, and the other components in (4) are zero. Hence from (4) we see that $v_3 = c = +1$. This proves Theorem 1.

For a left-handed coordinate system, $\mathbf{i} \times \mathbf{j} = -\mathbf{k}$ (see Fig. 186b in Sec. 9.3), resulting in $c = -1$. This proves the statement right after formula (2). ■

Section 9.9, page 416

PROOF OF THE INVARIANCE OF THE CURL

This proof will follow from two theorems (A and B), which we prove first.

THEOREM A

Transformation Law for Vector Components

For any vector **v** *the components* v_1, v_2, v_3 *and* $v_1{}^*, v_2{}^*, v_3{}^*$ *in any two systems of Cartesian coordinates* x_1, x_3, x_3 *and* $x_1{}^*, x_2{}^*, x_3{}^*$, *respectively, are related by*

$$(1)\qquad \begin{aligned} v_1{}^* &= c_{11}v_1 + c_{12}v_2 + c_{13}v_3 \\ v_2{}^* &= c_{21}v_1 + c_{22}v_2 + c_{23}v_3 \\ v_3{}^* &= c_{31}v_1 + c_{32}v_2 + c_{33}v_3, \end{aligned}$$

and conversely

$$(2)\qquad \begin{aligned} v_1 &= c_{11}v_1{}^* + c_{21}v_2{}^* + c_{31}v_3{}^* \\ v_2 &= c_{12}v_1{}^* + c_{22}v_2{}^* + c_{32}v_3{}^* \\ v_3 &= c_{13}v_1{}^* + c_{23}v_2{}^* + c_{33}v_3{}^* \end{aligned}$$

with coefficients

$$(3)\qquad \begin{matrix} c_{11} = \mathbf{i}^*\bullet\mathbf{i} & c_{12} = \mathbf{i}^*\bullet\mathbf{j} & c_{13} = \mathbf{i}^*\bullet\mathbf{k} \\ c_{21} = \mathbf{j}^*\bullet\mathbf{i} & c_{22} = \mathbf{j}^*\bullet\mathbf{j} & c_{23} = \mathbf{j}^*\bullet\mathbf{k} \\ c_{31} = \mathbf{k}^*\bullet\mathbf{i} & c_{32} = \mathbf{k}^*\bullet\mathbf{j} & c_{33} = \mathbf{k}^*\bullet\mathbf{k} \end{matrix}$$

satisfying

$$(4)\qquad \sum_{j=1}^{3} c_{kj}c_{mj} = \delta_{km} \qquad (k, m = 1, 2, 3),$$

where the **Kronecker delta**[2] *is given by*

$$\delta_{km} = \begin{cases} 0 & (k \neq m) \\ 1 & (k = m) \end{cases}$$

and **i, j, k** *and* **i*, j*, k*** *denote the unit vectors in the positive* x_1-, x_2-, x_3- *and* $x_1{}^*$-, $x_2{}^*$-, $x_3{}^*$-*directions, respectively.*

---

[2]LEOPOLD KRONECKER (1823–1891), German mathematician at Berlin, who made important contributions to algebra, group theory, and number theory.

We shall keep our discussion completely independent of Chap. 7, but readers familiar with matrices should recognize that we are dealing with **orthogonal transformations and matrices** and that our present theorem follows from Theorem 2 in Sec. 8.3.

PROOF The representation of $\mathbf{v}$ in the two systems are

$$\text{(5)} \quad \text{(a)} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \qquad \text{(b)} \quad \mathbf{v} = v_1^*\mathbf{i}^* + v_2^*\mathbf{j}^* + v_3^*\mathbf{k}^*.$$

Since $\mathbf{i}^* \bullet \mathbf{i}^* = 1$, $\mathbf{i}^* \bullet \mathbf{j}^* = 0$, $\mathbf{i}^* \bullet \mathbf{k}^* = 0$, we get from (5b) simply $\mathbf{i}^* \bullet \mathbf{v} = v_1^*$ and from this and (5a)

$$v_1^* = \mathbf{i}^* \bullet \mathbf{v} = \mathbf{i}^* \bullet v_1\mathbf{i} + \mathbf{i}^* \bullet v_2\mathbf{j} + \mathbf{i}^* \bullet v_3\mathbf{k} = v_1\mathbf{i}^* \bullet \mathbf{i} + v_2\mathbf{i}^* \bullet \mathbf{j} + v_3\mathbf{i}^* \bullet \mathbf{k}.$$

Because of (3), this is the first formula in (1), and the other two formulas are obtained similarly, by considering $\mathbf{j}^* \bullet \mathbf{v}$ and then $\mathbf{k}^* \bullet \mathbf{v}$. Formula (2) follows by the same idea, taking $\mathbf{i} \bullet \mathbf{v} = v_1$ from (5a) and then from (5b) and (3)

$$v_1 = \mathbf{i} \bullet \mathbf{v} = v_1^*\mathbf{i} \bullet \mathbf{i}^* + v_2^*\mathbf{i} \bullet \mathbf{j}^* + v_3^*\mathbf{i} \bullet \mathbf{k}^* = c_{11}v_1^* + c_{21}v_2^* + c_{31}v_3^*,$$

and similarly for the other two components.

We prove (4). We can write (1) and (2) briefly as

$$\text{(6)} \qquad \text{(a)} \quad v_j = \sum_{m=1}^{3} c_{mj}v_m^*, \qquad \text{(b)} \quad v_k^* = \sum_{j=1}^{3} c_{kj}v_j.$$

Substituting v_j into v_k^*, we get

$$v_k^* = \sum_{j=1}^{3} c_{kj} \sum_{m=1}^{3} c_{mj}v_m^* = \sum_{m=1}^{3} v_m^* \left(\sum_{j=1}^{3} c_{kj}c_{mj} \right),$$

where $k = 1, 2, 3$. Taking $k = 1$, we have

$$v_1^* = v_1^* \left(\sum_{j=1}^{3} c_{1j}c_{1j} \right) + v_2^* \left(\sum_{j=1}^{3} c_{1j}c_{2j} \right) + v_3^* \left(\sum_{j=1}^{3} c_{1j}c_{3j} \right).$$

For this to hold for *every* vector $\mathbf{v}$, the first sum must be 1 and the other two sums 0. This proves (4) with $k = 1$ for $m = 1, 2, 3$. Taking $k = 2$ and then $k = 3$, we obtain (4) with $k = 2$ and 3, for $m = 1, 2, 3$. ■

THEOREM B

Transformation Law for Cartesian Coordinates

The transformation of any Cartesian $x_1x_2x_3$-coordinate system into any other Cartesian $x_1^*x_2^*x_3^*$-coordinate system is of the form

$$\text{(7)} \qquad x_m^* = \sum_{j=1}^{3} c_{mj}x_j + b_m, \quad m = 1, 2, 3,$$

with coefficients (3) *and constants* b_1, b_2, b_3; *conversely,*

$$\text{(8)} \qquad x_k = \sum_{n=1}^{3} c_{nk}x_n^* + \tilde{b}_k, \qquad k = 1, 2, 3.$$

Theorem B follows from Theorem A by noting that the most general transformation of a Cartesian coordinate system into another such system may be decomposed into a transformation of the type just considered and a translation; and under a translation, corresponding coordinates differ merely by a constant.

PROOF OF THE INVARIANCE OF THE CURL

We write again x_1, x_2, x_3 instead of x, y, z, and similarly x_1^*, x_2^*, x_3^* for other Cartesian coordinates, assuming that both systems are right-handed. Let a_1, a_2, a_3 denote the components of curl **v** in the $x_1x_2x_3$-coordinates, as given by (1), Sec. 9.9, with

$$x = x_1, \qquad y = x_2, \qquad z = x_3.$$

Similarly, let a_1^*, a_2^*, a_3^* denote the components of curl **v** in the $x_1^*x_2^*x_3^*$-coordinate system. We prove that the length and direction of curl **v** are independent of the particular choice of Cartesian coordinates, as asserted. We do this by showing that the components of curl **v** satisfy the transformation law (2), which is characteristic of vector components. We consider a_1. We use (6a), and then the chain rule for functions of several variables (Sec. 9.6). This gives

$$a_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = \sum_{m=1}^{3} \left(c_{m3} \frac{\partial v_m^*}{\partial x_2} - c_{m2} \frac{\partial v_m^*}{\partial x_3} \right)$$

$$= \sum_{m=1}^{3} \sum_{j=1}^{3} \left(c_{m3} \frac{\partial v_m^*}{\partial x_j^*} \frac{\partial x_j^*}{\partial x_2} - c_{m2} \frac{\partial v_m^*}{\partial x_j^*} \frac{\partial x_j^*}{\partial x_3} \right).$$

From this and (7) we obtain

$$a_1 = \sum_{m=1}^{3} \sum_{j=1}^{3} (c_{m3}c_{j2} - c_{m2}c_{j3}) \frac{\partial v_m^*}{\partial x_j^*}$$

$$= (c_{33}c_{22} - c_{32}c_{23}) \left(\frac{\partial v_3^*}{\partial x_2^*} - \frac{\partial v_2^*}{\partial x_3^*} \right) + \cdots$$

$$= (c_{33}c_{22} - c_{32}c_{23})a_1^* + (c_{13}c_{32} - c_{12}c_{33})a_2^* + (c_{23}c_{12} - c_{22}c_{13})a_3^*.$$

Note what we did. The double sum had $3 \times 3 = 9$ terms, 3 of which were zero (when $m = j$), and the remaining 6 terms we combined in pairs as we needed them in getting a_1^*, a_2^*, a_3^*.

We now use (3), Lagrange's identity (see Team Project 24 in Problem Set 9.3) and $\mathbf{k}^* \times \mathbf{j}^* = -\mathbf{i}^*$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$. Then

$$c_{33}c_{22} - c_{32}c_{23} = (\mathbf{k}^* \cdot \mathbf{k})(\mathbf{j}^* \cdot \mathbf{j}) - (\mathbf{k}^* \cdot \mathbf{j})(\mathbf{j}^* \cdot \mathbf{k})$$

$$= (\mathbf{k}^* \times \mathbf{j}^*) \cdot (\mathbf{k} \times \mathbf{j}) = \mathbf{i}^* \cdot \mathbf{i} = c_{11}, \qquad \text{etc.}$$

Hence $a_1 = c_{11}a_1^* + c_{21}a_2^* + c_{31}a_3^*$. This is of the form of the first formula in (2) in Theorem A, and the other two formulas of the form (2) are obtained similarly. This proves the theorem for right-handed systems. If the $x_1x_2x_3$-coordinates are left-handed, then $\mathbf{k} \times \mathbf{j} = +\mathbf{i}$, but then there is a minus sign in front of the determinant in (1), Sec. 9.9. ■

Section 10.2, pages 426–427

PROOF OF THEOREM 1, PART (b) We prove that if

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz) \tag{1}$$

with continuous F_1, F_2, F_3 in a domain D is independent of path in D, then $F = \operatorname{grad} f$ in D for some f; in components

$$F_1 = \frac{\partial f}{\partial x}, \qquad F_2 = \frac{\partial f}{\partial y}, \qquad F_3 = \frac{\partial f}{\partial z}. \tag{2$'$}$$

We choose any fixed A: (x_0, y_0, z_0) in D and any B: (x, y, z) in D and define f by

$$f(x, y, z) = f_0 + \int_A^B (F_1\,dx^* + F_2\,dy^* + F_3\,dz^*) \tag{3}$$

with any constant f_0 and any path from A to B in D. Since A is fixed and we have independence of path, the integral depends only on the coordinates x, y, z, so that (3) defines a function $f(x, y, z)$ in D. We show that $\mathbf{F} = \operatorname{grad} f$ with this f, beginning with the first of the three relations (2$'$). Because of independence of path we may integrate from A to B_1: (x_1, y, z) and then parallel to the x-axis along the segment B_1B in Fig. 561 with B_1 chosen so that the whole segment lies in D. Then

$$f(x, y, z) = f_0 + \int_A^{B_1} (F_1\,dx^* + F_2\,dy^* + F_3\,dz^*) + \int_{B_1}^{B} (F_1\,dx^* + F_2\,dy^* + F_3\,dz^*).$$

We now take the partial derivative with respect to x on both sides. On the left we get $\partial f/\partial x$. We show that on the right we get F_1. The derivative of the first integral is zero because A: (x_0, y_0, z_0) and B_1: (x_1, y, z) do not depend on x. We consider the second integral. Since on the segment B_1B, both y and z are constant, the terms $F_2\,dy^*$ and

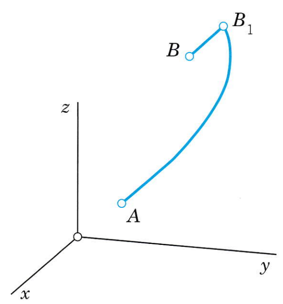


Fig. 561. Proof of Theorem 1

$F_3\,dz^*$ do not contribute to the derivative of the integral. The remaining part can be written as a definite integral,

$$\int_{B_1}^{B} F_1\,dx^* = \int_{x_1}^{x} F_1(x^*, y, z)\,dx^*.$$

Hence its partial derivative with respect to x is $F_1(x, y, z)$, and the first of the relations (2′) is proved. The other two formulas in (2′) follow by the same argument. ■

Section 13.4, page 620

PROOF OF THEOREM 2 Cauchy–Riemann Equations

We prove that Cauchy–Riemann equations

$$(1) \qquad u_x = v_y, \qquad u_y = -v_x$$

are sufficient for a complex function $f(z) = u(x, y) + iv(x, y)$ to be analytic; precisely, *if the real part u and the imaginary part v of $f(z)$ satisfy* (1) *in a domain D in the complex plane and if the partial derivatives in* (1) *are* ***continuous*** *in D, then $f(z)$ is analytic in D.*

In this proof we write $\Delta z = \Delta x + i\Delta y$ and $\Delta f = f(z + \Delta z) - f(z)$. The idea of proof is as follows.

(a) We express Δf in terms of first partial derivatives of u and v, by applying the mean value theorem of Sec. 9.6.

(b) We get rid of partial derivatives with respect to y by applying the Cauchy–Riemann equations.

(c) We let Δz approach zero and show that then $\Delta f/\Delta z$ as obtained approaches a limit, which is equal to $u_x + iv_x$, the right side of (4) in Sec. 13.4, regardless of the way of approach to zero.

(a) Let P: (x, y) be any fixed point in D. Since D is a domain, it contains a neighborhood of P. We can choose a point Q: $(x + \Delta x, y + \Delta y)$ in this neighborhood such that the straight-line segment PQ is in D. Because of our continuity assumptions we may apply the mean value theorem in Sec. 9.6. This yields

$$u(x + \Delta x, y + \Delta y) - u(x, y) = (\Delta x)u_x(M_1) + (\Delta y)u_y(M_1)$$
$$v(x + \Delta x, y + \Delta y) - v(x, y) = (\Delta x)v_x(M_2) + (\Delta y)v_y(M_2)$$

where M_1 and M_2 ($\neq M_1$ in general!) are suitable points on that segment. The first line is Re Δf and the second is Im Δf, so that

$$\Delta f = (\Delta x)u_x(M_1) + (\Delta y)u_y(M_1) + i\big[(\Delta x)v_x(M_2) + (\Delta y)v_y(M_2)\big].$$

(b) $u_y = -v_x$ and $v_y = u_x$ by the Cauchy–Riemann equations, so that

$$\Delta f = (\Delta x)u_x(M_1) - (\Delta y)v_x(M_1) + i\big[(\Delta x)v_x(M_2) + (\Delta y)u_x(M_2)\big].$$

Also $\Delta z = \Delta x + i\Delta y$, so that we can write $\Delta x = \Delta z - i\Delta y$ in the first term and $\Delta y = (\Delta z - \Delta x)/i = -i(\Delta z - \Delta x)$ in the second term. This gives

$$\Delta f = (\Delta z - i\Delta y)u_x(M_1) + i(\Delta z - \Delta x)v_x(M_1) + i[(\Delta x)v_x(M_2) + (\Delta y)u_x(M_2)].$$

By performing the multiplications and reordering we obtain

$$\begin{aligned}\Delta f = (\Delta z)u_x(M_1) - i\Delta y\{u_x(M_1) - u_x(M_2)\}\\ + i[(\Delta z)v_x(M_1) - \Delta x\{v_x(M_1) - v_x(M_2)\}].\end{aligned}$$

Division by Δz now yields

$$\text{(A)}\quad \frac{\Delta f}{\Delta z} = u_x(M_1) + iv_x(M_1) - \frac{i\Delta y}{\Delta z}\{u_x(M_1) - u_x(M_2)\} - \frac{i\Delta x}{\Delta z}\{v_x(M_1) - v_x(M_2)\}.$$

(c) We finally let Δz approach zero and note that $|\Delta y/\Delta z| \leqq 1$ and $|\Delta x/\Delta z| \leqq 1$ in (A). Then Q: $(x + \Delta x, y + \Delta y)$ approaches P: (x, y), so that M_1 and M_2 must approach P. Also, since the partial derivatives in (A) are assumed to be continuous, they approach their value at P. In particular, the differences in the braces $\{\cdots\}$ in (A) approach zero. Hence the limit of the right side of (A) exists and is independent of the path along which $\Delta z \to 0$. We see that this limit equals the right side of (4) in Sec. 13.4. This means that $f(z)$ is analytic at every point z in D, and the proof is complete. ■

Section 14.2, pages 647–648

GOURSAT'S PROOF OF CAUCHY'S INTEGRAL THEOREM Goursat proved Cauchy's integral theorem without assuming that $f'(z)$ is continuous, as follows.

We start with the case when C is the boundary of a triangle. We orient C counterclockwise. By joining the midpoints of the sides we subdivide the triangle into four congruent triangles (Fig. 562). Let $C_{\text{I}}, C_{\text{II}}, C_{\text{III}}, C_{\text{IV}}$ denote their boundaries. We claim that (see Fig. 562).

$$\text{(1)}\qquad \oint_C f\,dz = \oint_{C_{\text{I}}} f\,dz + \oint_{C_{\text{II}}} f\,dz + \oint_{C_{\text{III}}} f\,dz + \oint_{C_{\text{IV}}} f\,dz.$$

Indeed, on the right we integrate along each of the three segments of subdivision in both possible directions (Fig. 562), so that the corresponding integrals cancel out in pairs, and the sum of the integrals on the right equals the integral on the left. We now pick an integral on the right that is biggest in absolute value and call its path C_1. Then, by the triangle inequality (Sec. 13.2),

$$\left|\oint_C f\,dz\right| \leqq \left|\oint_{C_{\text{I}}} f\,dz\right| + \left|\oint_{C_{\text{II}}} f\,dz\right| + \left|\oint_{C_{\text{III}}} f\,dz\right| + \left|\oint_{C_{\text{IV}}} f\,dz\right| \leqq 4\left|\oint_{C_1} f\,dz\right|.$$

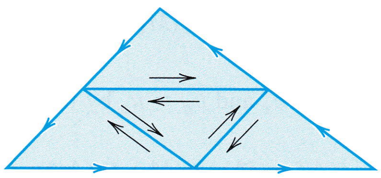


Fig. 562. Proof of Cauchy's integral theorem

We now subdivide the triangle bounded by C_1 as before and select a triangle of subdivision with boundary C_2 for which

$$\left|\oint_{C_1} f\,dz\right| \leqq 4\left|\oint_{C_2} f\,dz\right|. \qquad \text{Then} \qquad \left|\oint_{C} f\,dz\right| \leqq 4^2\left|\oint_{C_2} f\,dz\right|.$$

Continuing in this fashion, we obtain a sequence of triangles $T_1, T_2, \cdots$ with boundaries $C_1, C_2, \cdots$ that are similar and such that T_n lies in T_m when $n > m$, and

$$\left|\oint_{C} f\,dz\right| \leqq 4^n\left|\oint_{C_n} f\,dz\right|, \qquad n = 1, 2, \cdots. \tag{2}$$

Let z_0 be the point that belongs to all these triangles. Since f is differentiable at $z = z_0$, the derivative $f'(z_0)$ exists. Let

$$h(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0). \tag{3}$$

Solving this algebraically for $f(z)$ we have

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + h(z)(z - z_0).$$

Integrating this over the boundary C_n of the triangle T_n gives

$$\oint_{C_n} f(z)\,dz = \oint_{C_n} f(z_0)\,dz + \oint_{C_n} (z - z_0)f'(z_0)\,dz + \oint_{C_n} h(z)(z - z_0)\,dz.$$

Since $f(z_0)$ and $f'(z_0)$ are constants and C_n is a closed path, the first two integrals on the right are zero, as follows from Cauchy's proof, which is applicable because the integrands do have continuous derivatives (0 and *const*, respectively). We thus have

$$\oint_{C_n} f(z)\,dz = \oint_{C_n} h(z)(z - z_0)\,dz.$$

Since $f'(z_0)$ is the limit of the difference quotient in (3), for given $\epsilon > 0$ we can find a $\delta > 0$ such that

$$|h(z)| < \epsilon \qquad \text{when} \qquad |z - z_0| < \delta. \tag{4}$$

We may now take n so large that the triangle T_n lies in the disk $|z - z_0| < \delta$. Let L_n be the length of C_n. Then $|z - z_0| < L_n$ for all z on C_n and z_0 in T_n. From this and (4) we have $|h(z)(z - z_0)| < \epsilon L_n$. The *ML*-inequality in Sec. 14.1 now gives

$$\left|\oint_{C_n} f(z)\,dz\right| = \left|\oint_{C_n} h(z)(z - z_0)\,dz\right| \leqq \epsilon L_n \cdot L_n = \epsilon L_n^2. \tag{5}$$

Now denote the length of C by L. Then the path C_1 has the length $L_1 = L/2$, the path C_2 has the length $L_2 = L_1/2 = L/4$, etc., and C_n has the length $L_n = L/2^n$. Hence $L_n^2 = L^2/4^n$. From (2) and (5) we thus obtain

$$\left|\oint_{C} f\,dz\right| \leqq 4^n\left|\oint_{C_n} f\,dz\right| \leqq 4^n\epsilon L_n^2 = 4^n\epsilon\frac{L^2}{4^n} = \epsilon L^2.$$

By choosing ϵ (> 0) sufficiently small we can make the expression on the right as small as we please, while the expression on the left is the definite value of an integral. Consequently, this value must be zero, and the proof is complete.

The proof for *the case in which C is the boundary of a polygon* follows from the previous proof by subdividing the polygon into triangles (Fig. 563). The integral corresponding to each such triangle is zero. The sum of these integrals is equal to the integral over C, because we integrate along each segment of subdivision in both directions, the corresponding integrals cancel out in pairs, and we are left with the integral over C.

The case of a general simple closed path C can be reduced to the preceding one by inscribing in C a closed polygon P of chords, which approximates C "sufficiently accurately," and it can be shown that there is a polygon P such that the integral over P differs from that over C by less than any preassigned positive real number $\tilde{\epsilon}$, no matter how small. The details of this proof are somewhat involved and can be found in Ref. [D6] listed in App. 1. ■

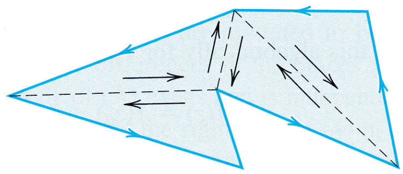


Fig. 563. Proof of Cauchy's integral theorem for a polygon

Section 15.1, page 667

PROOF OF THEOREM 4 Cauchy's Convergence Principle for Series

(a) In this proof we need two concepts and a theorem, which we list first.

1. A **bounded sequence** $s_1, s_2, \cdots$ is a sequence whose terms all lie in a disk of (sufficiently large, finite) radius K with center at the origin; thus $|s_n| < K$ for all n.

2. A **limit point** a of a sequence $s_1, s_2, \cdots$ is a point such that, given an $\epsilon > 0$, there are infinitely many terms satisfying $|s_n - a| < \epsilon$. (Note that this does *not* imply convergence, since there may still be infinitely many terms that do not lie within that circle of radius ϵ and center a.)

Example: $\frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \cdots$ has the limit points 0 and 1 and diverges.

3. A bounded sequence in the complex plane has at least one limit point. (Bolzano–Weierstrass theorem; proof below. Recall that "sequence" always mean *infinite* sequence.)

(b) We now turn to the actual proof that $z_1 + z_2 + \cdots$ converges if and only if for every $\epsilon > 0$ we can find an N such that

$$|z_{n+1} + \cdots + z_{n+p}| < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots. \tag{1}$$

Here, by the definition of partial sums,

$$s_{n+p} - s_n = z_{n+1} + \cdots + z_{n+p}.$$

Writing $n + p = r$, we see from this that (1) is equivalent to

$$(1^*) \qquad |s_r - s_n| < \epsilon \qquad \text{for all } r > N \text{ and } n > N.$$

Suppose that $s_1, s_2, \cdots$ converges. Denote its limit by s. Then for a given $\epsilon > 0$ we can find an N such that

$$|s_n - s| < \frac{\epsilon}{2} \qquad \text{for every } n > N.$$

Hence, if $r > N$ and $n > N$, then by the triangle inequality (Sec. 13.2),

$$|s_r - s_n| = |(s_r - s) - (s_n - s)| \leqq |s_r - s| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

that is, (1*) holds.

(c) Conversely, assume that $s_1, s_2, \cdots$ satisfies (1*). We first prove that then the sequence must be bounded. Indeed, choose a fixed ϵ and a fixed $n = n_0 > N$ in (1*). Then (1*) implies that all s_r with $r > N$ lie in the disk of radius ϵ and center s_{n_0} and only *finitely many terms* $s_1, \cdots, s_N$ may not lie in this disk. Clearly, we can now find a circle so large that this disk and these finitely many terms all lie within this new circle. Hence the sequence is bounded. By the Bolzano–Weierstrass theorem, it has at least one limit point, call it s.

We now show that the sequence is convergent with the limit s. Let $\epsilon > 0$ be given. Then there is an N^* such that $|s_r - s_n| < \epsilon/2$ for all $r > N^*$ and $n > N^*$, by (1*). Also, by the definition of a limit point, $|s_n - s| < \epsilon/2$ for *infinitely many n*, so that we can find and fix an $n > N^*$ such that $|s_n - s| < \epsilon/2$. Together, for *every* $r > N^*$,

$$|s_r - s| = |(s_r - s_n) + (s_n - s)| \leqq |s_r - s_n| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

that is, the sequence $s_1, s_2, \cdots$ is convergent with the limit s. ■

THEOREM

Bolzano–Weierstrass Theorem[3]

A bounded infinite sequence $z_1, z_2, z_3, \cdots$ in the complex plane has at least one limit point.

PROOF It is obvious that we need both conditions: a finite sequence cannot have a limit point, and the sequence $1, 2, 3, \cdots$, which is infinite but not bounded, has no limit point. To prove the theorem, consider a bounded infinite sequence $z_1, z_2, \cdots$ and let K be such that $|z_n| < K$ for all n. If only finitely many values of the z_n are different, then, since the sequence is infinite, some number z must occur infinitely many times in the sequence, and, by definition, this number is a limit point of the sequence.

We may now turn to the case when the sequence contains infinitely many *different* terms. We draw a large square Q_0 that contains all z_n. We subdivide Q_0 into four congruent squares, which we number 1, 2, 3, 4. Clearly, at least one of these squares (each taken

[3]BERNARD BOLZANO (1781–1848), Austrian mathematician and professor of religious studies, was a pioneer in the study of point sets, the foundation of analysis, and mathematical logic.
For Weierstrass, see Sec. 15.5.

with its complete boundary) must contain infinitely many terms of the sequence. The square of this type with the lowest number (1, 2, 3, or 4) will be denoted by Q_1. This is the first step. In the next step we subdivide Q_1 into four congruent squares and select a square Q_2 by the same rule, and so on. This yields an infinite sequence of squares Q_0, $Q_1, Q_2, \cdots, Q_n, \cdots$ with the property that the side of Q_n approaches zero as n approaches infinity, and Q_m contains all Q_n with $n > m$. It is not difficult to see that the number which belongs to all these squares,[4] call it $z = a$, is a limit point of the sequence. In fact, given an $\epsilon > 0$, we can choose an N so large that the side of the square Q_N is less than ϵ and, since Q_N contains infinitely many z_n, we have $|z_n - a| < \epsilon$ for infinitely many n. This completes the proof. ■

Section 15.3, pages 681–682

PART (b) OF THE PROOF OF THEOREM 5

We have to show that

$$\sum_{n=2}^{\infty} a_n \left[\frac{(z + \Delta z)^n - z^n}{\Delta z} - nz^{n-1}\right]$$

$$= \sum_{n=2}^{\infty} a_n \,\Delta z\big[(z + \Delta z)^{n-2} + 2z(z + \Delta z)^{n-3} + \cdots + (n-1)z^{n-2}\big],$$

thus,

$$\frac{(z + \Delta z)^n - z^n}{\Delta z} - nz^{n-1}$$

$$= \Delta z\big[(z + \Delta z)^{n-2} + 2z(z + \Delta z)^{n-3} + \cdots + (n-1)z^{n-2}\big].$$

If we set $z + \Delta z = b$ and $z = a$, thus $\Delta z = b - a$, this becomes simply

$$\text{(7a)} \qquad \frac{b^n - a^n}{b - a} - na^{n-1} = (b - a)A_n \qquad (n = 2, 3, \cdots),$$

where A_n is the expression in the brackets on the right,

$$\text{(7b)} \qquad A_n = b^{n-2} + 2ab^{n-3} + 3a^2b^{n-4} + \cdots + (n-1)a^{n-2};$$

thus, $A_2 = 1$, $A_3 = b + 2a$, etc. We prove (7) by induction. When $n = 2$, then (7) holds, since then

$$\frac{b^2 - a^2}{b - a} - 2a = \frac{(b + a)(b - a)}{b - a} - 2a = b - a = (b - a)A_2.$$

Assuming that (7) holds for $n = k$, we show that it holds for $n = k + 1$. By adding and subtracting a term in the numerator and then dividing we first obtain

$$\frac{b^{k+1} - a^{k+1}}{b - a} = \frac{b^{k+1} - ba^k + ba^k - a^{k+1}}{b - a} = b\,\frac{b^k - a^k}{b - a} + a^k.$$

[4]The fact that such a unique number $z = a$ exists seems to be obvious, but it actually follows from an axiom of the real number system, the so-called *Cantor–Dedekind axiom:* see footnote 3 in App. A3.3.

By the induction hypothesis, the right side equals $b[(b - a)A_k + ka^{k-1}] + a^k$. Direct calculation shows that this is equal to

$$(b - a)\{bA_k + ka^{k-1}\} + aka^{k-1} + a^k.$$

From (7b) with $n = k$ we see that the expression in the braces $\{\cdots\}$ equals

$$b^{k-1} + 2ab^{k-2} + \cdots + (k - 1)ba^{k-2} + ka^{k-1} = A_{k+1}.$$

Hence our result is

$$\frac{b^{k+1} - a^{k+1}}{b - a} = (b - a)A_{k+1} + (k + 1)a^k.$$

Taking the last term to the left, we obtain (7) with $n = k + 1$. This proves (7) for any integer $n \geqq 2$ and completes the proof. ■

Section 18.2, page 754

ANOTHER PROOF OF THEOREM 1 ***without the use of a harmonic conjugate***

We show that if $w = u + iv = f(z)$ is analytic and maps a domain D conformally onto a domain D^* and $\Phi^*(u, v)$ is harmonic in D^*, then

$$\Phi(x, y) = \Phi^*(u(x, y), v(x, y)) \tag{1}$$

is harmonic in D, that is, $\nabla^2\Phi = 0$ in D. We make no use of a harmonic conjugate of Φ^*, but use straightforward differentiation. By the chain rule,

$$\Phi_x = \Phi_u{}^* u_x + \Phi_v{}^* v_x.$$

We apply the chain rule again, underscoring the terms that will drop out when we form $\nabla^2\Phi$:

$$\begin{aligned}\Phi_{xx} &= \underline{\Phi_u{}^* u_{xx}} + (\Phi^*_{uu}u_x + \underline{\Phi^*_{uv}v_x})u_x \\ &\quad + \underline{\Phi_v{}^* v_{xx}} + (\underline{\Phi^*_{vu}u_x} + \Phi^*_{vv}v_x)v_x.\end{aligned}$$

Φ_{yy} is the same with each x replaced by y. We form the sum $\nabla^2\Phi$. In it, $\Phi^*_{vu} = \Phi^*_{uv}$ is multiplied by

$$u_x v_x + u_y v_y$$

which is 0 by the Cauchy–Riemann equations. Also $\nabla^2 u = 0$ and $\nabla^2 v = 0$. There remains

$$\nabla^2\Phi = \Phi^*_{uu}(u_x{}^2 + u_y{}^2) + \Phi^*_{vv}(v_x{}^2 + v_y{}^2).$$

By the Cauchy–Riemann equations this becomes

$$\nabla^2\Phi = (\Phi^*_{uu} + \Phi^*_{vv})(u_x{}^2 + v_x{}^2)$$

and is 0 since Φ^* is harmonic. ■

APPENDIX 5

Tables

For Tables of Laplace transforms see Secs. 6.8 and 6.9.
For Tables of Fourier transforms see Sec. 11.10.

If you have a Computer Algebra System (CAS), you may not need the present tables, but you may still find them convenient from time to time.

Table A1 Bessel Functions

For more extensive tables see Ref. [GR1] in App. 1.

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	−0.2601	0.3391	6.0	0.1506	−0.2767
0.1	0.9975	0.0499	3.1	−0.2921	0.3009	6.1	0.1773	−0.2559
0.2	0.9900	0.0995	3.2	−0.3202	0.2613	6.2	0.2017	−0.2329
0.3	0.9776	0.1483	3.3	−0.3443	0.2207	6.3	0.2238	−0.2081
0.4	0.9604	0.1960	3.4	−0.3643	0.1792	6.4	0.2433	−0.1816
0.5	0.9385	0.2423	3.5	−0.3801	0.1374	6.5	0.2601	−0.1538
0.6	0.9120	0.2867	3.6	−0.3918	0.0955	6.6	0.2740	−0.1250
0.7	0.8812	0.3290	3.7	−0.3992	0.0538	6.7	0.2851	−0.0953
0.8	0.8463	0.3688	3.8	−0.4026	0.0128	6.8	0.2931	−0.0652
0.9	0.8075	0.4059	3.9	−0.4018	−0.0272	6.9	0.2981	−0.0349
1.0	0.7652	0.4401	4.0	−0.3971	−0.0660	7.0	0.3001	−0.0047
1.1	0.7196	0.4709	4.1	−0.3887	−0.1033	7.1	0.2991	0.0252
1.2	0.6711	0.4983	4.2	−0.3766	−0.1386	7.2	0.2951	0.0543
1.3	0.6201	0.5220	4.3	−0.3610	−0.1719	7.3	0.2882	0.0826
1.4	0.5669	0.5419	4.4	−0.3423	−0.2028	7.4	0.2786	0.1096
1.5	0.5118	0.5579	4.5	−0.3205	−0.2311	7.5	0.2663	0.1352
1.6	0.4554	0.5699	4.6	−0.2961	−0.2566	7.6	0.2516	0.1592
1.7	0.3980	0.5778	4.7	−0.2693	−0.2791	7.7	0.2346	0.1813
1.8	0.3400	0.5815	4.8	−0.2404	−0.2985	7.8	0.2154	0.2014
1.9	0.2818	0.5812	4.9	−0.2097	−0.3147	7.9	0.1944	0.2192
2.0	0.2239	0.5767	5.0	−0.1776	−0.3276	8.0	0.1717	0.2346
2.1	0.1666	0.5683	5.1	−0.1443	−0.3371	8.1	0.1475	0.2476
2.2	0.1104	0.5560	5.2	−0.1103	−0.3432	8.2	0.1222	0.2580
2.3	0.0555	0.5399	5.3	−0.0758	−0.3460	8.3	0.0960	0.2657
2.4	0.0025	0.5202	5.4	−0.0412	−0.3453	8.4	0.0692	0.2708
2.5	−0.0484	0.4971	5.5	−0.0068	−0.3414	8.5	0.0419	0.2731
2.6	−0.0968	0.4708	5.6	0.0270	−0.3343	8.6	0.0146	0.2728
2.7	−0.1424	0.4416	5.7	0.0599	−0.3241	8.7	−0.0125	0.2697
2.8	−0.1850	0.4097	5.8	0.0917	−0.3110	8.8	−0.0392	0.2641
2.9	−0.2243	0.3754	5.9	0.1220	−0.2951	8.9	−0.0653	0.2559

$J_0(x) = 0$ for $x = 2.40483, 5.52008, 8.65373, 11.7915, 14.9309, 18.0711, 21.2116, 24.3525, 27.4935, 30.6346$

$J_1(x) = 0$ for $x = 3.83171, 7.01559, 10.1735, 13.3237, 16.4706, 19.6159, 22.7601, 25.9037, 29.0468, 32.1897$

Table A1 ***(continued)***

x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$
0.0	$(-\infty)$	$(-\infty)$	2.5	0.498	0.146	5.0	−0.309	0.148
0.5	−0.445	−1.471	3.0	0.377	0.325	5.5	−0.339	−0.024
1.0	0.088	−0.781	3.5	0.189	0.410	6.0	−0.288	−0.175
1.5	0.382	−0.412	4.0	−0.017	0.398	6.5	−0.173	−0.274
2.0	0.510	−0.107	4.5	−0.195	0.301	7.0	−0.026	−0.303

Table A2 **Gamma Function** [see (24) in App. A3.1]

α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$
1.00	1.000 000	1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384
1.02	0.988 844	1.22	0.913 106	1.42	0.886 356	1.62	0.895 924	1.82	0.936 845
1.04	0.978 438	1.24	0.908 521	1.44	0.885 805	1.64	0.898 642	1.84	0.942 612
1.06	0.968 744	1.26	0.904 397	1.46	0.885 604	1.66	0.901 668	1.86	0.948 687
1.08	0.959 725	1.28	0.900 718	1.48	0.885 747	1.68	0.905 001	1.88	0.955 071
1.10	0.951 351	1.30	0.897 471	1.50	0.886 227	1.70	0.908 639	1.90	0.961 766
1.12	0.943 590	1.32	0.894 640	1.52	0.887 039	1.72	0.912 581	1.92	0.968 774
1.14	0.936 416	1.34	0.892 216	1.54	0.888 178	1.74	0.916 826	1.94	0.976 099
1.16	0.929 803	1.36	0.890 185	1.56	0.889 639	1.76	0.921 375	1.96	0.983 743
1.18	0.923 728	1.38	0.888 537	1.58	0.891 420	1.78	0.926 227	1.98	0.991 708
1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384	2.00	1.000 000

Table A3 **Factorial Function and Its Logarithm with Base 10**

n	$n!$	log ($n!$)	n	$n!$	log ($n!$)	n	$n!$	log ($n!$)
1	1	0.000 000	6	720	2.857 332	11	39 916 800	7.601 156
2	2	0.301 030	7	5 040	3.702 431	12	479 001 600	8.680 337
3	6	0.778 151	8	40 320	4.605 521	13	6 227 020 800	9.794 280
4	24	1.380 211	9	362 880	5.559 763	14	87 178 291 200	10.940 408
5	120	2.079 181	10	3 628 800	6.559 763	15	1 307 674 368 000	12.116 500

Table A4 **Error Function, Sine and Cosine Integrals** [see (35), (40), (42) in App. A3.1]

x	erf x	Si(x)	ci(x)	x	erf x	Si(x)	ci(x)
0.0	0.0000	0.0000	∞	2.0	0.9953	1.6054	−0.4230
0.2	0.2227	0.1996	1.0422	2.2	0.9981	1.6876	−0.3751
0.4	0.4284	0.3965	0.3788	2.4	0.9993	1.7525	−0.3173
0.6	0.6039	0.5881	0.0223	2.6	0.9998	1.8004	−0.2533
0.8	0.7421	0.7721	−0.1983	2.8	0.9999	1.8321	−0.1865
1.0	0.8427	0.9461	−0.3374	3.0	1.0000	1.8487	−0.1196
1.2	0.9103	1.1080	−0.4205	3.2	1.0000	1.8514	−0.0553
1.4	0.9523	1.2562	−0.4620	3.4	1.0000	1.8419	0.0045
1.6	0.9763	1.3892	−0.4717	3.6	1.0000	1.8219	0.0580
1.8	0.9891	1.5058	−0.4568	3.8	1.0000	1.7934	0.1038
2.0	0.9953	1.6054	−0.4230	4.0	1.0000	1.7582	0.1410

Table A5 Binomial Distribution

Probability function $f(x)$ [see (2), Sec. 24.7] and distribution function $F(x)$

		$p = 0.1$		$p = 0.2$		$p = 0.3$		$p = 0.4$		$p = 0.5$	
n	x	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
		0.		**0.**		**0.**		**0.**		**0.**	
1	0	9000	0.9000	8000	0.8000	7000	0.7000	6000	0.6000	5000	0.5000
	1	1000	1.0000	2000	1.0000	3000	1.0000	4000	1.0000	5000	1.0000
	0	8100	0.8100	6400	0.6400	4900	0.4900	3600	0.3600	2500	0.2500
2	1	1800	0.9900	3200	0.9600	4200	0.9100	4800	0.8400	5000	0.7500
	2	0100	1.0000	0400	1.0000	0900	1.0000	1600	1.0000	2500	1.0000
	0	7290	0.7290	5120	0.5120	3430	0.3430	2160	0.2160	1250	0.1250
3	1	2430	0.9720	3840	0.8960	4410	0.7840	4320	0.6480	3750	0.5000
	2	0270	0.9990	0960	0.9920	1890	0.9730	2880	0.9360	3750	0.8750
	3	0010	1.0000	0080	1.0000	0270	1.0000	0640	1.0000	1250	1.0000
	0	6561	0.6561	4096	0.4096	2401	0.2401	1296	0.1296	0625	0.0625
	1	2916	0.9477	4096	0.8192	4116	0.6517	3456	0.4752	2500	0.3125
4	2	0486	0.9963	1536	0.9728	2646	0.9163	3456	0.8208	3750	0.6875
	3	0036	0.9999	0256	0.9984	0756	0.9919	1536	0.9744	2500	0.9375
	4	0001	1.0000	0016	1.0000	0081	1.0000	0256	1.0000	0625	1.0000
	0	5905	0.5905	3277	0.3277	1681	0.1681	0778	0.0778	0313	0.0313
	1	3281	0.9185	4096	0.7373	3602	0.5282	2592	0.3370	1563	0.1875
5	2	0729	0.9914	2048	0.9421	3087	0.8369	3456	0.6826	3125	0.5000
	3	0081	0.9995	0512	0.9933	1323	0.9692	2304	0.9130	3125	0.8125
	4	0005	1.0000	0064	0.9997	0284	0.9976	0768	0.9898	1563	0.9688
	5	0000	1.0000	0003	1.0000	0024	1.0000	0102	1.0000	0313	1.0000
	0	5314	0.5314	2621	0.2621	1176	0.1176	0467	0.0467	0156	0.0156
	1	3543	0.8857	3932	0.6554	3025	0.4202	1866	0.2333	0938	0.1094
	2	0984	0.9841	2458	0.9011	3241	0.7443	3110	0.5443	2344	0.3438
6	3	0146	0.9987	0819	0.9830	1852	0.9295	2765	0.8208	3125	0.6563
	4	0012	0.9999	0154	0.9984	0595	0.9891	1382	0.9590	2344	0.8906
	5	0001	1.0000	0015	0.9999	0102	0.9993	0369	0.9959	0938	0.9844
	6	0000	1.0000	0001	1.0000	0007	1.0000	0041	1.0000	0156	1.0000
	0	4783	0.4783	2097	0.2097	0824	0.0824	0280	0.0280	0078	0.0078
	1	3720	0.8503	3670	0.5767	2471	0.3294	1306	0.1586	0547	0.0625
	2	1240	0.9743	2753	0.8520	3177	0.6471	2613	0.4199	1641	0.2266
7	3	0230	0.9973	1147	0.9667	2269	0.8740	2903	0.7102	2734	0.5000
	4	0026	0.9998	0287	0.9953	0972	0.9712	1935	0.9037	2734	0.7734
	5	0002	1.0000	0043	0.9996	0250	0.9962	0774	0.9812	1641	0.9375
	6	0000	1.0000	0004	1.0000	0036	0.9998	0172	0.9984	0547	0.9922
	7	0000	1.0000	0000	1.0000	0002	1.0000	0016	1.0000	0078	1.0000
	0	4305	0.4305	1678	0.1678	0576	0.0576	0168	0.0168	0039	0.0039
	1	3826	0.8131	3355	0.5033	1977	0.2553	0896	0.1064	0313	0.0352
	2	1488	0.9619	2936	0.7969	2965	0.5518	2090	0.3154	1094	0.1445
	3	0331	0.9950	1468	0.9437	2541	0.8059	2787	0.5941	2188	0.3633
8	4	0046	0.9996	0459	0.9896	1361	0.9420	2322	0.8263	2734	0.6367
	5	0004	1.0000	0092	0.9988	0467	0.9887	1239	0.9502	2188	0.8555
	6	0000	1.0000	0011	0.9999	0100	0.9987	0413	0.9915	1094	0.9648
	7	0000	1.0000	0001	1.0000	0012	0.9999	0079	0.9993	0313	0.9961
	8	0000	1.0000	0000	1.0000	0001	1.0000	0007	1.0000	0039	1.0000

Table A6 Poisson Distribution

Probability function $f(x)$ [see (5), Sec. 24.7] and distribution function $F(x)$

x	$\mu = 0.1$ $f(x)$	$F(x)$	$\mu = 0.2$ $f(x)$	$F(x)$	$\mu = 0.3$ $f(x)$	$F(x)$	$\mu = 0.4$ $f(x)$	$F(x)$	$\mu = 0.5$ $f(x)$	$F(x)$
	0.		0.		0.		0.		0.	
0	9048	0.9048	8187	0.8187	7408	0.7408	6703	0.6703	6065	0.6065
1	0905	0.9953	1637	0.9825	2222	0.9631	2681	0.9384	3033	0.9098
2	0045	0.9998	0164	0.9989	0333	0.9964	0536	0.9921	0758	0.9856
3	0002	1.0000	0011	0.9999	0033	0.9997	0072	0.9992	0126	0.9982
4	0000	1.0000	0001	1.0000	0003	1.0000	0007	0.9999	0016	0.9998
5							0001	1.0000	0002	1.0000

x	$\mu = 0.6$ $f(x)$	$F(x)$	$\mu = 0.7$ $f(x)$	$F(x)$	$\mu = 0.8$ $f(x)$	$F(x)$	$\mu = 0.9$ $f(x)$	$F(x)$	$\mu = 1$ $f(x)$	$F(x)$
	0.		0.		0.		0.		0.	
0	5488	0.5488	4966	0.4966	4493	0.4493	4066	0.4066	3679	0.3679
1	3293	0.8781	3476	0.8442	3595	0.8088	3659	0.7725	3679	0.7358
2	0988	0.9769	1217	0.9659	1438	0.9526	1647	0.9371	1839	0.9197
3	0198	0.9966	0284	0.9942	0383	0.9909	0494	0.9865	0613	0.9810
4	0030	0.9996	0050	0.9992	0077	0.9986	0111	0.9977	0153	0.9963
5	0004	1.0000	0007	0.9999	0012	0.9998	0020	0.9997	0031	0.9994
6			0001	1.0000	0002	1.0000	0003	1.0000	0005	0.9999
7									0001	1.0000

x	$\mu = 1.5$ $f(x)$	$F(x)$	$\mu = 2$ $f(x)$	$F(x)$	$\mu = 3$ $f(x)$	$F(x)$	$\mu = 4$ $f(x)$	$F(x)$	$\mu = 5$ $f(x)$	$F(x)$
	0.		0.		0.		0.		0.	
0	2231	0.2231	1353	0.1353	0498	0.0498	0183	0.0183	0067	0.0067
1	3347	0.5578	2707	0.4060	1494	0.1991	0733	0.0916	0337	0.0404
2	2510	0.8088	2707	0.6767	2240	0.4232	1465	0.2381	0842	0.1247
3	1255	0.9344	1804	0.8571	2240	0.6472	1954	0.4335	1404	0.2650
4	0471	0.9814	0902	0.9473	1680	0.8153	1954	0.6288	1755	0.4405
5	0141	0.9955	0361	0.9834	1008	0.9161	1563	0.7851	1755	0.6160
6	0035	0.9991	0120	0.9955	0504	0.9665	1042	0.8893	1462	0.7622
7	0008	0.9998	0034	0.9989	0216	0.9881	0595	0.9489	1044	0.8666
8	0001	1.0000	0009	0.9998	0081	0.9962	0298	0.9786	0653	0.9319
9			0002	1.0000	0027	0.9989	0132	0.9919	0363	0.9682
10					0008	0.9997	0053	0.9972	0181	0.9863
11					0002	0.9999	0019	0.9991	0082	0.9945
12					0001	1.0000	0006	0.9997	0034	0.9980
13							0002	0.9999	0013	0.9993
14							0001	1.0000	0005	0.9998
15									0002	0.9999
16									0000	1.0000

Table A7 Normal Distribution

Values of the distribution function $\Phi(z)$ [see (3), Sec. 24.8]. $\Phi(-z) = 1 - \Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
	0.		**0.**		**0.**		**0.**		**0.**		**0.**
0.01	5040	0.51	6950	1.01	8438	1.51	9345	2.01	9778	2.51	9940
0.02	5080	0.52	6985	1.02	8461	1.52	9357	2.02	9783	2.52	9941
0.03	5120	0.53	7019	1.03	8485	1.53	9370	2.03	9788	2.53	9943
0.04	5160	0.54	7054	1.04	8508	1.54	9382	2.04	9793	2.54	9945
0.05	5199	0.55	7088	1.05	8531	1.55	9394	2.05	9798	2.55	9946
0.06	5239	0.56	7123	1.06	8554	1.56	9406	2.06	9803	2.56	9948
0.07	5279	0.57	7157	1.07	8577	1.57	9418	2.07	9808	2.57	9949
0.08	5319	0.58	7190	1.08	8599	1.58	9429	2.08	9812	2.58	9951
0.09	5359	0.59	7224	1.09	8621	1.59	9441	2.09	9817	2.59	9952
0.10	5398	0.60	7257	1.10	8643	1.60	9452	2.10	9821	2.60	9953
0.11	5438	0.61	7291	1.11	8665	1.61	9463	2.11	9826	2.61	9955
0.12	5478	0.62	7324	1.12	8686	1.62	9474	2.12	9830	2.62	9956
0.13	5517	0.63	7357	1.13	8708	1.63	9484	2.13	9834	2.63	9957
0.14	5557	0.64	7389	1.14	8729	1.64	9495	2.14	9838	2.64	9959
0.15	5596	0.65	7422	1.15	8749	1.65	9505	2.15	9842	2.65	9960
0.16	5636	0.66	7454	1.16	8770	1.66	9515	2.16	9846	2.66	9961
0.17	5675	0.67	7486	1.17	8790	1.67	9525	2.17	9850	2.67	9962
0.18	5714	0.68	7517	1.18	8810	1.68	9535	2.18	9854	2.68	9963
0.19	5753	0.69	7549	1.19	8830	1.69	9545	2.19	9857	2.69	9964
0.20	5793	0.70	7580	1.20	8849	1.70	9554	2.20	9861	2.70	9965
0.21	5832	0.71	7611	1.21	8869	1.71	9564	2.21	9864	2.71	9966
0.22	5871	0.72	7642	1.22	8888	1.72	9573	2.22	9868	2.72	9967
0.23	5910	0.73	7673	1.23	8907	1.73	9582	2.23	9871	2.73	9968
0.24	5948	0.74	7704	1.24	8925	1.74	9591	2.24	9875	2.74	9969
0.25	5987	0.75	7734	1.25	8944	1.75	9599	2.25	9878	2.75	9970
0.26	6026	0.76	7764	1.26	8962	1.76	9608	2.26	9881	2.76	9971
0.27	6064	0.77	7794	1.27	8980	1.77	9616	2.27	9884	2.77	9972
0.28	6103	0.78	7823	1.28	8997	1.78	9625	2.28	9887	2.78	9973
0.29	6141	0.79	7852	1.29	9015	1.79	9633	2.29	9890	2.79	9974
0.30	6179	0.80	7881	1.30	9032	1.80	9641	2.30	9893	2.80	9974
0.31	6217	0.81	7910	1.31	9049	1.81	9649	2.31	9896	2.81	9975
0.32	6255	0.82	7939	1.32	9066	1.82	9656	2.32	9898	2.82	9976
0.33	6293	0.83	7967	1.33	9082	1.83	9664	2.33	9901	2.83	9977
0.34	6331	0.84	7995	1.34	9099	1.84	9671	2.34	9904	2.84	9977
0.35	6368	0.85	8023	1.35	9115	1.85	9678	2.35	9906	2.85	9978
0.36	6406	0.86	8051	1.36	9131	1.86	9686	2.36	9909	2.86	9979
0.37	6443	0.87	8078	1.37	9147	1.87	9693	2.37	9911	2.87	9979
0.38	6480	0.88	8106	1.38	9162	1.88	9699	2.38	9913	2.88	9980
0.39	6517	0.89	8133	1.39	9177	1.89	9706	2.39	9916	2.89	9981
0.40	6554	0.90	8159	1.40	9192	1.90	9713	2.40	9918	2.90	9981
0.41	6591	0.91	8186	1.41	9207	1.91	9719	2.41	9920	2.91	9982
0.42	6628	0.92	8212	1.42	9222	1.92	9726	2.42	9922	2.92	9982
0.43	6664	0.93	8238	1.43	9236	1.93	9732	2.43	9925	2.93	9983
0.44	6700	0.94	8264	1.44	9251	1.94	9738	2.44	9927	2.94	9984
0.45	6736	0.95	8289	1.45	9265	1.95	9744	2.45	9929	2.95	9984
0.46	6772	0.96	8315	1.46	9279	1.96	9750	2.46	9931	2.96	9985
0.47	6808	0.97	8340	1.47	9292	1.97	9756	2.47	9932	2.97	9985
0.48	6844	0.98	8365	1.48	9306	1.98	9761	2.48	9934	2.98	9986
0.49	6879	0.99	8389	1.49	9319	1.99	9767	2.49	9936	2.99	9986
0.50	6915	1.00	8413	1.50	9332	2.00	9772	2.50	9938	3.00	9987

Table A8 Normal Distribution

Values of z for given values of $\Phi(z)$ [see (3), Sec. 24.8] and $D(z) = \Phi(z) - \Phi(-z)$
Example: $z = 0.279$ if $\Phi(z) = 61\%$; $z = 0.860$ if $D(z) = 61\%$.

%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$	%	$z(\Phi)$	$z(D)$
1	−2.326	0.013	41	−0.228	0.539	81	0.878	1.311
2	−2.054	0.025	42	−0.202	0.553	82	0.915	1.341
3	−1.881	0.038	43	−0.176	0.568	83	0.954	1.372
4	−1.751	0.050	44	−0.151	0.583	84	0.994	1.405
5	−1.645	0.063	45	−0.126	0.598	85	1.036	1.440
6	−1.555	0.075	46	−0.100	0.613	86	1.080	1.476
7	−1.476	0.088	47	−0.075	0.628	87	1.126	1.514
8	−1.405	0.100	48	−0.050	0.643	88	1.175	1.555
9	−1.341	0.113	49	−0.025	0.659	89	1.227	1.598
10	−1.282	0.126	50	0.000	0.674	90	1.282	1.645
11	−1.227	0.138	51	0.025	0.690	91	1.341	1.695
12	−1.175	0.151	52	0.050	0.706	92	1.405	1.751
13	−1.126	0.164	53	0.075	0.722	93	1.476	1.812
14	−1.080	0.176	54	0.100	0.739	94	1.555	1.881
15	−1.036	0.189	55	0.126	0.755	95	1.645	1.960
16	−0.994	0.202	56	0.151	0.772	96	1.751	2.054
17	−0.954	0.215	57	0.176	0.789	97	1.881	2.170
18	−0.915	0.228	58	0.202	0.806	97.5	1.960	2.241
19	−0.878	0.240	59	0.228	0.824	98	2.054	2.326
20	−0.842	0.253	60	0.253	0.842	99	2.326	2.576
21	−0.806	0.266	61	0.279	0.860	99.1	2.366	2.612
22	−0.772	0.279	62	0.305	0.878	99.2	2.409	2.652
23	−0.739	0.292	63	0.332	0.896	99.3	2.457	2.697
24	−0.706	0.305	64	0.358	0.915	99.4	2.512	2.748
25	−0.674	0.319	65	0.385	0.935	99.5	2.576	2.807
26	−0.643	0.332	66	0.412	0.954	99.6	2.652	2.878
27	−0.613	0.345	67	0.440	0.974	99.7	2.748	2.968
28	−0.583	0.358	68	0.468	0.994	99.8	2.878	3.090
29	−0.553	0.372	69	0.496	1.015	99.9	3.090	3.291
30	−0.524	0.385	70	0.524	1.036			
31	−0.496	0.399	71	0.553	1.058	99.91	3.121	3.320
32	−0.468	0.412	72	0.583	1.080	99.92	3.156	3.353
33	−0.440	0.426	73	0.613	1.103	99.93	3.195	3.390
34	−0.412	0.440	74	0.643	1.126	99.94	3.239	3.432
35	−0.385	0.454	75	0.674	1.150	99.95	3.291	3.481
36	−0.358	0.468	76	0.706	1.175	99.96	3.353	3.540
37	−0.332	0.482	77	0.739	1.200	99.97	3.432	3.615
38	−0.305	0.496	78	0.772	1.227	99.98	3.540	3.719
39	−0.279	0.510	79	0.806	1.254	99.99	3.719	3.891
40	−0.253	0.524	80	0.842	1.282			

Table A9 ***t*-Distribution**

Values of z for given values of the distribution function $F(z)$ (see (8) in Sec. 25.3). Example: For 9 degrees of freedom, $z = 1.83$ when $F(z) = 0.95$.

$F(z)$	Number of Degrees of Freedom									
	1	2	3	4	5	6	7	8	9	10
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.32	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.26	0.26
0.7	0.73	0.62	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54
0.8	1.38	1.06	0.98	0.94	0.92	0.91	0.90	0.89	0.88	0.88
0.9	3.08	1.89	1.64	1.53	1.48	1.44	1.41	1.40	1.38	1.37
0.95	6.31	2.92	2.35	2.13	2.02	1.94	1.89	1.86	1.83	1.81
0.975	12.7	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.23
0.99	31.8	6.96	4.54	3.75	3.36	3.14	3.00	2.90	2.82	2.76
0.995	63.7	9.92	5.84	4.60	4.03	3.71	3.50	3.36	3.25	3.17
0.999	318.3	22.3	10.2	7.17	5.89	5.21	4.79	4.50	4.30	4.14

$F(z)$	Number of Degrees of Freedom									
	11	12	13	14	15	16	17	18	19	20
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26
0.7	0.54	0.54	0.54	0.54	0.54	0.54	0.53	0.53	0.53	0.53
0.8	0.88	0.87	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.86
0.9	1.36	1.36	1.35	1.35	1.34	1.34	1.33	1.33	1.33	1.33
0.95	1.80	1.78	1.77	1.76	1.75	1.75	1.74	1.73	1.73	1.72
0.975	2.20	2.18	2.16	2.14	2.13	2.12	2.11	2.10	2.09	2.09
0.99	2.72	2.68	2.65	2.62	2.60	2.58	2.57	2.55	2.54	2.53
0.995	3.11	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86	2.85
0.999	4.02	3.93	3.85	3.79	3.73	3.69	3.65	3.61	3.58	3.55

$F(z)$	Number of Degrees of Freedom									
	22	24	26	28	30	40	50	100	200	∞
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.25	0.25	0.25	0.25
0.7	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.52
0.8	0.86	0.86	0.86	0.85	0.85	0.85	0.85	0.85	0.84	0.84
0.9	1.32	1.32	1.31	1.31	1.31	1.30	1.30	1.29	1.29	1.28
0.95	1.72	1.71	1.71	1.70	1.70	1.68	1.68	1.66	1.65	1.65
0.975	2.07	2.06	2.06	2.05	2.04	2.02	2.01	1.98	1.97	1.96
0.99	2.51	2.49	2.48	2.47	2.46	2.42	2.40	2.36	2.35	2.33
0.995	2.82	2.80	2.78	2.76	2.75	2.70	2.68	2.63	2.60	2.58
0.999	3.50	3.47	3.43	3.41	3.39	3.31	3.26	3.17	3.13	3.09

Table A10 Chi-square Distribution

Values of x for given values of the distribution function $F(z)$ (see Sec. 25.3 before (17)). Example: For 3 degrees of freedom, $z = 11.34$ when $F(z) = 0.99$.

$F(z)$	Number of Degrees of Freedom									
	1	2	3	4	5	6	7	8	9	10
0.005	0.00	0.01	0.07	0.21	0.41	0.68	0.99	1.34	1.73	2.16
0.01	0.00	0.02	0.11	0.30	0.55	0.87	1.24	1.65	2.09	2.56
0.025	0.00	0.05	0.22	0.48	0.83	1.24	1.69	2.18	2.70	3.25
0.05	0.00	0.10	0.35	0.71	1.15	1.64	2.17	2.73	3.33	3.94
0.95	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92	18.31
0.975	5.02	7.38	9.35	11.14	12.83	14.45	16.01	17.53	19.02	20.48
0.99	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67	23.21
0.995	7.88	10.60	12.84	14.86	16.75	18.55	20.28	21.95	23.59	25.19

$F(z)$	Number of Degrees of Freedom									
	11	12	13	14	15	16	17	18	19	20
0.005	2.60	3.07	3.57	4.07	4.60	5.14	5.70	6.26	6.84	7.43
0.01	3.05	3.57	4.11	4.66	5.23	5.81	6.41	7.01	7.63	8.26
0.025	3.82	4.40	5.01	5.63	6.26	6.91	7.56	8.23	8.91	9.59
0.05	4.57	5.23	5.89	6.57	7.26	7.96	8.67	9.39	10.12	10.85
0.95	19.68	21.03	22.36	23.68	25.00	26.30	27.59	28.87	30.14	31.41
0.975	21.92	23.34	24.74	26.12	27.49	28.85	30.19	31.53	32.85	34.17
0.99	24.72	26.22	27.69	29.14	30.58	32.00	33.41	34.81	36.19	37.57
0.995	26.76	28.30	29.82	31.32	32.80	34.27	35.72	37.16	38.58	40.00

$F(z)$	Number of Degrees of Freedom									
	21	22	23	24	25	26	27	28	29	30
0.005	8.0	8.6	9.3	9.9	10.5	11.2	11.8	12.5	13.1	13.8
0.01	8.9	9.5	10.2	10.9	11.5	12.2	12.9	13.6	14.3	15.0
0.025	10.3	11.0	11.7	12.4	13.1	13.8	14.6	15.3	16.0	16.8
0.05	11.6	12.3	13.1	13.8	14.6	15.4	16.2	16.9	17.7	18.5
0.95	32.7	33.9	35.2	36.4	37.7	38.9	40.1	41.3	42.6	43.8
0.975	35.5	36.8	38.1	39.4	40.6	41.9	43.2	44.5	45.7	47.0
0.99	38.9	40.3	41.6	43.0	44.3	45.6	47.0	48.3	49.6	50.9
0.995	41.4	42.8	44.2	45.6	46.9	48.3	49.6	51.0	52.3	53.7

$F(z)$	Number of Degrees of Freedom							
	40	50	60	70	80	90	100	> 100 (Approximation)
0.005	20.7	28.0	35.5	43.3	51.2	59.2	67.3	$\frac{1}{2}(h - 2.58)^2$
0.01	22.2	29.7	37.5	45.4	53.5	61.8	70.1	$\frac{1}{2}(h - 2.33)^2$
0.025	24.4	32.4	40.5	48.8	57.2	65.6	74.2	$\frac{1}{2}(h - 1.96)^2$
0.05	26.5	34.8	43.2	51.7	60.4	69.1	77.9	$\frac{1}{2}(h - 1.64)^2$
0.95	55.8	67.5	79.1	90.5	101.9	113.1	124.3	$\frac{1}{2}(h + 1.64)^2$
0.975	59.3	71.4	83.3	95.0	106.6	118.1	129.6	$\frac{1}{2}(h + 1.96)^2$
0.99	63.7	76.2	88.4	100.4	112.3	124.1	135.8	$\frac{1}{2}(h + 2.33)^2$
0.995	66.8	79.5	92.0	104.2	116.3	128.3	140.2	$\frac{1}{2}(h + 2.58)^2$

In the last column, $h = \sqrt{2m - 1}$, where m is the number of degrees of freedom.

Table A11 F-Distribution with (m, n) Degrees of Freedom

Values of z for which the distribution function $F(z)$ [see (13), Sec. 25.4] has the value **0.95**

Example: For (7, 4) d.f., $z = 6.09$ if $F(z) = 0.95$.

n	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$
1	161	200	216	225	230	234	237	239	241
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
32	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.19
34	4.13	3.28	2.88	2.65	2.49	2.38	2.29	2.23	2.17
36	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.15
38	4.10	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

Table A11 F-Distribution with (*m, n*) Degrees of Freedom *(continued)*

Values of z for which the distribution function $F(z)$ [see (13), Sec. 25.4] has the value **0.95**

n	$m = 10$	$m = 15$	$m = 20$	$m = 30$	$m = 40$	$m = 50$	$m = 100$	∞
1	242	246	248	250	251	252	253	254
2	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5
3	8.79	8.70	8.66	8.62	8.59	8.58	8.55	8.53
4	5.96	5.86	5.80	5.75	5.72	5.70	5.66	5.63
5	4.74	4.62	4.56	4.50	4.46	4.44	4.41	4.37
6	4.06	3.94	3.87	3.81	3.77	3.75	3.71	3.67
7	3.64	3.51	3.44	3.38	3.34	3.32	3.27	3.23
8	3.35	3.22	3.15	3.08	3.04	3.02	2.97	2.93
9	3.14	3.01	2.94	2.86	2.83	2.80	2.76	2.71
10	2.98	2.85	2.77	2.70	2.66	2.64	2.59	2.54
11	2.85	2.72	2.65	2.57	2.53	2.51	2.46	2.40
12	2.75	2.62	2.54	2.47	2.43	2.40	2.35	2.30
13	2.67	2.53	2.46	2.38	2.34	2.31	2.26	2.21
14	2.60	2.46	2.39	2.31	2.27	2.24	2.19	2.13
15	2.54	2.40	2.33	2.25	2.20	2.18	2.12	2.07
16	2.49	2.35	2.28	2.19	2.15	2.12	2.07	2.01
17	2.45	2.31	2.23	2.15	2.10	2.08	2.02	1.96
18	2.41	2.27	2.19	2.11	2.06	2.04	1.98	1.92
19	2.38	2.23	2.16	2.07	2.03	2.00	1.94	1.88
20	2.35	2.20	2.12	2.04	1.99	1.97	1.91	1.84
22	2.30	2.15	2.07	1.98	1.94	1.91	1.85	1.78
24	2.25	2.11	2.03	1.94	1.89	1.86	1.80	1.73
26	2.22	2.07	1.99	1.90	1.85	1.82	1.76	1.69
28	2.19	2.04	1.96	1.87	1.82	1.79	1.73	1.65
30	2.16	2.01	1.93	1.84	1.79	1.76	1.70	1.62
32	2.14	1.99	1.91	1.82	1.77	1.74	1.67	1.59
34	2.12	1.97	1.89	1.80	1.75	1.71	1.65	1.57
36	2.11	1.95	1.87	1.78	1.73	1.69	1.62	1.55
38	2.09	1.94	1.85	1.76	1.71	1.68	1.61	1.53
40	2.08	1.92	1.84	1.74	1.69	1.66	1.59	1.51
50	2.03	1.87	1.78	1.69	1.63	1.60	1.52	1.44
60	1.99	1.84	1.75	1.65	1.59	1.56	1.48	1.39
70	1.97	1.81	1.72	1.62	1.57	1.53	1.45	1.35
80	1.95	1.79	1.70	1.60	1.54	1.51	1.43	1.32
90	1.94	1.78	1.69	1.59	1.53	1.49	1.41	1.30
100	1.93	1.77	1.68	1.57	1.52	1.48	1.39	1.28
150	1.89	1.73	1.64	1.54	1.48	1.44	1.34	1.22
200	1.88	1.72	1.62	1.52	1.46	1.41	1.32	1.19
1000	1.84	1.68	1.58	1.47	1.41	1.36	1.26	1.08
∞	1.83	1.67	1.57	1.46	1.39	1.35	1.24	1.00

Table A11 F-Distribution with (*m*, *n*) Degrees of Freedom *(continued)*

Values of z for which the distribution function $F(z)$ [see (13), Sec. 25.4] has the value **0.99**

n	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$
1	4052	4999	5403	5625	5764	5859	5928	5981	6022
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
32	7.50	5.34	4.46	3.97	3.65	3.43	3.26	3.13	3.02
34	7.44	5.29	4.42	3.93	3.61	3.39	3.22	3.09	2.98
36	7.40	5.25	4.38	3.89	3.57	3.35	3.18	3.05	2.95
38	7.35	5.21	4.34	3.86	3.54	3.32	3.15	3.02	2.92
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64
90	6.93	4.85	4.01	3.54	3.23	3.01	2.84	2.72	2.61
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
150	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50
1000	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

Table A11 F-Distribution with (*m*, *n*) Degrees of Freedom *(continued)*

Values of z for which the distribution function $F(z)$ [see (13), Sec. 25.4] has the value **0.99**

n	$m = 10$	$m = 15$	$m = 20$	$m = 30$	$m = 40$	$m = 50$	$m = 100$	∞
1	6056	6157	6209	6261	6287	6303	6334	6366
2	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5
3	27.2	26.9	26.7	26.5	26.4	26.4	26.2	26.1
4	14.5	14.2	14.0	13.8	13.7	13.7	13.6	13.5
5	10.1	9.72	9.55	9.38	9.29	9.24	9.13	9.02
6	7.87	7.56	7.40	7.23	7.14	7.09	6.99	6.88
7	6.62	6.31	6.16	5.99	5.91	5.86	5.75	5.65
8	5.81	5.52	5.36	5.20	5.12	5.07	4.96	4.86
9	5.26	4.96	4.81	4.65	4.57	4.52	4.42	4.31
10	4.85	4.56	4.41	4.25	4.17	4.12	4.01	3.91
11	4.54	4.25	4.10	3.94	3.86	3.81	3.71	3.60
12	4.30	4.01	3.86	3.70	3.62	3.57	3.47	3.36
13	4.10	3.82	3.66	3.51	3.43	3.38	3.27	3.17
14	3.94	3.66	3.51	3.35	3.27	3.22	3.11	3.00
15	3.80	3.52	3.37	3.21	3.13	3.08	2.98	2.87
16	3.69	3.41	3.26	3.10	3.02	2.97	2.86	2.75
17	3.59	3.31	3.16	3.00	2.92	2.87	2.76	2.65
18	3.51	3.23	3.08	2.92	2.84	2.78	2.68	2.57
19	3.43	3.15	3.00	2.84	2.76	2.71	2.60	2.49
20	3.37	3.09	2.94	2.78	2.69	2.64	2.54	2.42
22	3.26	2.98	2.83	2.67	2.58	2.53	2.42	2.31
24	3.17	2.89	2.74	2.58	2.49	2.44	2.33	2.21
26	3.09	2.81	2.66	2.50	2.42	2.36	2.25	2.13
28	3.03	2.75	2.60	2.44	2.35	2.30	2.19	2.06
30	2.98	2.70	2.55	2.39	2.30	2.25	2.13	2.01
32	2.93	2.65	2.50	2.34	2.25	2.20	2.08	1.96
34	2.89	2.61	2.46	2.30	2.21	2.16	2.04	1.91
36	2.86	2.58	2.43	2.26	2.18	2.12	2.00	1.87
38	2.83	2.55	2.40	2.23	2.14	2.09	1.97	1.84
40	2.80	2.52	2.37	2.20	2.11	2.06	1.94	1.80
50	2.70	2.42	2.27	2.10	2.01	1.95	1.82	1.68
60	2.63	2.35	2.20	2.03	1.94	1.88	1.75	1.60
70	2.59	2.31	2.15	1.98	1.89	1.83	1.70	1.54
80	2.55	2.27	2.12	1.94	1.85	1.79	1.65	1.49
90	2.52	2.24	2.09	1.92	1.82	1.76	1.62	1.46
100	2.50	2.22	2.07	1.89	1.80	1.74	1.60	1.43
150	2.44	2.16	2.00	1.83	1.73	1.66	1.52	1.33
200	2.41	2.13	1.97	1.79	1.69	1.63	1.48	1.28
1000	2.34	2.06	1.90	1.72	1.61	1.54	1.38	1.11
∞	2.32	2.04	1.88	1.70	1.59	1.52	1.36	1.00

Table A12 Distribution Function $F(x) = P(T \leqq x)$ of the Random Variable T in Section 25.8

x	$n = 3$
	0.
0	167
1	500

x	$n = 4$
	0.
0	042
1	167
2	375

x	$n = 5$
	0.
0	008
1	042
2	117
3	242
4	408

x	$n = 6$
	0.
0	001
1	008
2	028
3	068
4	136
5	235
6	360
7	500

x	$n = 7$
	0.
1	001
2	005
3	015
4	035
5	068
6	119
7	191
8	281
9	386
10	500

x	$n = 8$
	0.
2	001
3	003
4	007
5	016
6	031
7	054
8	089
9	138
10	199
11	274
12	360
13	452

x	$n = 9$
	0.
4	001
5	003
6	006
7	012
8	022
9	038
10	060
11	090
12	130
13	179
14	238
15	306
16	381
17	460

x	$n = 10$
	0.
6	001
7	002
8	005
9	008
10	014
11	023
12	036
13	054
14	078
15	108
16	146
17	190
18	242
19	300
20	364
21	431
22	500

x	$n = 11$
	0.
8	001
9	002
10	003
11	005
12	008
13	013
14	020
15	030
16	043
17	060
18	082
19	109
20	141
21	179
22	223
23	271
24	324
25	381
26	440
27	500

x	$n = 20$
	0.
50	001
51	002
52	002
53	003
54	004
55	005
56	006
57	007
58	008
59	010
60	012
61	014
62	017
63	020
64	023
65	027
66	032
67	037
68	043
69	049
70	056
71	064
72	073
73	082
74	093
75	104
76	117
77	130
78	144
79	159
80	176
81	193
82	211
83	230
84	250
85	271
86	293
87	315
88	339
89	362
90	387
91	411
92	436
93	462
94	487

x	$n = 19$
	0.
43	001
44	002
45	002
46	003
47	003
48	004
49	005
50	006
51	008
52	010
53	012
54	014
55	017
56	021
57	025
58	029
59	034
60	040
61	047
62	054
63	062
64	072
65	082
66	093
67	105
68	119
69	133
70	149
71	166
72	184
73	203
74	223
75	245
76	267
77	290
78	314
79	339
80	365
81	391
82	418
83	445
84	473
85	500

x	$n = 18$
	0.
38	001
39	002
40	003
41	003
42	004
43	005
44	007
45	009
46	011
47	013
48	016
49	020
50	024
51	029
52	034
53	041
54	048
55	056
56	066
57	076
58	088
59	100
60	115
61	130
62	147
63	165
64	184
65	205
66	227
67	250
68	275
69	300
70	327
71	354
72	383
73	411
74	441
75	470
76	500

x	$n = 17$
	0.
32	001
33	002
34	002
35	003
36	004
37	005
38	007
39	009
40	011
41	014
42	017
43	021
44	026
45	032
46	038
47	046
48	054
49	064
50	076
51	088
52	102
53	118
54	135
55	154
56	174
57	196
58	220
59	245
60	271
61	299
62	328
63	358
64	388
65	420
66	452
67	484

x	$n = 16$
	0.
27	001
28	002
29	002
30	003
31	004
32	006
33	008
34	010
35	013
36	016
37	021
38	026
39	032
40	039
41	048
42	058
43	070
44	083
45	097
46	114
47	133
48	153
49	175
50	199
51	225
52	253
53	282
54	313
55	345
56	378
57	412
58	447
59	482

x	$n = 15$
	0.
23	001
24	002
25	003
26	004
27	006
28	008
29	010
30	014
31	018
32	023
33	029
34	037
35	046
36	057
37	070
38	084
39	101
40	120
41	141
42	164
43	190
44	218
45	248
46	279
47	313
48	349
49	385
50	423
51	461
52	500

x	$n = 14$
	0.
18	001
19	002
20	002
21	003
22	005
23	007
24	010
25	013
26	018
27	024
28	031
29	040
30	051
31	063
32	079
33	096
34	117
35	140
36	165
37	194
38	225
39	259
40	295
41	334
42	374
43	415
44	457
45	500

x	$n = 13$
	0.
14	001
15	001
16	002
17	003
18	005
19	007
20	011
21	015
22	021
23	029
24	038
25	050
26	064
27	082
28	102
29	126
30	153
31	184
32	218
33	255
34	295
35	338
36	383
37	429
38	476

x	$n = 12$
	0.
11	001
12	002
13	003
14	004
15	007
16	010
17	016
18	022
19	031
20	043
21	058
22	076
23	098
24	125
25	155
26	190
27	230
28	273
29	319
30	369
31	420
32	473

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Some Constants

$$e = 2.71828\ 18284\ 59045\ 23536$$
$$\sqrt{e} = 1.64872\ 12707\ 00128\ 14685$$
$$e^2 = 7.38905\ 60989\ 30650\ 22723$$

$$\pi = 3.14159\ 26535\ 89793\ 23846$$
$$\pi^2 = 9.86960\ 44010\ 89358\ 61883$$
$$\sqrt{\pi} = 1.77245\ 38509\ 05516\ 02730$$

$$\log_{10} \pi = 0.49714\ 98726\ 94133\ 85435$$
$$\ln \pi = 1.14472\ 98858\ 49400\ 17414$$
$$\log_{10} e = 0.43429\ 44819\ 03251\ 82765$$
$$\ln 10 = 2.30258\ 50929\ 94045\ 68402$$

$$\sqrt{2} = 1.41421\ 35623\ 73095\ 04880$$
$$\sqrt[3]{2} = 1.25992\ 10498\ 94873\ 16477$$
$$\sqrt{3} = 1.73205\ 08075\ 68877\ 29353$$
$$\sqrt[3]{3} = 1.44224\ 95703\ 07408\ 38232$$
$$\ln 2 = 0.69314\ 71805\ 59945\ 30942$$
$$\ln 3 = 1.09861\ 22886\ 68109\ 69140$$

$$\gamma = 0.57721\ 56649\ 01532\ 86061$$
$$\ln \gamma = -0.54953\ 93129\ 81644\ 82234$$
(see Sec. 5.6)

$$1^\circ = 0.01745\ 32925\ 19943\ 29577 \text{ rad}$$
$$1 \text{ rad} = 57.29577\ 95130\ 82320\ 87680^\circ = 57^\circ 17' 44.806''$$

Polar Coordinates

$$x = r \cos \theta \qquad y = r \sin \theta$$
$$r = \sqrt{x^2 + y^2} \qquad \tan \theta = \frac{y}{x}$$
$$dx\, dy = r\, dr\, d\theta$$

Series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m \qquad (|x| < 1)$$
$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$
$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$
$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$$
$$\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m} \qquad (|x| < 1)$$
$$\arctan x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1} \qquad (|x| < 1)$$

Greek Alphabet

α	Alpha	ν	Nu
β	Beta	ξ	Xi
γ, Γ	Gamma	o	Omicron
δ, Δ	Delta	π	Pi
ϵ, ε	Epsilon	ρ	Rho
ζ	Zeta	σ, Σ	Sigma
η	Eta	τ	Tau
$\theta, \vartheta, \Theta$	Theta	υ, Υ	Upsilon
ι	Iota	ϕ, φ, Φ	Phi
κ	Kappa	χ	Chi
λ, Λ	Lambda	ψ, Ψ	Psi
μ	Mu	ω, Ω	Omega

Vectors

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
$$\text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$
$$\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$