Collection of exercises in SYMMETRIC TRANSFORMATIONS for Programme Primary school teachers

Leni Lvovská

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Inspiration is a state in which the soul perceives impressions more vividly, understands and classifies ideas, and thus better explains them. It is just as necessary in geometry as in poetry.

Alexander Sergejevič Puškin (1799 - 1837)[11]

Introduction

This book of exercises was developed in order to support the teaching of geometry to future elementary school teachers. This book offers the students a set of solved tasks as well as a number of further exercises specifically chosen for the topics of *Symmetric transformations*, where study materials are missing to a similar extent and students are forced to choose tasks from several different sources. At the same time, it follows the latest trend of cross-curricular subject and points out in the presented examples and exercises the connection of geometry with other subjects and especially with the world around us.

Many tasks work with the magnetic kit Geomag. If you do not have it, these tasks can be demonstrated using skewers and balls of modeling. For the creation of illustrations, the GeoGebra software was used. It is therefore easy to use the GeoGebra tutorial software directly in the classroom or on a standalone task. There are direct references to selected dynamic applets and stepped constructions for specific constructions.

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Many thanks to Helena Durnová for the preparation of the English version of this text and Pavel Kříž for the support of the typesetting in the LaTeXsystem.

1 Basic properties of symmetries

Mapping in a plane assigns each point X of the plane exactly one point X' of that plane. We call the point X preimage and the point X' image.
Symmetrical transformation in geometry is such a mapping between Euclidian space that preserves the lengths. A symmetrical mapping of a space onto itself is called a symmetry.

- Symmetry preserves the lengths, i.e. for any two points X, Y and their images X', Y' the equality $XY \cong X'Y'$ holds.
- By composing two symmetries, a symmetry is created again.
- Each symmetry is an ijective mapping.
- An inversion of symmetry is also a symmetry.
- Identity is a symmetry.
- All symmetries of an Euclidian space with the operation of transformation composition form a group of symmetries, the so-called euclidian group.
- If A, B, C and A', B', C' are two triplets of points which do not lie on the same line and if $AB \cong A'B'$, $BC \cong B'C'$ a $AC \cong A'C'$, then there is only one symmetry in the plane in which the image of A is A', the image of B is B', and the image of C is C' (the so-called theorem of definiry a symmetry in a plane).

In this text we will only deal with symmetries in a plane.

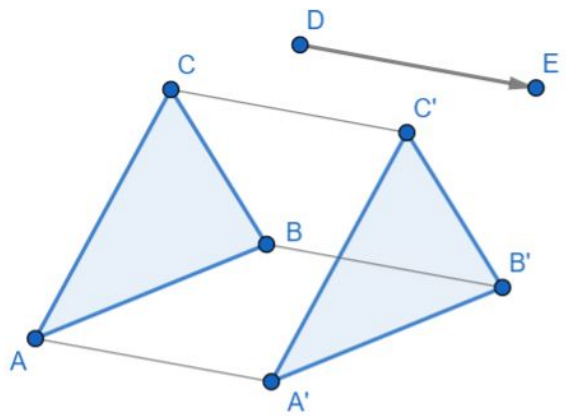
2 Basic types of symmetries in the plane

To clarify the terms we use in the tasks bellow, we give an overview of the basic symmetries in the plane and their properties.

translational symmetry

All points of the plane are shifted in the same direction by the same distance and the distance is given by the oriented line, resp. vector. The image is uniquely determined by the offset vector.

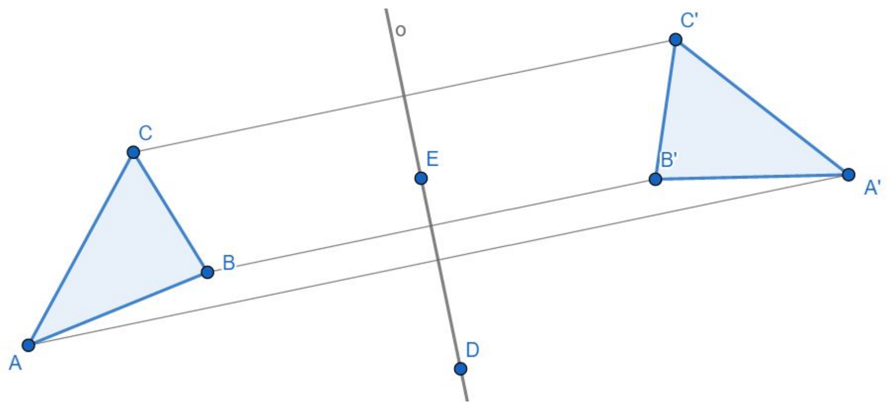
$$\mathcal{T}(\overrightarrow{DE}) : \triangle ABC \mapsto \triangle A'B'C'$$



reflectional symmetry

A transformation given by the symmetry axis, which divides the plane into two half-planes. The corresponding points lie on the perpendicular to the axis of symmetry in opposite half-planes and are equidistant from the axis.

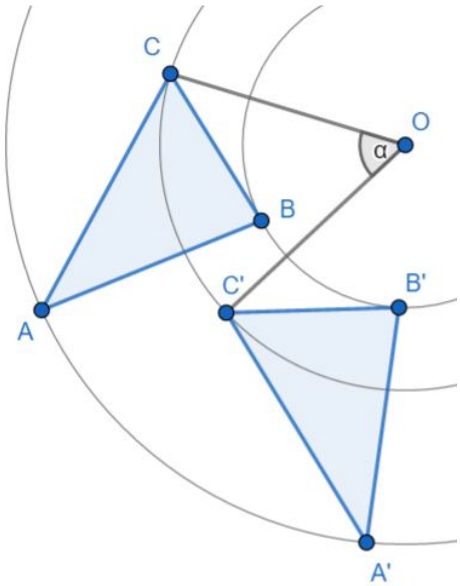
$$\mathcal{O}(\leftrightarrow DE) : \triangle ABC \mapsto \triangle A'B'C'$$



rotational symmetry

All points in the plane are rotated around a fixed point (center of rotation) in the same direction by the same angle (angle of rotation).

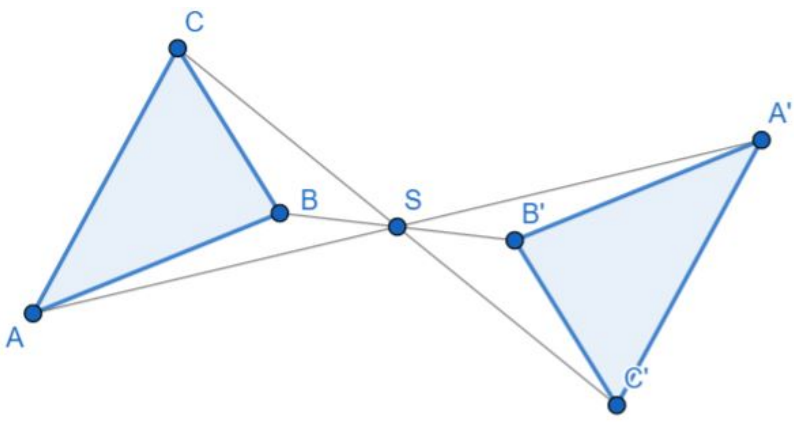
$$\mathcal{R}(O, \alpha) : \triangle ABC \mapsto \triangle A'B'C'$$



point reflection symmetry

Point reflection symmetry in a plane is a special case of rotation - rotation by 180 degrees around the point of symmetry.

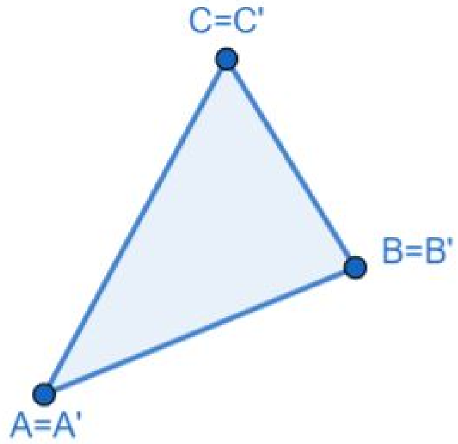
$$\mathcal{S}(S) : \triangle ABC \mapsto \triangle A'B'C'$$



identitity

A mapping in which each point maps into itself. It can be considered as a zero-length translational symmetry or a zero-angle rotation.

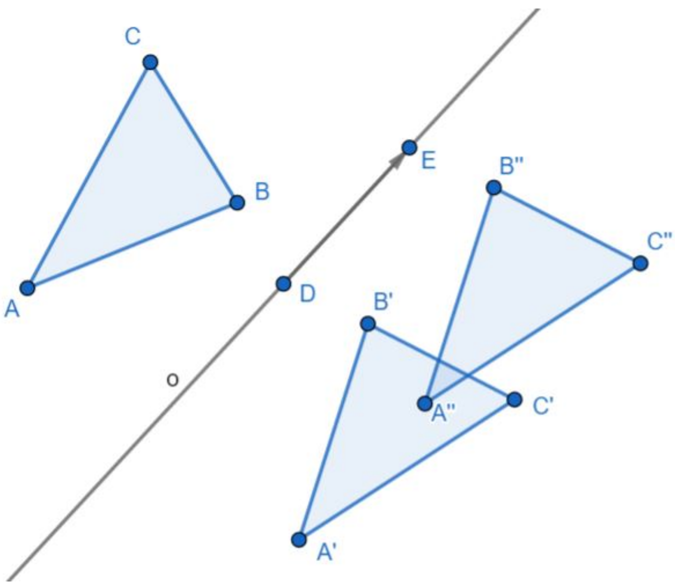
$$\mathcal{I}\lceil : \triangle ABC \mapsto \triangle A'B'C'$$



glide reflection symmetry

Glide reflection symmetry is the composition of reflectional symmetry and translation in the direction of the axis.

$$\mathcal{T}(\overrightarrow{DE})\mathcal{O}(o) : \triangle ABC \mapsto \triangle A''B''C''$$



Example 2.1. Decompose the given symmetry in the plane into direct, i.e. preserving orientation and indirect, i.e. not preserving orientation.

Sollution: Direct symmetry - translational symmetry, rotational symm., identity. Indirect symmetry - reflectional symm., glide reflectional symm.

Example 2.2. Which of the capital letters of the alphabet A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z are reflectionally symmetrical? Divide the letters into groups according to the number of symmetry axis..

Solution: 1 axis: A, B, C, D, E, K, M, T, U V, W, Y; 2 axes: H, I, X, O.

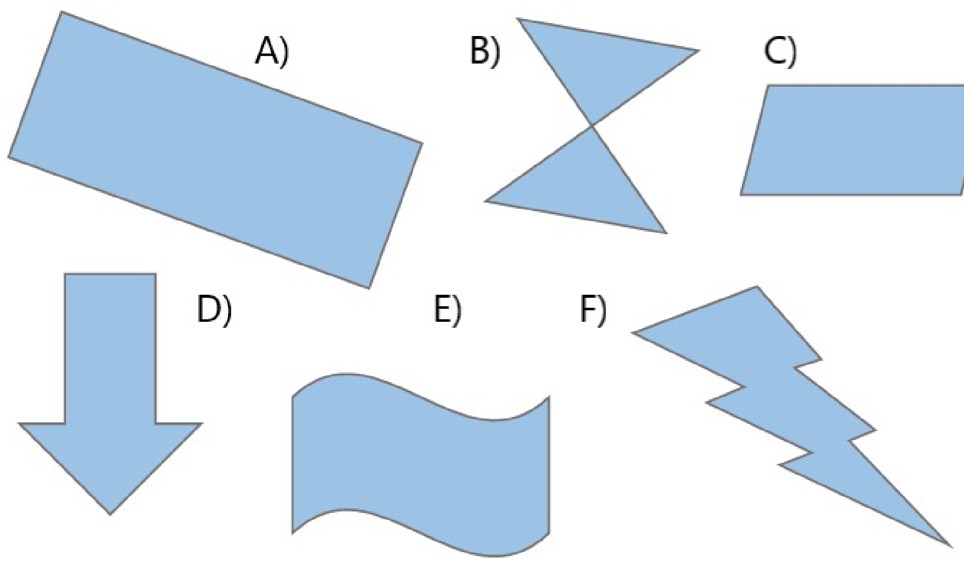
Example 2.3. The figure shows six planar shapes:

a) Which of the figures below are reflectionally symmetrical?

b) For reflectionally symmetric figures, determine all their axes of symmetry.

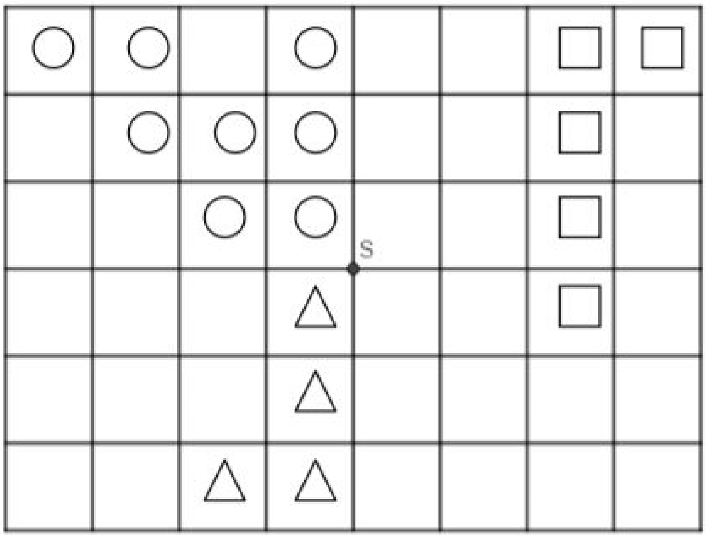
c) Draw a figure in the plane with exactly four axes of symmetry.

c) Draw a figure in the plane which has an infinite number of axes of symmetry.



Solution: A) 2 axes of symmetry, B) 2 axes of symmetry, D) one axis of symmetry. Square is an example of figure with four axes of symmetry, A circle is an example of figure with an infinite number of axes of symmetry.

Excercise 2.4. Draw the image so that it is point reflection symmetrical with point of symmetry S.

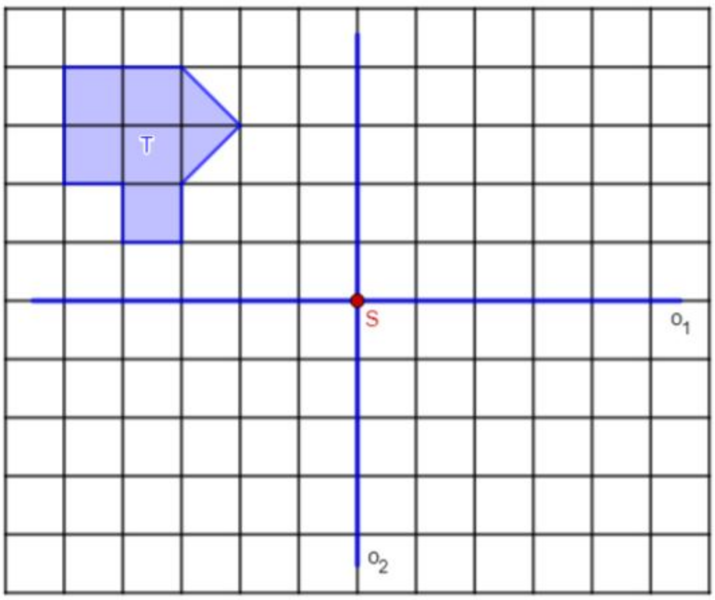


Excercise 2.5. Redraw formation T in a square grid:

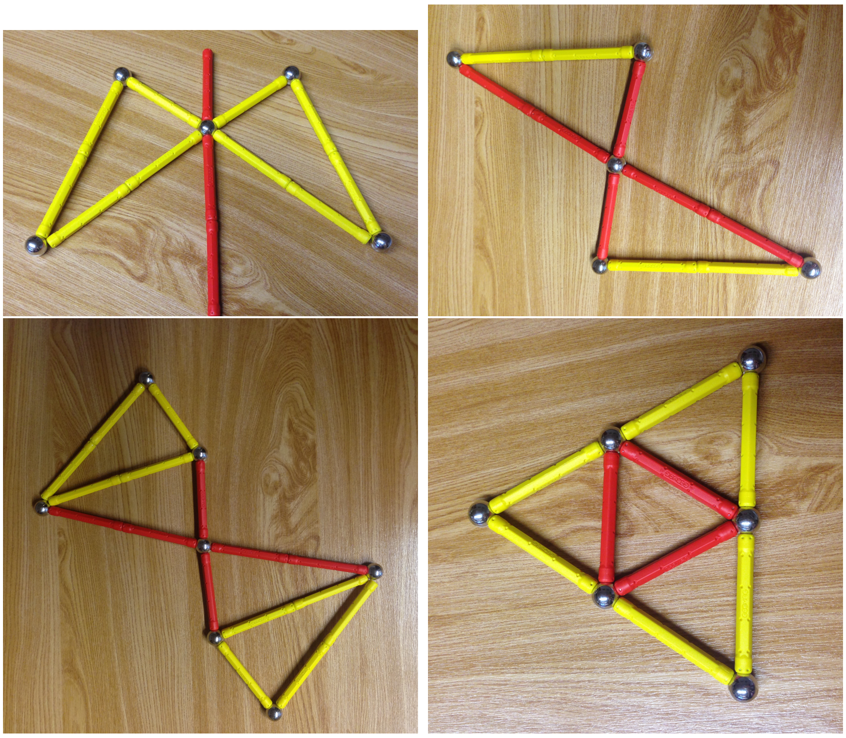
a) in reflectional symmetry with the axis o_1 and label it T_1,

b) in reflectional symmetry with the axis o_2 and label it T_2,

c) in point reflection symmetry with the point S and label it T_3,



Excercise 2.6. Discuss the images. Which transformations are demonstrated by these models? What are the defining elements and self-associated points of these transformations? Create similar models.



Excercise 2.7. Draw the image of the line segment AB in reflectional symmetry with the axis o, such that the axis:

a) intersect with the line segment AB in one point which is not one of its end points,

b) intersect with the line segment AB in its center and is perpendicular to it,

c) is parallel to the line segment AB,

d) intersect with the line segment AB at point B and is not perpendicular to the line segment AB.

Excercise 2.8. How many axes of symmetry each of these objects have: a) line segment, b) half-line, c) line.

Excercise 2.9. Construct an image of circle k in reflectional symmetry with the axis o, such that:

a) passes through the center S of the circle K,

b) does not intersect with the circle K,

c) intersect with the circle K, but not through its center.

Excercise 2.10. An quadrilateral $ABCD$ is given. Construct a quadrilateral which is reflectionally symmetry with $ABCD$ and maps A into center of BC.

Excercise 2.11. Which of the following formations are point reflectional symmetrical:

a) line segment,

b) half-line,

c) rectangle,

d) circle,

e) square,

f) rhombus,

g) a parallerogram,

h) a triangle with sides of different size,

i) an equilateral triangle,

j) an isosceles triangle.

Draw the images, if so, specify the point of reflection.

Excercise 2.12. In point reflection determined by the point S determine the image of the line segment AB, when the point of reflection

a) does not lie on the line AB,

b) coincident with the point A,

c) lies on the line segment AB, $S \neq A \neq B$.

Excercise 2.13. Draw the angle AVB, $|\sphericalangle AVB| = 45°$. Determine its image in point reflection with center a) V, b) A, c) $S \notin \sphericalangle AVB$.

Excercise 2.14. In rotational symmetry given by point S and the angle $\beta = 45°$ determine the image of the line segment AB, which goes through S.

Excercise 2.15. In rotational symmetry given by point M and the angle $\gamma = 60°$ determine the image of the line p, which does not go through center of rotation.

Excercise 2.16. In rotational symmetry given by point R and the angle $\sigma = -60°$ determine the image of the circle k.

Excercise 2.17. Draw any triangle ABC. Construct triangle $A'B'C'$ as a point reflection

a) with the center at point S, which is the center of the side AC,

b) in which the A is the image of point B.

Excercise 2.18. Two perpendicular line, k and l, are given. Construct their images in point reflection with the center S, if:

a) S is the the intersection of k and l,

b) S lies on the strait line k and it does not coincide with the intersection k and l.

Excercise 2.19. Two perpendicular line, a and b, are given. Determine a translational symmetry $\mathcal{T}$, which maps the lines a and b into the lines a' and b' such that, the intersections of a, b, a' and b' formed a) square, b) rectangle.

Excercise 2.20. Three different points A, B, C, not lying on a single straight line are given. In the translational symmetry determined by the vector AB construct the image of:

a) line segment AC,

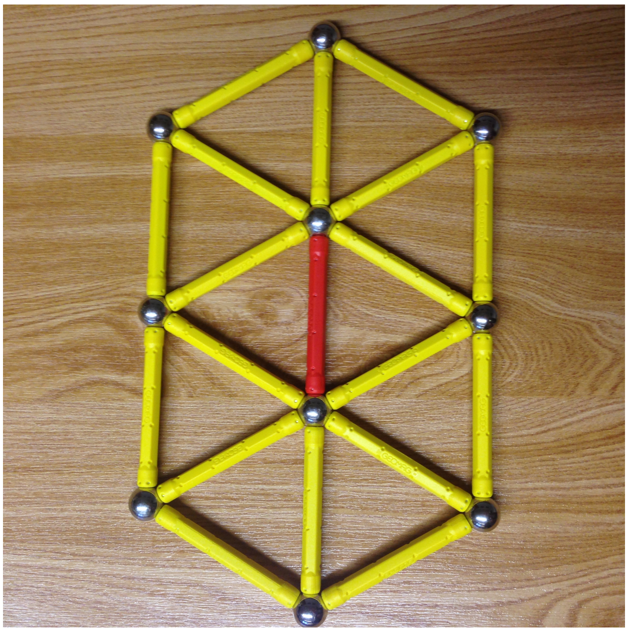
b) line BC,

c) line AB.

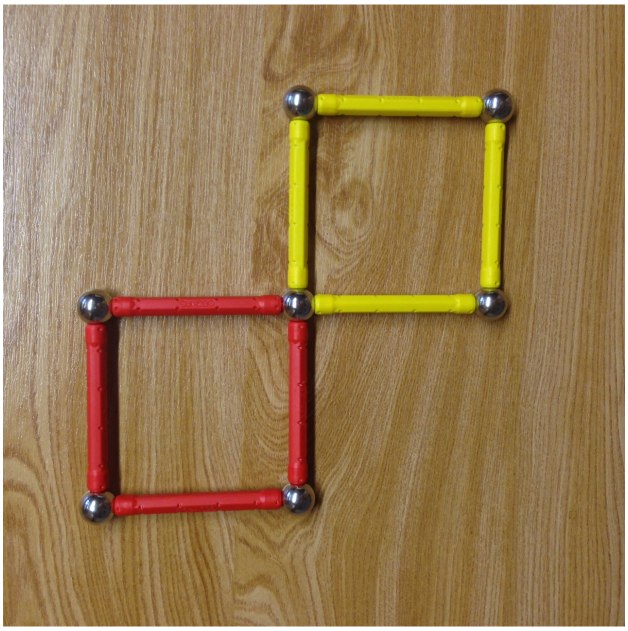
Excercise 2.21. The square $ABCD$, $a = 5\,\mathrm{cm}$, is given. The diagonals of the square intersect at a point S. Construct the circle k, defined by the point S and radius $2\,\mathrm{cm}$. The point X lies on the line AS, $|AX| = 8\,\mathrm{cm}$. Transform the image in the rotational symmetry determined by A and angle $45°$.

Excercise 2.22. Construct a regular hexagon $ABCDEF$ and its image $A'B'C'D'E'F'$ in the translational symmetry specified by the vector SD, where S is the center of symmetry of the hexagon $ABCDEF$. Which shape is formed by the union of the hexagons $ABCDEF$ and $A'B'C'D'E'F'$? Model the task using the Geomag kit.

Solution: The hexagon $ABB'A'F'F$ is formed by the union of the hexagon $ABCDEF$ and the hexagon $A'B'C'D'E'F'$, but $ABB'A'F'F$ is not regular hexagon. Vector SD is sign by red stick in the model.



Excercise 2.23. Specify all the symmetries which convert the yellow square into the red one. Draw images and label vertices, such that appropriated to each transformation.



Example 2.24. The isosceles trapezoid $ABCD$ is given, $|AB| = 2|CD|$, $AB \parallel CD$, S is center of line segment AB. Specify the symmetries which reproduce :

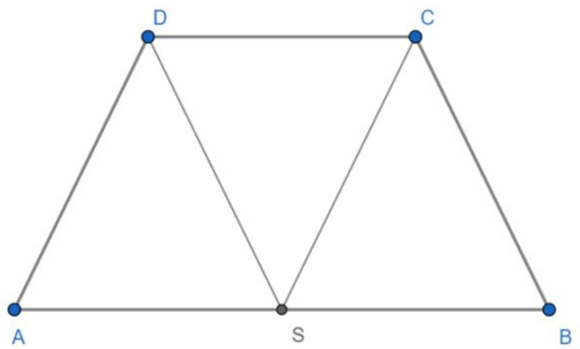
a) $\triangle ASD \mapsto \triangle BSC$,

b) $\triangle ASD \mapsto \triangle SBC$,

c) $\triangle SBC \mapsto \triangle ASD$,

d) $\triangle DCS \mapsto \triangle DCS$,

Solution:



a) reflectional symmetry $\mathcal{O}(\leftrightarrow AB)$: $\triangle ASD \mapsto \triangle BSC$,

b) translational symmetry $\mathcal{T}(\overrightarrow{AS})$: $\triangle ASD \mapsto \triangle SBC$,

c) translational symmetry $\mathcal{T}(\overrightarrow{SA})$:$\triangle SBC \mapsto \triangle ASD$,

d) identity $\mathcal{I}d$: $\triangle DCS \mapsto \triangle DCS$,

Excercise 2.25. The regular octagon $ABCDEFGH$ is given. Specify the symmetries which reproduce:

a) $\triangle ADE \mapsto \triangle BGF$,

b) $\triangle ACD \mapsto \triangle EGH$,

c) $\triangle ACD \mapsto \triangle DEG$,

d) $\triangle ACD \mapsto \triangle GED$,

Example 2.26. Draw the image if the regular pentagon $ABCDE$ in:

a) the point reflectional symmetry with the point D,

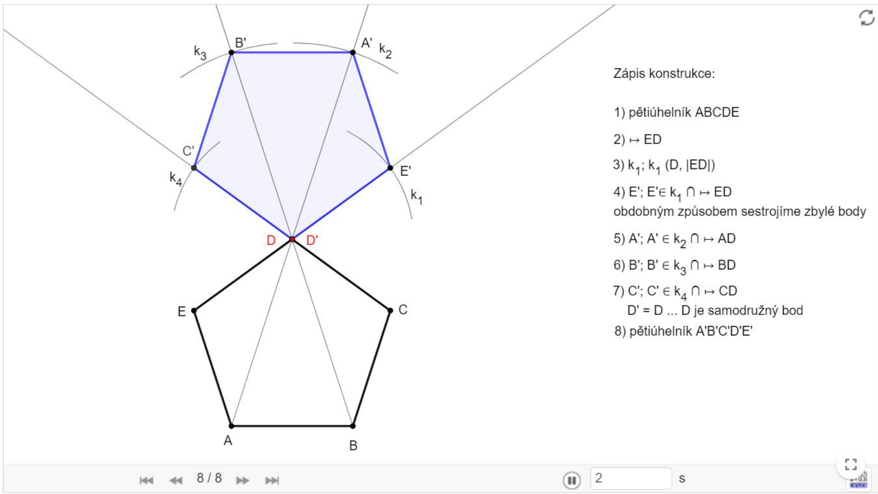
b) the reflectional symmetry with the axis AC,

c) the translational symmetry given by vector CE,

d) the rotational symmetry given by point C and the angle $60°$.

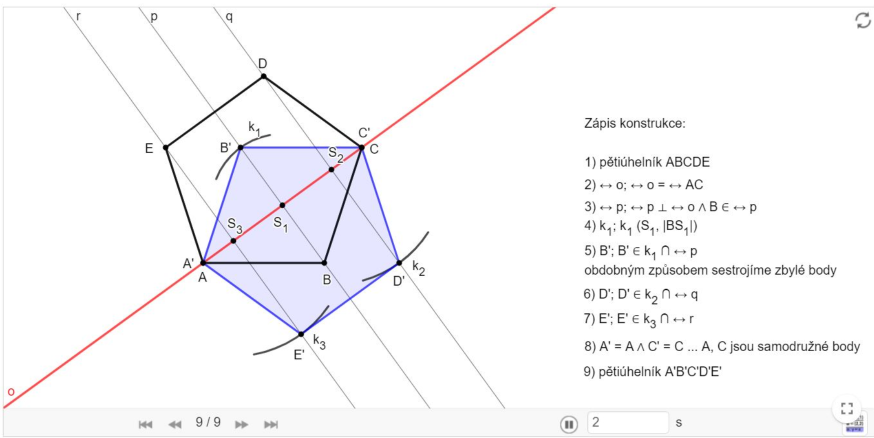
Solution:

a) The pentagon $ABCDE$ in the point reflectional symmetry with the point D.



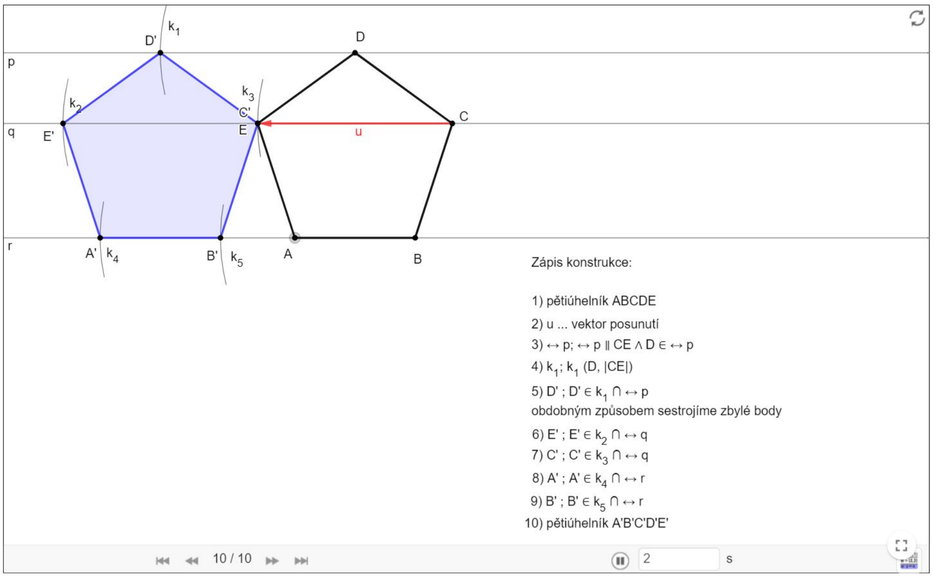
The construction step by step:
`https://www.geogebra.org/m/saxej6u9#material/kzb9kwdm`

b) The pentagon $ABCDE$ in the reflectional symmetry with the axis AC.



The construction step by step:
`https://www.geogebra.org/m/saxej6u9#material/xmygytx8`

c) The pentagon $ABCDE$ in the translational symmetry given by $\overrightarrow{CE}$.



The construction step by step:
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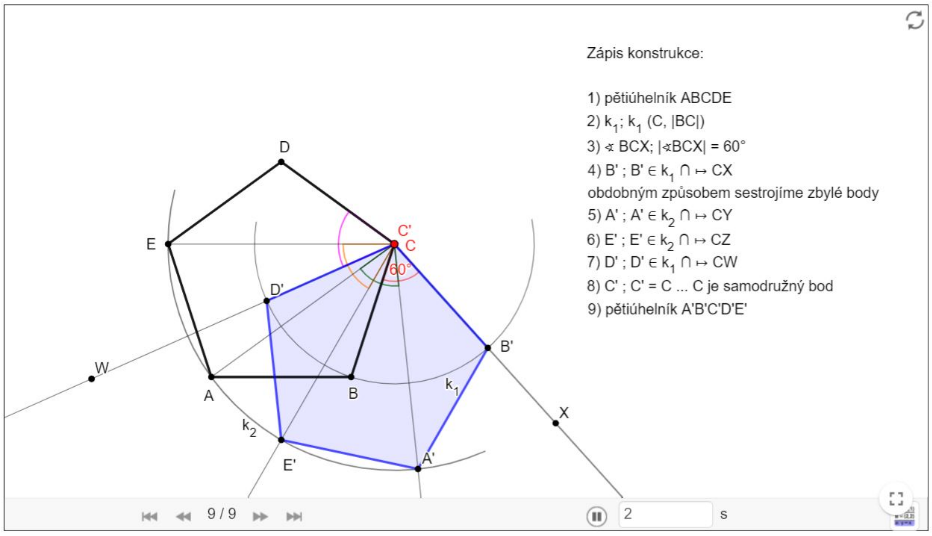
d) The pentagon $ABCDE$ in the rotational symmetry given by point C and the angle 60°.



The construction step by step:
`https://www.geogebra.org/m/saxej6u9#material/tpjxeurj`

Example 2.27. Specify all the symmetries which reproduce (map onto itself):

a) an equilateral triangle,

b) a regular pentagon,

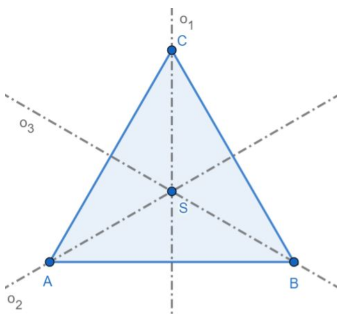
c) a regular hexagon.

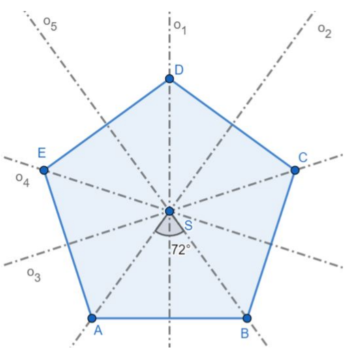
Solution: In any symmetry which reproduce any of regular n-gons, the center of the circumcircle is a self-asociated point Using the properties of regular n-gons and symmetries in the plane, we obtain the following results:

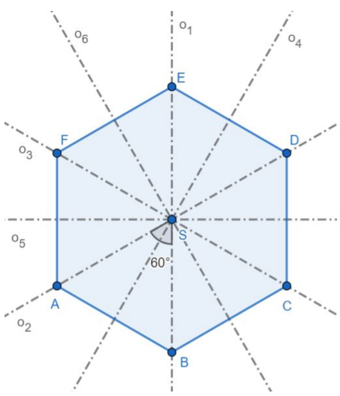
a) There are six symmetries of an equilaterla triangle that map it onto itself: $\mathcal{I}d$, $\mathcal{O}_1(o_1)$, $\mathcal{O}_2(o_2)$, $\mathcal{O}_3(o_3)$, $\mathcal{R}_1(S,120°)$, $\mathcal{R}_2(S,240°)$, i. e. identity, three reflectional symmetries with axes being the axes of the three sides, two rotations with the center in the circumcenter of the triangel and the corresponding angle.

b) There are ten symmetries of a regular pentagon that map it onto itself: $\mathcal{I}d$, $\mathcal{O}_1(o_1)$, $\mathcal{O}_2(o_2)$, $\mathcal{O}_3(o_3)$, $\mathcal{O}_4(o_4)$, $\mathcal{O}_5(o_5)$, $\mathcal{R}_1(S,72°)$, $\mathcal{R}_2(S,144°)$, $\mathcal{R}_3(S,216°)$, $\mathcal{R}_4(S,288°)$, i. e. identity, five reflectional symmetries with axes being the axes of the its sides, four rotations with their center in circumcenter and the corresponding angles.

c) There are 12 symmetries a regular hexagon that map it onto itself: $\mathcal{I}d$, $\mathcal{S}(S)$, $\mathcal{O}_1(o_1)$, $\mathcal{O}_2(o_2)$, $\mathcal{O}_3(o_3)$, $\mathcal{O}_4(o_4)$, $\mathcal{O}_5(o_5)$, $\mathcal{O}_6(o_6)$, $\mathcal{R}_1(S,60°)$, $\mathcal{R}_2(S,120°)$, $\mathcal{R}_3(S,240°)$, $\mathcal{R}_4(S,300°)$, i. e. identity, point reflection symmetry, six reflectional symmetries — three with axes being the axes of its sides and three with the axes incident with the pairs of opposite vertices, four rotations with their center in circumcenter and the corresponding angles.





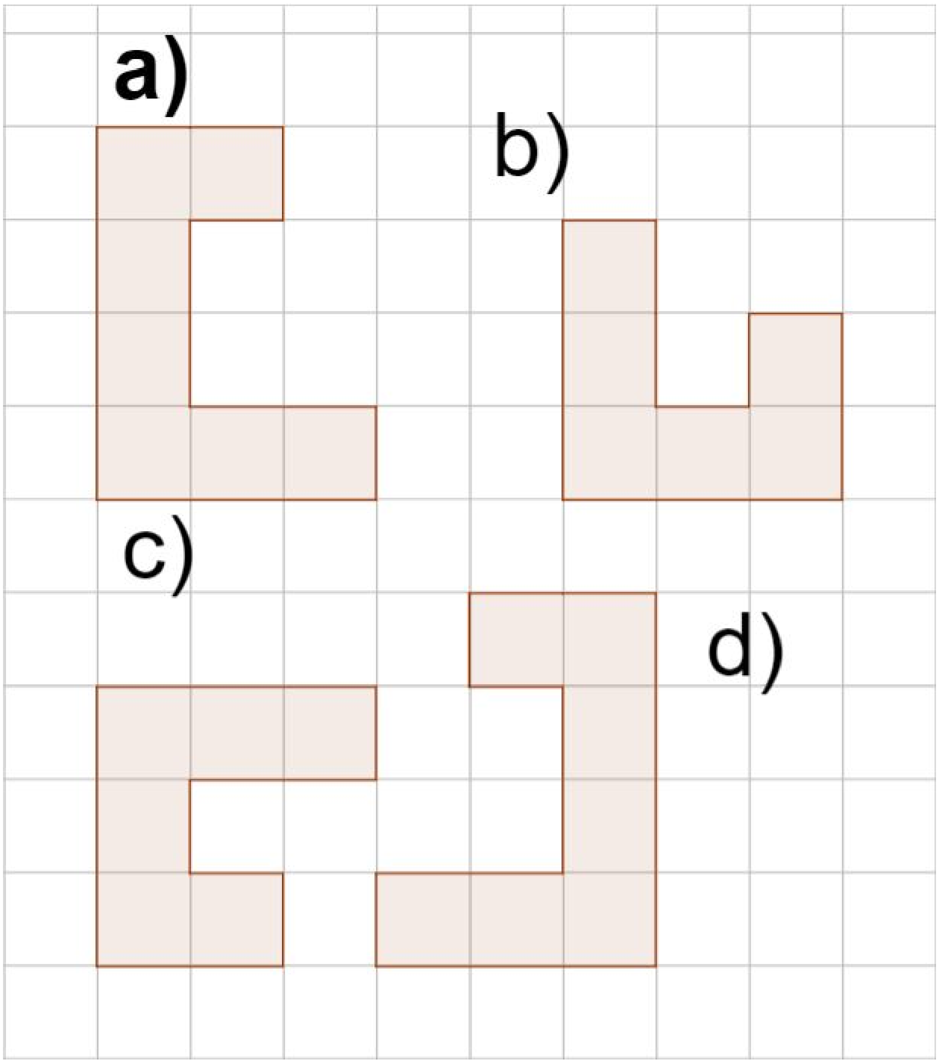


Excercise 2.28. In the picture you can see four object in a square network.

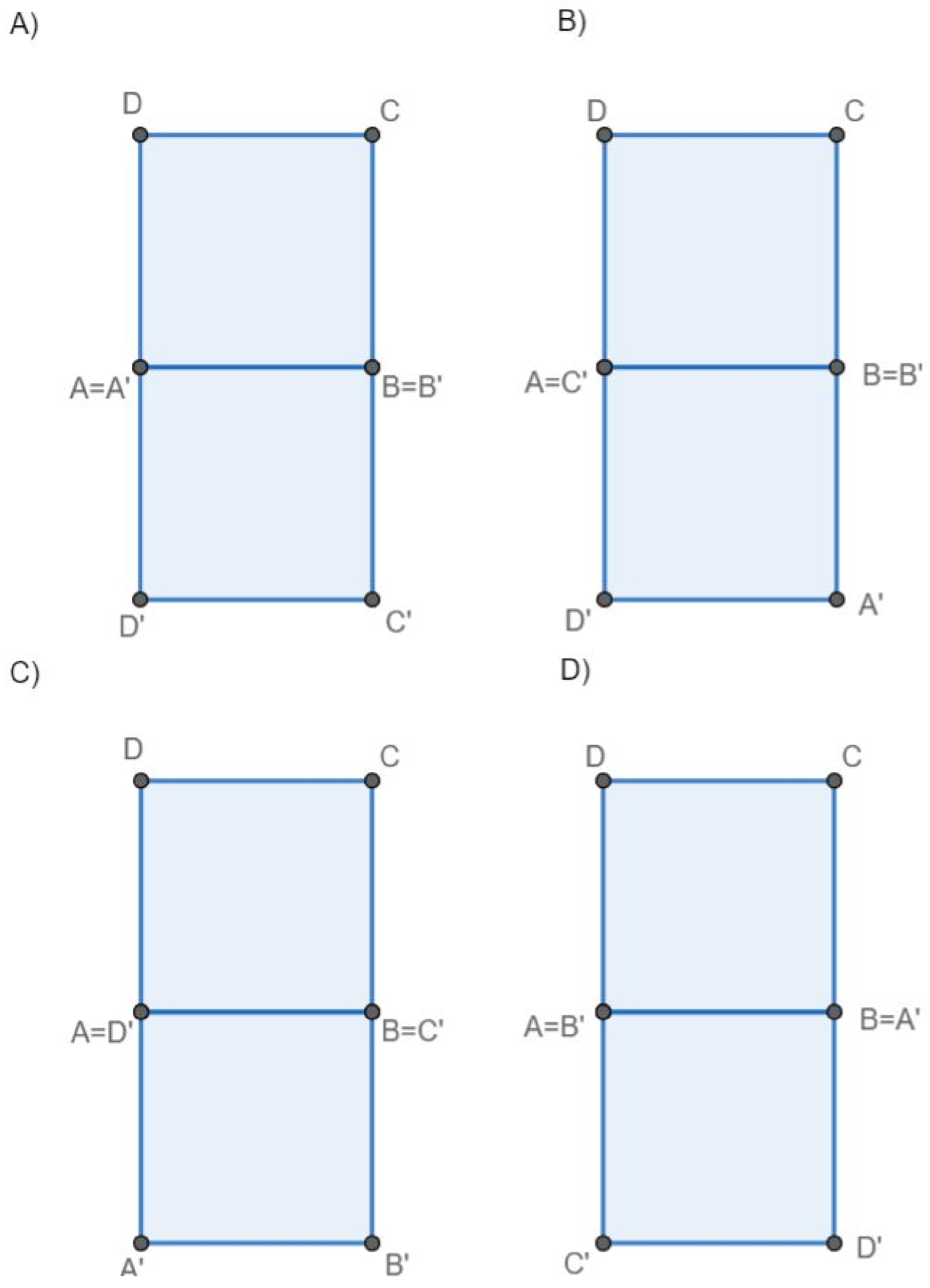
a) Decide between which shapes there is symmetry in the figures.

b) Determine the found symmetry into direct and indirect.

c) Determine the symmetry and defining elements.



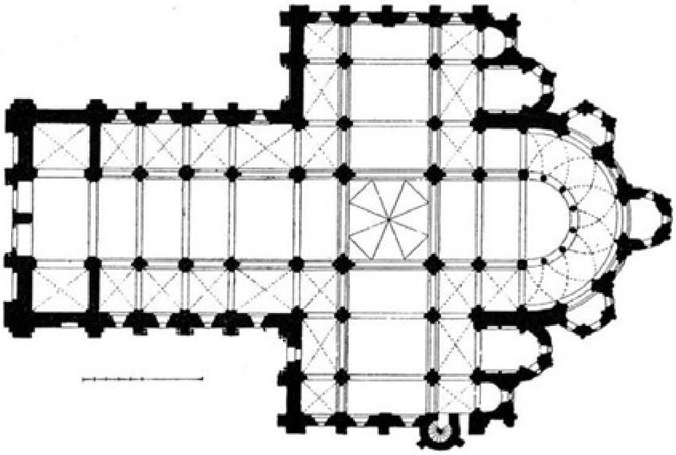
Example 2.29. Specify all the symmetries which convert the square $ABCD$ into $A'B'C'D'$. U každého zobrazení zapište i jeho určiújící prvky.



Solution: A) the reflectional symmetry with the axis AB, B) the rotational symmetry given by point B and the angle $90°$, C) the translational symmetry given by $\overrightarrow{DA}$, D) the point reflectional symmetry with the point at the center of line segment AB.

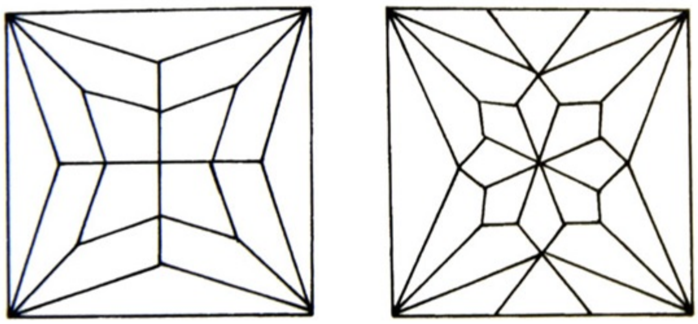
Excercise 2.30. Sestrojte kosočtverec $ABCD$ tak, aby $|AB| = |BD|$. Specify all the symmetries which convert equilateral triangle ABD into triangle, which creates together with triangle ABD rhombus $ABCD$.

Example 2.31. The picture shows a plan of the Romanesque church of Sainte Foy in Conques, France. Find the symmetries in the image and draw them: , e.i. redraw the section of plan a) in translational symmetry, b) in rotational symmetry, c) in point reflection symmetry d) in reflectional symmetry.



Solution: See the appendix at the end of the text.

Example 2.32. The picture shows the star vault of the Renaissance castle Náměšť na Hané. Find all symmetry axes in both vaults and mark the center of symmetry. Try to suggest the coloring of the image so that: a) 1 axis , b) 2 axes, c) 3 axes, d) 4 axes souměrnosti of reflectional symmetry



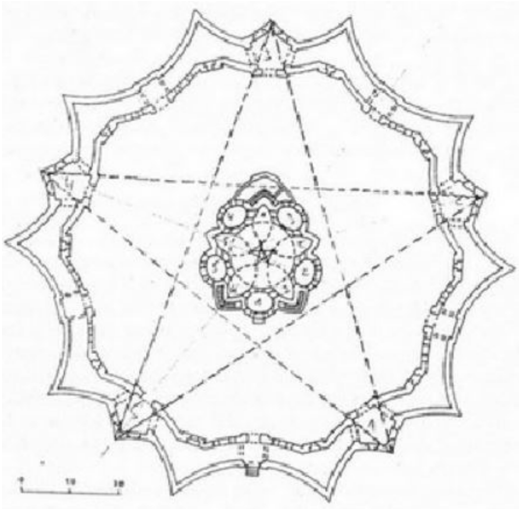
Solution: Example of coloring the first vault in the appendix at the end of the text.

Excercise 2.33. In the picture you can see a flor plan of the church of St. John of Nepomuk at Zelená hora in the shape of a five-pointed star:

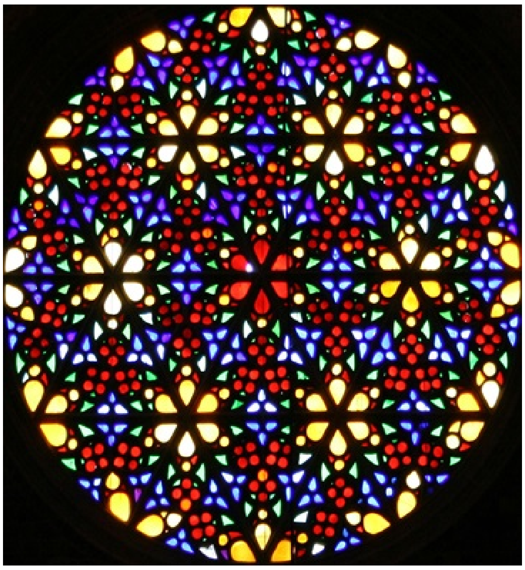
a) Find all axes of symmetry.

b) By how many degrees do we need to turn the shape, so that one point of the star maps into the neighbouring one?

c) Choose one axis and color the flor plan it according to it.



Excercise 2.34. In the picture you can see the rosette in the La Seu cathedral in Mallorce. In it, find a translational symmetry, a rotational symmetry, a point reflection symmetry and a reflectional symmetry.

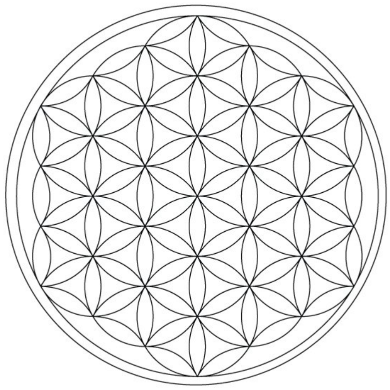


Excercise 2.35. In the picture you can see the mandala well known as *Flower of Life*.

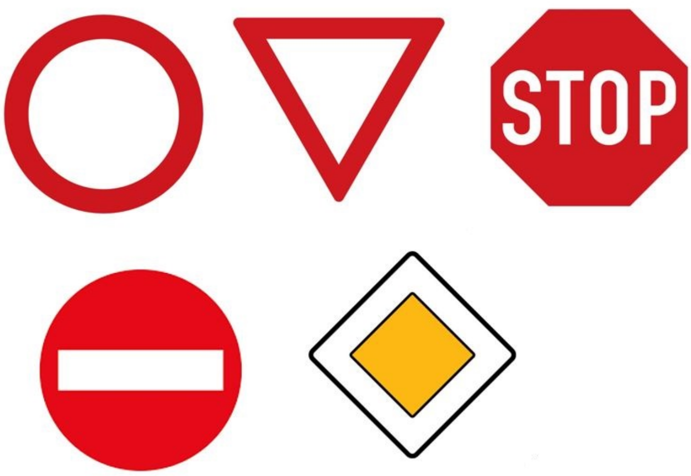
a) Find all axes of symmetry.

b) Color the mandala so that the number of symmetry axes changes to 2.

c) Is it possible to color the mandala to be a point reflection symmetrical but not a reflection symmetrical? Justify.



Example 2.36. Name the road signs in the picture and specify the number of symmetry axes. For your homework, shoot additional tags on the road and group them by number of symmetry axes.



Solution: See the appendix at the end of the text.

3 Composition of symmetries

Example 3.1. Complete the following statement correctly:

1. The composition of (two) translational symmetries is. . . . .
2. The composition of two point reflection symmetries is. . . . .
3. The composition of two rotations with a common center is . . . .
4. The composition of two reflectional symmetries with a common axis is. . . . .
5. The composition of two reflectional symmetries with different parallel axes is . . . .
6. The composition of two reflectional symmetries with intersecting axes is. . . . .

For each of your statements, draw a suitable picture.

Solution: 1) The composition of (two) translational symmetries is *a translational symmetry.* 2) The composition of two point reflection symmetries is *a translational symmetry.* 3) The composition of two rotations with a common center is *a rotational symmetry with the same center.* 4) The composition of two reflectional symmetries with a common axis is *an identity.* 5) The composition of two reflectional symmetries with different parallel axes is *a translational symmetry.* 6) The composition of two reflectional symmetries with intersecting axes is *a rotational symmetry with the center in the intersection of the axes.*

Excercise 3.2. An equilateral triangle ABC is given. S_1, S_2 and S_3 are the centers of its sides AB, BC, CD. Determine the image of ABC in $F = \mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3$, where $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are point reflections with centers S_1, S_2, S_3, respectively. Determmine the resulting mapping F.

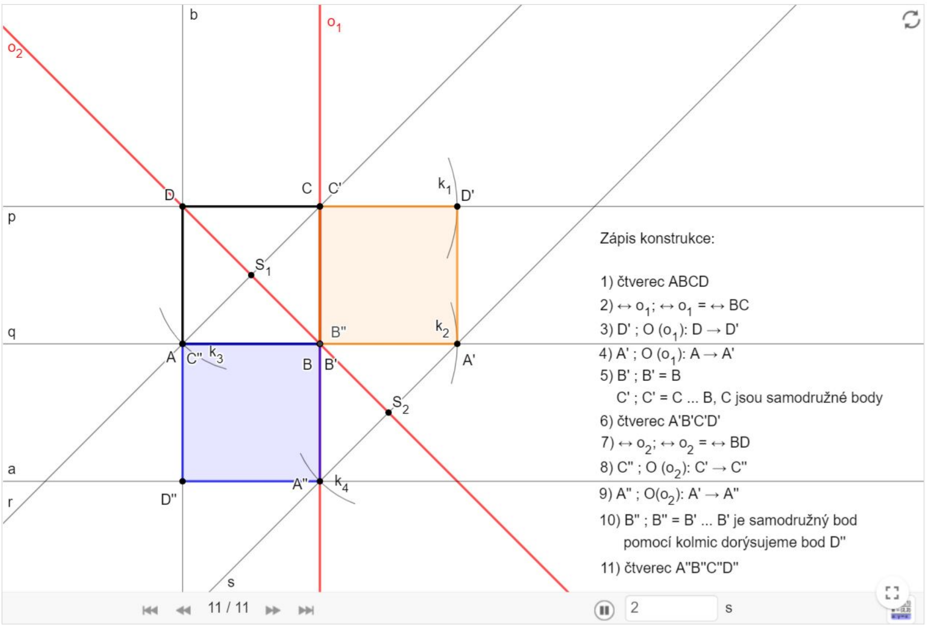
Excercise 3.3. The square $ABCD$ map first in the translational symmetry $\mathcal{T}(\overrightarrow{CD})$ and the image $A'B'C'D'$ then in rotation $\mathcal{R}$ specified by the D and angle $90°$. Determine the composition $f = \mathcal{R} \circ \mathcal{T}$.

Excercise 3.4. The rhombus $ABCD$ map first in the reflectional symmetry $\mathcal{O}$ with the axis DC and the image $A'B'C'D'$ then in the rotation $\mathcal{R}$ specified by the point C and angle $\alpha = |\sphericalangle DCB|$. Determine the composition $f = \mathcal{R} \circ \mathcal{O}$.

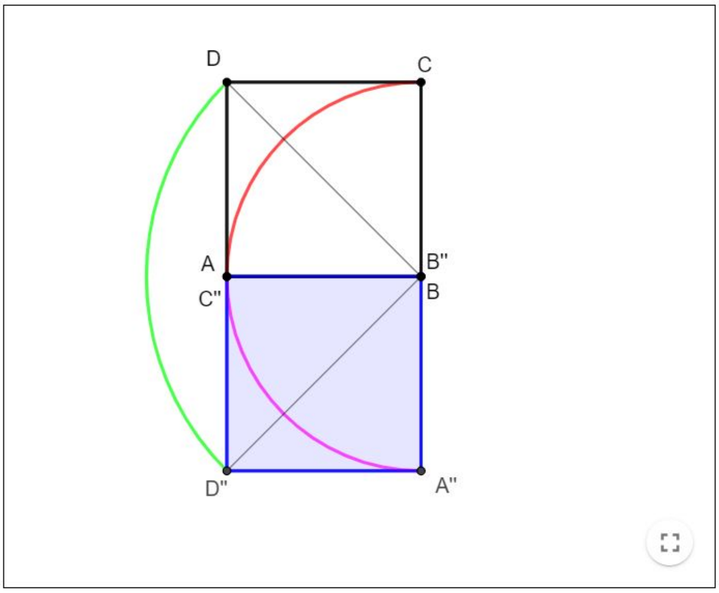
Example 3.5. The square $ABCD$ map first in the reflectional symmetry $\mathcal{O}_1$ with the axis BC and the image $A'B'C'D'$ in the reflectional symmetry $\mathcal{O}_2$ with the axis BD. Determine the composition $f = \mathcal{O}_2 \circ \mathcal{O}_1$.

Solution:

Resultin mapping is $\mathcal{R}(B, +90°)$.



The construction step by step:
`https://www.geogebra.org/m/saxej6u9#material/p59bhmen`



Excercise 3.6. A pair of congruent triangles, ABC and KLM, is given of the triangle KLM maps onto the corresponding vertex of the triangle ABC. (Make use of the fact that the pair preimage- image uniquely defines the reflectional symmetry.) Construct two reflectional symmetries in sucha way that the resulting composition of these symmetries maps the triangle KLM onto the triangle ABC.

Hint: Use two reflectional symmetries. Choose the first symmetry so that a chosen point

Choose the second reflectional symmetry in such a way that the image created by the first symmetry would map onto the triangle ABC.

4 Proof tasks

Example 4.1. In planar symmetry, let a point A maps into point A', a point B maps into point B', and a point C maps point C'. Prove that if point C lies between points A and B, then point C' lies between points A' and B'.

Solution: From the definition of plane symmetry, the following relations are valid for points A, B, C and their images A', B', C':

$$A'C' \cong AC, \quad B'C' \cong BC, \quad A'B' \cong AB \tag{1}$$

Figure 1

If point C is between points A and B, then $AC + CB = AB$ valid. Due to (1) $A'C' + C'B' = A'B'$ is also valid, i.e. the point C' lies between the points A' and B'.

The statement we are proving also follows from the theorem which says that in a symmetry in a plane, the image of the line segment AB is a line segment $A'B'$ of the same length as AB.

That proof is part of the proof of that theorem.

Example 4.2. Z is a plane symmetry that has two different self-associated points. Does this symmetry still have any self-associated points?

Solution: Let us denote the self-associated points as A, B, and their images in the given **symmetry / mapping** as A', B'. According to the task assigned, $A = A'$, $B = B'$. The image of the line segment AB is thus the line segment AB. If point X lies between the points A, B, then point X' lies between points A', B', i.e. also between points A, B and the following holds: $AX \cong AX'$ (also $BX \cong BX'$), i.e. $X = X'$. The answer to the question is thus positive. Furthermore, it is evident that in mapping Z each point of the line segment AB is self-associated.

By analogy, it is easy to show that not only the points of the line segment AB are self-associated, but also all the points of the straight line **containing** AB are self-associated: If point Y lies on the straight line AB and if it e.g. holds that point B lies between points A, Y, then point B lies between points A, Y'. As $BY' \cong BY'$, then $Y = Y'$.

In the case when the considered point of the straight line AB lies on the half-line opposite to the half-line AB, we proceed in a similar way. Any symmetry Z that has at least two self-associated points therefore has infinitely many self-associated points; all the points of the straight line determined by two self-associated points are self-associated.

Example 4.3. Prove that any symmetry in a plane in which there are three non-collinear self-associated points, every point is self-associated, i.e. such a symmetry is identity.

Solution: As follows from the results stated in example 4.2, each point of the straight lines AB, BC, AC is self-associated. If X is any point in the plane not lying on any of these straight lines, we can always construct a straight line that passes through X and intersects at least two of these lines in two different points (**prove**). THese points (in fig. **??** denoted as Y, Z) are self-associated, and thus also the point X is, as follows from 4.2, is self-associated, and hence each point of the plane is self-associated and the given mapping is an identity.

Remark: From the solutions of examples 4.2 and 4.3, it is evident that if a symmetry in a plane has two different self-associated points, then such mapping is either reflectional symmetry with the axis given by these points, or an identity (in the cases when the mapping has another self-associated point, not lying on the straight line given by the two self-associated points).

Example 4.4. Two different points P, P' are given. Determine at least one symmetry in which point P' is the image of P.

Solution: Based on the properties of the individual types of symmetries in a plane it is evident that point P' is the image of P in:

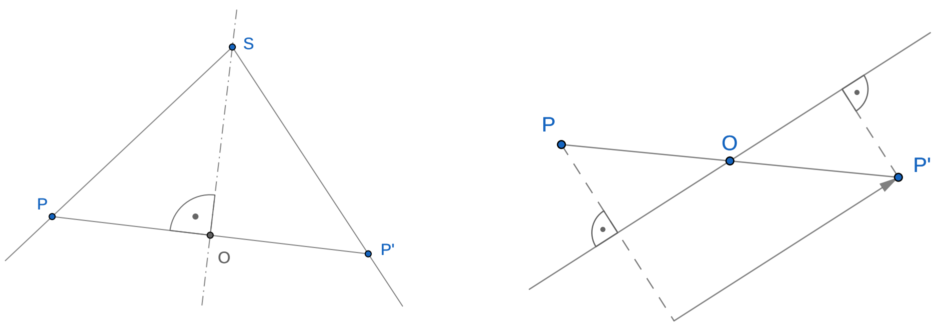
a) reflectional symmetry with the axis being the axis of the line segment PP' (Fig. 2a),

b) point reflection symmetry with the center in the midpoint O of the line PP',

c) rotational symmetry with the center S lying on the axis of the line segment PP' and oriented angle $\sphericalangle PSP'$ (S is an arbitrary point of the axis of the line segment PP'; ifi $S \in PP'$, i.e. $S = 0$, the corresponding rotation is the point reflection symmetry - see b) (Fig. 2a),

d) translational symmetry determined by the ordered pair P, P',

e) glide reflectional symmetry, composed of the reflectional symmetry with the axis o incident with the midpoint of the line segment PP', with $o \not\parallel PP'$, $o \not\perp PP'$, and translational symmetry in the direction of this axis, with the length of translation equal to the length of the orthogonal projection of the line segment PP' onto the line o (Fig. 2b).



(a) (b)

Figure 2

Remark: Notice the connections between the results of this example and the theorem about the determination of a symmetry in a plane.

Example 4.5. Two equal circles $k_1(S_1, r)$, $k_2(S_2, r)$ are given, intersecting in points X, Y. Determine at least one symmetry in which the circle k_2 is the image of circle k_1.

Solution: In any symmetry, the image of the circle $k(S, r)$ is the circle with equal radius r, whose center is in the image of point S. The image of the given circle k_1 is a circle k_2 in any symmetry in which the point S_2 is the image of the point S_1. To solve the task, we may thus use the results of example 4.4.

Example 4.6. Show that the composition of two reflection symmetries with the axes perpendicular to each other is the point reflection symmetry with the center in the intersection of the two axes.

Solution: Let us denote the considered reflection symmetries O_1, O_2, their axes o_1, o_2, the intersections of these axes as S, the mapping composed of these symmestries as Z, i.e. $Z = O_1 \circ O_2$. Let u denote the image of the point X in O_1 as X_1, the image of the point X_1 in O_2 as X', i.e. the image of the point X in Z is the point X'.

The mapping Z will be the point reflection symmetry with the center in S if and only if for each point X of the plane and its image X' in that mapping, the point S is the midpoint of the line segment XX'.

a) If $X = S$, then $X = X' = S$, because S is a self-associated point of the mapping Z.

b) Let $X \in o_1$ and $X \neq S$. Then $X = X_1$ and the point S is the midpoint of XX'. (Fig. 3)

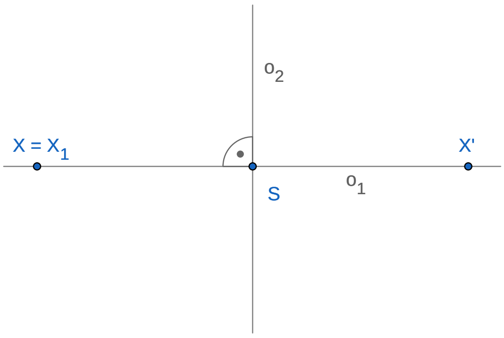


Figure 3

c) Let $X \in o_2$ and $X \neq S$. Then $X_1 \in o_2$, S is the midpoint of XX_1 a $X_1 = X'$. It thus holds that S is the midpoint of XX'. (Fig. 4)

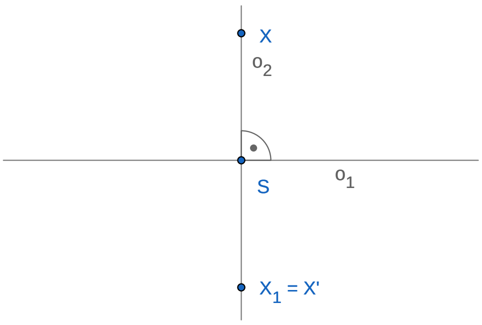


Figure 4

d) Let $X \notin o_1$ and $X \notin o_2$. Then $X_1 \notin o_2$. Denote the intersection of the straigth line XX_1 with the axis o_1 as X_0. The points X, S, X_1 do not lie on the same straight line. For the triangle XSX_1, the following holds: $XS \cong SX_1$ and the line o_1 is its axis of symmetry, i.e. $\sphericalangle XSX_0 \cong \sphericalangle X_0SX_1$. Denote further the intersection of the line X_1X' with the axis o_2 as X''. The points X_1, S, X' also do not lie on the same straight line. For the triangle X_1SX' the following holds: $X_1S \cong X'S$ and o_2 is its axis of symmetry, i.e. $\sphericalangle X_1SX'' \cong \sphericalangle X''SX'$. The angle $\sphericalangle XSX'$ is the graphic sum of the angles $\sphericalangle XSX_1$, $\sphericalangle X_1SX'$, i.e.

$$\sphericalangle XSX' = \sphericalangle XSX_1 + \sphericalangle X_1SX' = 2(\sphericalangle X_0SX_1 + \sphericalangle X_1SX'') = 2R.$$

The angle XSX' is thus straight, which meanst that the points X, S, X' lie on a single straight line. It further holds that $XS \cong X_1S \cong X'S$, i.e. $XS \cong X'S$. Thus the point S is the midpoint of the line segment XX'.

For all the cases that can occur for the position of an arbitrary point X of the given plane with respect to the straight lines o_1, o_2 we thus conclude that the point S is the midpoint of the line segment XX'. The relevant mappint $Z = O_1 \circ O_2$ is thus the point reflection symmetry with the center S.

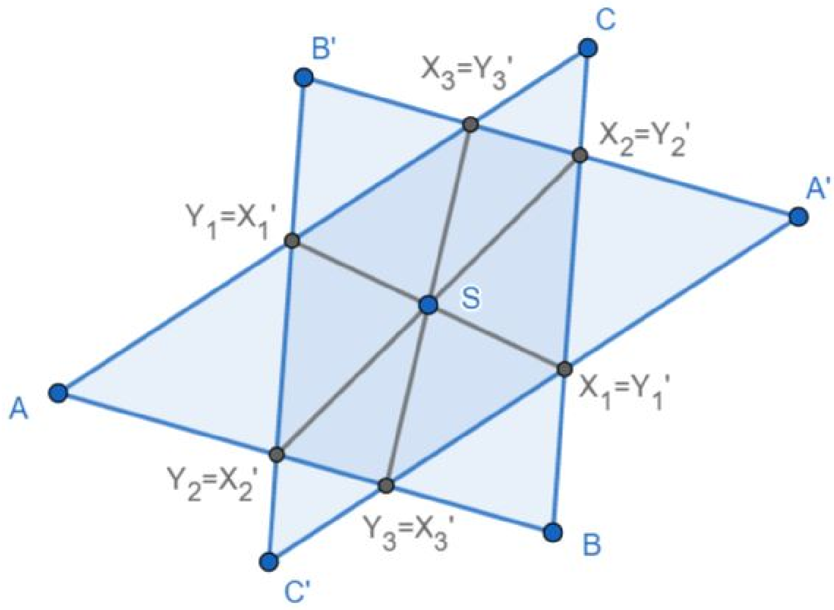
5 Constructive tasks

Example 5.1. Given a point S lying inside the given triangle ABC. Construct a rung XY of the triangle ABC, which is bisected by the point S.

Solution: The endpoints of the sought line segment XY lie on the border of the given triangle and point-reflectional with the center S. Thus:

$$\mathcal{S}(S) : X \mapsto X' = Y.$$

We do not know their position, but we know the object, on which the source as well as the image lie. We thus map triangle ABC in point reflectional symmetry with the center S and the endpoints of the rung will lie at the intersection of ABC and $A'B'C'$.



Conclusion: Depending on the position of S in the triangle, the task may have 1, 2, or 3 solution. (Draw all the cases.)

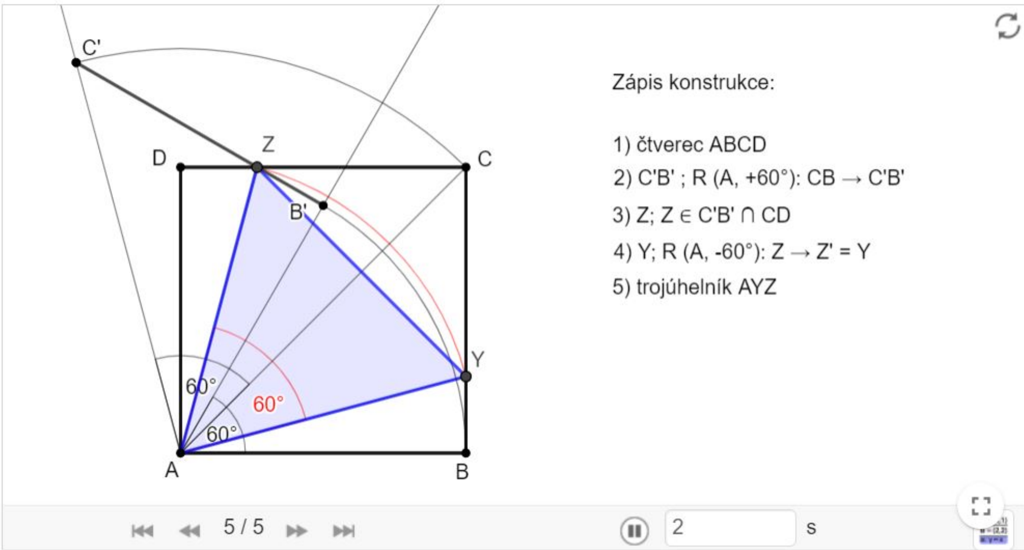
Excercise 5.2. Let the circle $k(S; r)$ be given, $r = 3\,\text{cm}$, and a point A such that $|SA| = 1,5\,\text{cm}$. Construct all the chords XY of the circle k that are 5,5 cm long and pass through A.

Excercise 5.3. A line segment AA_1 with the length $5\,\text{cm}$ is given. Construct all the triangles ABC, for which AA_1 is the median and at the same time $b = 6\,\text{cm}$ a těžnice $t_b = 6\,\text{cm}$.

Excercise 5.4. A line segment AA_1, $|AA_1| = t_a$ is given. Construct all the triangles ABC, for which AA_1 is the median t_a and whose other medians are t_b, t_c.

Excercise 5.5. Construct a trapezoid $ABCD$ if the lengths of both its bases a, c, and both its diagonal e, f are given.

Example 5.6. Into the square $ABCD$ inscribe an equilateral triangle AYZ such that $Y \in BC$ and $Z \in CD$.

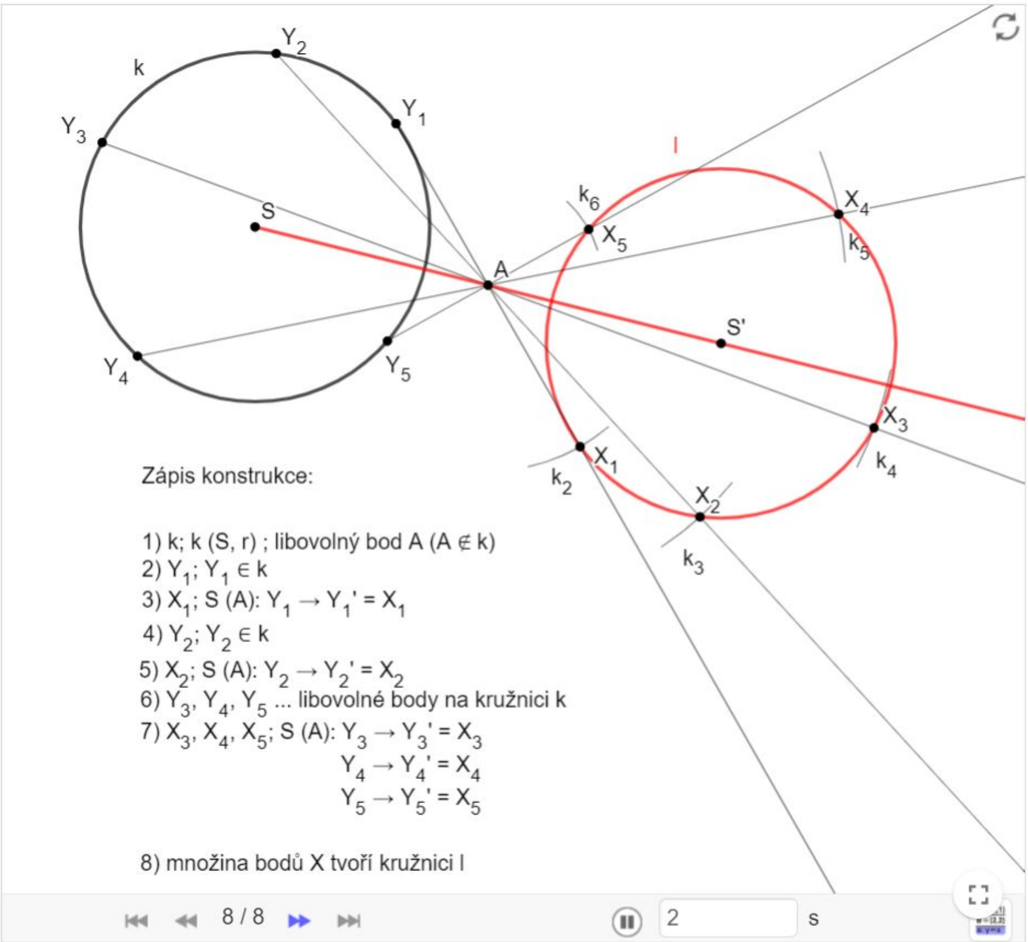


The construction step by step:
`https://www.geogebra.org/m/saxej6u9#material/p59bhmen`

Excercise 5.7. Two concentric circles $k(S; 2cm)$, $l(S; 3cm)$ are given and point A such that $|SA| = 2,3\,\text{cm}$. Construct all equilateral triangles ABC, for which $B \in k$, $C \in l$.

Excercise 5.8. A straight line a and a point $A \in a$ are given, further another straight line s is given, $s \neq a$. Construct a regular hexagon $ABCDEF$ with the center $S \in s$ and the side $AB \in a$.

Example 5.9. A circle $k(S; r)$ and point A, not lying on k, are given. Determine the set of points X such that the point A is the midpoint of XY, while Y lies on the circle k.



Konstrukce "krok po kroku":
`https://www.geogebra.org/m/saxej6u9#material/p59bhmen`

Excercise 5.10. A square $ABCD$, a straight line p and point S, not lying on the straight line p are given. Construct a line segment XY such that the point S is its midpoint and point, the point X lied on the straight line p and the point Y lied on the square $ABCD$.

Excercise 5.11. Two intersecting lines a, b and line segment MN are given. Construct the square $ABCD$ with the side AB such that AB lead parallel to MN and $|AB| = |MN|$ and point A lies on the straight line a, the point B on the straight line b.

Excercise 5.12. A straight line p and two circles, k and l, lying in different half-planes determined by the line p. Construct a line segment XY perpendicular to the line p such that X lies on the circle k, and point Y on the circle l and the straight line p pass through the middpoint XY.

Excercise 5.13. Two intersecting lines a, b and point A, not incident with either of them, are given. Construct a square $ABCD$ such that the point B lies on a, the point D lies on b.

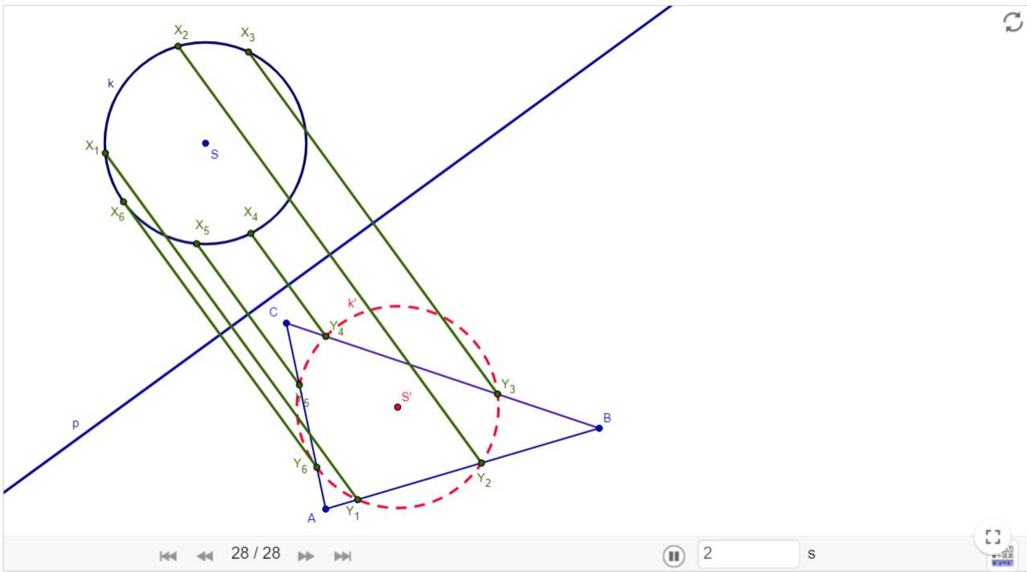
Excercise 5.14. A straight line p and points A, B, lying in the same half-plane determined by the line p, are given. On the line p, determine a point X such that the distance $|AX| + |XB|$ is minimal.

Excercise 5.15. Let two circles k, l be given, intersecting in points X, Y. Draw such a line through X which cuts chords of the same lengths on both circles (consider circles with different diameters).

Excercise 5.16. Let the straight line p and the circles k and l are given, $k(S;r)$, $l(O;\rho)$, $S \neq O$, $r > \rho$, $|S, \leftrightarrow p| = d_1$, $|O, \leftrightarrow p| = d_2$. Construct all the straight lines parallel to p, on which the circles k, l cut chord of the same length.

Excercise 5.17. Construct all the triangles ABC, if you know: $a+b+c = o$, kde $o = 12\,\text{cm}$, $\alpha = 45°$, $\beta = 75°$.

Example 5.18. The straight line p, the circle k and triangle ABC are given. Construct all the line segments XY such that X lies on the circle k, Y on the perimeter of the triangle ABC, the line segment XY is perpendicular to p and the center of the line segment XY lies on the straightline p.

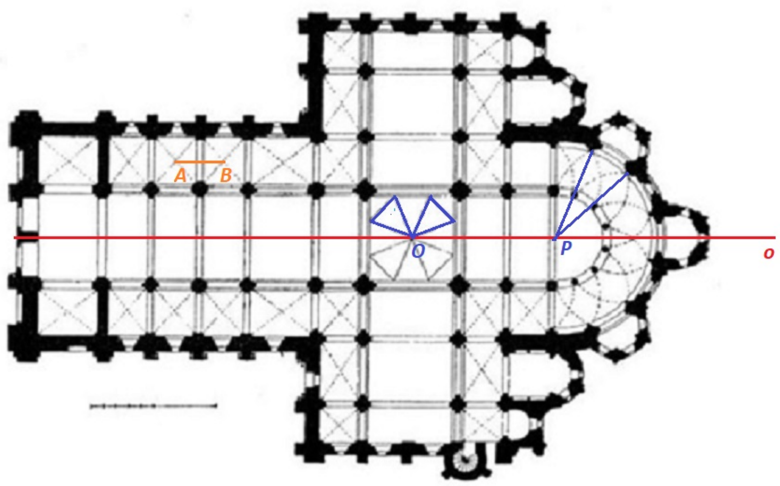


The construction step by step: `https://www.geogebra.org/m/HxhzBUjd`

Appendix 1

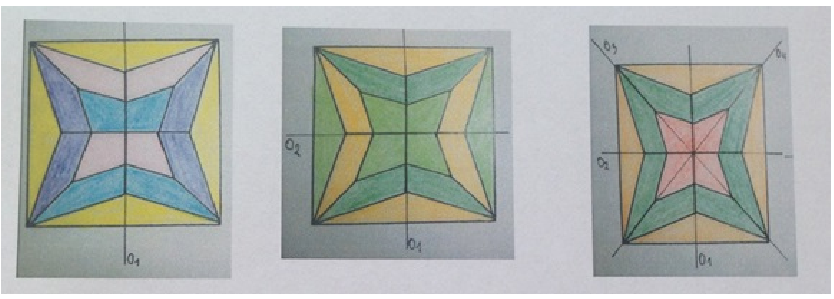
Solution to excercise 2.31

Examples of symmetries in the floor plan: reflectional symmetry with the axis o of the floor plan of the whole cathedral, rotational symmetry with the center O and angle$90°$, rotational symmetry with the center P and angle $\frac{180}{7}^{\circ}$, point reflection with the center O, translational symmetry given by the vector $\overrightarrow{AB}$, etc.



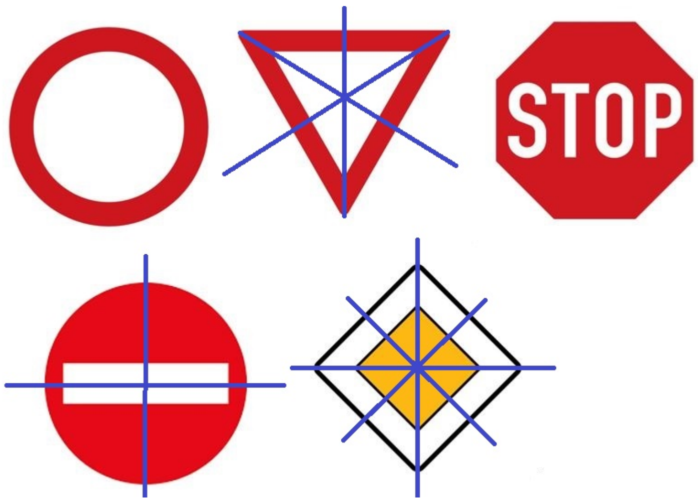
Solution to excercise 2.32

Examples of colority a) with one axis of symmetry, b) with two axes of symmetry, d) with four axes of symmetry. Case c) with two axes of symmetry cannot be coloured.



Solution to excercise 2.36

No entry of all vehicles (in both directions) – infinitely many axes, Give right of way – 3 osy, Stop, give right of way – no axis, No entry of all vehicles in one direction – 2 axes, Main road - 4 axes.



Appendix 2

Pass the exam (group activity)

The following material presents a leaf method that can be successfully used, for example, to repeat terms. Copy prepared terms - make as many copies of the entire set as there are groups. Cut them into leaves and give each set of terms one group.

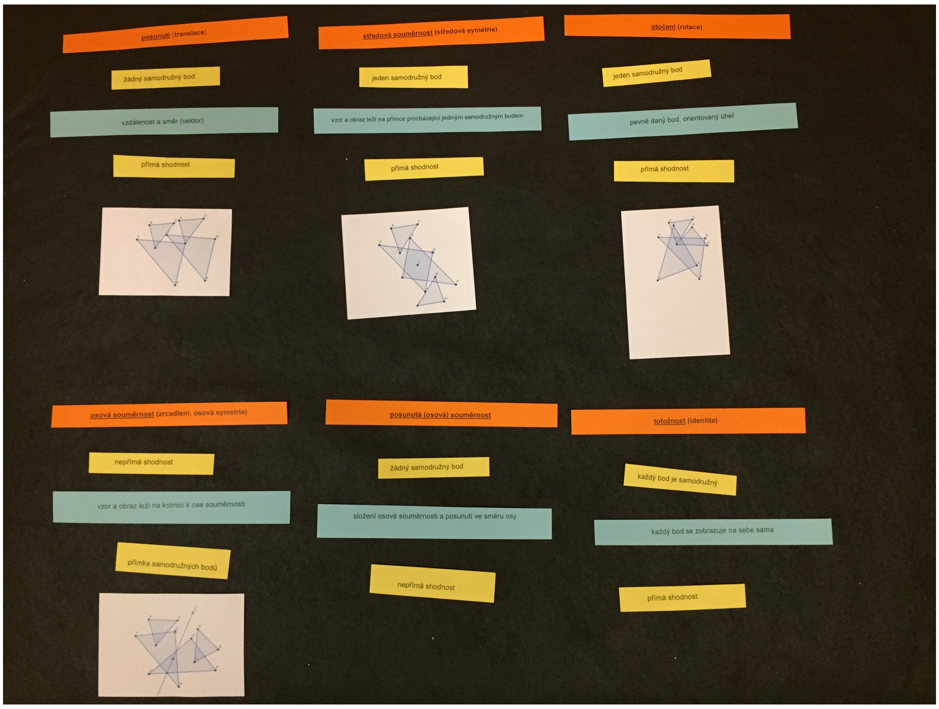
The goal is to first correctly assign the other terms from the entire set to all underlined terms. (For example, we can ask students to add every picture underlined with a suitable image.)

<u>glide reflection symmetry</u>
<u>reflectional symmetry</u>
<u>identity</u>
<u>point reflection symmetry</u>
<u>rotational symmetry</u>
<u>translational symmetry</u>

- oriented line, resp. vector
- fixed point, oriented angle
- the source as well as the image lie on the straight line which pass throught the self-associated point
- each point maps into itself
- the source as well as the image lie on the perpendicular to the symmetry axis
- the composition of reflectional symmetry and translation in the direction of the axis
- direct symmetry
- direct symmetry
- direct symmetry
- direct symmetry

- indirect symmetry
- indirect symmetry
- a straight line of self-associated points
- one self-associated point
- one self-associated point
- no self-associated point
- one self-associated point
- each point is a self-associated point

The following figure shows an example of the result of such a group activity:



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