Computation with Roman Numerals

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It is widely held that computation with Roman numerals is difficult if not impossible.[1] By presenting simple procedures for adding and multiplying with Roman numerals, we show that this common idea is mistaken. We also suggest that its acceptance arises from the mistaken belief that computations in different numeral systems will mirror one another, a belief which can be explained as depending upon a confusion of numerals with numbers.

I. Addition

The Roman and Arabic numeral systems differ in that the latter has symbols of different types for units ('0', '1', ..., '9') and indicates powers of ten by symbol position, while the former has symbols of different types for powers of ten ('I', 'X', 'C', ...)[2] and indicates units by symbol iteration.[3] For this reason, Roman numerals are *numeral-wise additive* in that each Roman numeral denotes the sum of the numbers denoted by the simple Roman numerals[4] occurring within it. For example, 'CXLVI' denotes the sum of C, XL, V, and I. That Roman numerals have this property suggests that one could add with them by simple concatenation,

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[1] *Cf.*, *e.g.*, K. MENNINGER, *Number Symbols and Number Words*, trans. by P. BRONEER (Cambridge, Mass.: M.I.T. Press, 1969), p. 298; C.M. TAISBAK, "Roman Numerals and the Abacus", *Classica et Mediaevalia*, **26** (1965), p. 153; and D.D. STREBE, *Elements of Modern Arithmetic* (Glenville, Il.: Scott, Freeman, and Co., 1971), p. 13.

It should perhaps be observed that two obvious methods for computing with Roman numerals are (1) translate the Roman numerals into Arabic, perform the appropriate Arabic computations, and translate the result into Roman numerals, and (2) translate the Roman numerals into tally marks, perform the appropriate tally computations, and translate the result into Roman numerals. What one wants, though, and what we provide, are simple and efficient procedures for computing with Roman numerals which are not parasitic upon computations in some other numeral system.

[2] For the purpose of this point we ignore Roman numerals denoting five times some power of ten (*i.e.*, 'V', 'L', ...) and the *subtractive numerals* (*i.e.*, 'IV', 'XL', ...; 'IX', 'XC', ...), which make it it possible to indicate units economically.

[3] Since the Roman system indicates powers of ten by the presence of numerals of the appropriate type (and not by numeral position), a symbol for zero is not needed for writing Roman numerals.

[4] A *simple* Roman numeral is either a numeral consisting of a single symbol or a subtractive numeral.

if one had transformation rules with which to produce a Roman numeral from the result. The procedure we now present is based on this idea.

Our procedure has two stages. The first (steps A and B) generates from two or more Roman addends a concatenation of simple Roman numerals having two properties:

(1) *hierarchical distribution:* no simple numeral stands to the left of any simple numeral denoting a larger number than it denotes.

(2) *additive adequacy:* the sum of the numbers denoted by the simple numerals in the concatenation is the sum of the numbers denoted by the original addends.

The second stage (step C) produces the Roman numeral for the sum of the denotations of the original addends by applying transformations which preserve these properties to the result of stage one.

Stage One

A. Concatenate the Roman addends, flagging the right-most simple numeral of each.

B. Regroup so that the resulting concatenation is hierarchically distributed. Eliminate flags.

Step A simply requires that we write Roman addends in rows rather than the columns we use for Arabic addends. We write Arabic addends in columns because this makes it (psychologically) easier for us to exploit the positional character of Arabic numerals when we add. Since Roman numerals are not positional, there is no reason to write Roman addends in this way. The device of flagging prevents possible ambiguities. It prevents one from reading, for example, 'IX' where 'I' is the last simple numeral of one addend and 'X' is the first simple numeral of the next, as the subtractive numeral for nine ('IX'). Step B eliminates the need for flags by hierarchically distributing the simple numerals in the result of step A, producing an additively adequate, hierarchically distributed concatenation of simple Roman numerals. Stage two is designed to produce a Roman numeral from this concatenation.

Stage Two

C. Apply these transformations[5] wherever possible to the result of step B (the order in which these transformations are applied is irrelevant):

1. $i\ i\ i\ i\ i \rightarrow v$
2. $i\ i\ i\ i \rightarrow iv$
3. $iv\ iv \rightarrow v\ i\ i\ i$
4. $v\ v \rightarrow x$

[5] If N denotes the non-negative integer n and R is the Roman numeral denoting r, we let $\ulcorner(\mathrm{R})_{\mathrm{N}}\urcorner$ denote the Roman numeral for $r \cdot 10^{n}$. *E.g.*, $(\mathrm{I})_1$ is 'X', and $(\mathrm{XIV})_2$ is 'MCD'. (Those unfamiliar with the

5. *ix ix* → *x v i i i*
6. *iv i* → *v*
7. *v iv* → *ix*
8. *ix iv* → *x i i i*
9. *ix i* → *x*
10. *ix v* → *x iv*

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square corners notation will find it explained in the section on "quasi-quotation" in W. V. D. QUINE, *Mathematical Logic*, rev. ed. [Cambridge, Mass.: Harvard University Press, 1951], pp. 33–7.)

Transformation C8 is to be read, "For each non-negative integer n, if the concatenation contains both an occurrence of $(\mathrm{IX})_n$ and an occurrence of $(\mathrm{IV})_n$, replace them with one occurrence of $(\mathrm{X})_n$ and three occurrences of $(\mathrm{I})_n$, hierarchically distributing them within the concatenation," and the other transformations in like fashion.

The use of these natural conventions makes possible not only the economical expression of these transformation rules (*e.g.*, C4 applied three times to the concatenation 'DDLLVV' would yield 'MCX') but also a surprisingly compact multiplication table (see the next section).

Note that, if one were to lay down as the first rule "expand all subtractive numerals," our addition procedure would be much simpler: only transformations C1, C2, and C4 would be needed.

Note further that a calculator has an effective procedure for determining the Roman numeral denoted by $\ulcorner(\mathrm{R})_\mathrm{N}\urcorner$, whenever $\ulcorner(\mathrm{R})_\mathrm{N}\urcorner$ denotes a Roman numeral that can be written with the simple Roman numerals at his command. He begins by writing the simple Roman numerals at his command in rows four columns wide, in order of increasing 'size':

Chart 0:	I	IV	V	IX
	X	XL	L	XC
	C	CD	D	CM
	M	...	...	...

He then constructs further charts, one for each row of Chart 0:

Chart 1:	$(\mathrm{I})_0$	$(\mathrm{IV})_0$	$(\mathrm{V})_0$	$(\mathrm{IX})_0$
	$(\mathrm{I})_1$	$(\mathrm{IV})_1$	$(\mathrm{V})_1$	$(\mathrm{IX})_1$
	$(\mathrm{I})_2$	$(\mathrm{IV})_2$	$(\mathrm{V})_2$	$(\mathrm{IX})_2$
	$(\mathrm{I})_3$	...	...	...
Chart 2:	...	...	...	...
	$(\mathrm{X})_0$	$(\mathrm{XL})_0$	$(\mathrm{L})_0$	$(\mathrm{XC})_0$
	$(\mathrm{X})_1$	$(\mathrm{XL})_1$	$(\mathrm{L})_1$	$(\mathrm{XC})_1$
	$(\mathrm{X})_2$	...	...	...
Chart 3:	...	...	...	...
	...	...	...	...
	$(\mathrm{C})_0$	$(\mathrm{CD})_0$	$(\mathrm{D})_0$	$(\mathrm{CM})_0$
	$(\mathrm{C})_1$	...	...	...
Chart 4:	...	...	...	...
	...	...	...	...
	...	...	...	...
	$(\mathrm{M})_0$	...	...	...

and so on. These charts give him a procedure for finding the Roman numeral denoted by $\ulcorner(\mathrm{R})_\mathrm{N}\urcorner$ if $\ulcorner(\mathrm{R})_\mathrm{N}\urcorner$ denotes a simple Roman numeral at his command: it is found in the position of Chart 0 corresponding to the position of $\ulcorner(\mathrm{R})_\mathrm{N}\urcorner$ on the chart on which it occurs. If $\ulcorner(\mathrm{R})_\mathrm{N}\urcorner$ denotes a non-simple numeral writeable with the simple Roman numerals at the calculator's command, he can also find the Roman numeral it denotes: it would be the (hierarchically distributed) concatenation of the simple numerals denoted by $\ulcorner(\mathrm{A})_\mathrm{N}\urcorner, \ldots, \ulcorner(\mathrm{K})_\mathrm{N}\urcorner$, where A, ..., K are the simple numerals occurring in R.

These transformations yield a Roman numeral from the concatenation of simple numerals resulting from stage one.[6] Since that concatenation is additively adequate and the transformation rules preserve additive adequacy, the Roman numeral resulting from stage two denotes the sum of the numbers denoted by the original addends.

Working through an example will help to make this procedure clear. Suppose one wished to compute the Roman numeral for the sum of CXLVIII, CXLIV, and LXXII. Then:

Step A yields	CXLVIII*CXLIV*LXXII
Step B yields	CCLXLXLXXVIVIIIII
Step C6 (thrice) yields[7]	CCL$\underline{\mathrm{LL}}$V$\underline{\mathrm{V}}$IIII
Step C3 (twice) yields	CC$\underline{\mathrm{C}}$L$\underline{\mathrm{X}}$IIII
Step C2 yields	CCCLX$\underline{\mathrm{IV}}$.

And this is the Roman numeral for the sum desired.

If the reader will work through this example and a few of his own, he will see that this procedure is simple and natural, though it might at first appear a bit complicated. With obvious shortcuts (eliminating step A, applying transformations as seems appropriate during concatenation, and using a blackboard to avoid having to re-write) one can, with a little practice, add several numbers dictated in Roman numerals almost as quickly as they can be dictated.[8]

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[6] A hierarchically distributed concatenation of simple Roman numerals is a Roman numeral if, and only if, for every non-negative integer n, it contains:

(a) at most three occurrences of $(\mathrm{I})_n$
(b) at most one occurrence of $(\mathrm{IV})_n$
(c) at most one occurrence of $(\mathrm{V})_n$
(d) at most one occurrence of $(\mathrm{IX})_n$
(e) no joint occurrences of $(\mathrm{IV})_n$ and $(\mathrm{I})_n$
(f) no joint occurrences of $(\mathrm{IV})_n$ and $(\mathrm{V})_n$
(g) no joint occurrences of $(\mathrm{IV})_n$ and $(\mathrm{IX})_n$
(h) no joint occurrences of $(\mathrm{IX})_n$ and $(\mathrm{I})_n$
(i) no joint occurrences of $(\mathrm{IX})_n$ and $(\mathrm{V})_n$.

It is easy to see that, if step C terminates, it terminates with a Roman numeral: C1 and C2 guarantee condition (a), C3 guarantees condition (b), ..., and C10 guarantees condition (i). That Step C does indeed terminate follows from the fact that each transformation, each time it is applied, yields a simple Roman numeral whose denotation is larger than the denotation of each simple numeral removed by that application of the transformation.

Some systems of Roman numerals allow subtractive numerals other than those we allow in $n \cdot 2$ above (*e.g.*, 'IC' or 'IIX'). Allowing such numerals would require appropriate but obvious changes in the conditions under which a concatenation of simple Roman numerals is a Roman numeral and in our addition and multiplication procedures.

[7] We have underlined the simple numerals which result from each set of transformations.

[8] Just as the procedure for Arabic addition makes possible a procedure for Arabic subtraction, so our procedure for Roman addition makes possible a procedure for Roman subtraction (consisting largely in *cancellation* facilitated by an analogue of *borrowing*) which, for reasons of space, we do not present. Similarly, our procedure for Roman multiplication in the next section makes possible a procedure for Roman division.

It might be thought that a weakness of the Roman numeral system is that one requires arbitrarily many symbols with which to denote arbitrarily large numbers. While this holds for the Roman system

II. Multiplication

Arabic multiplication takes advantage of the positional character of Arabic numerals and the procedure for Arabic addition. In like manner, our procedure for Roman multiplication exploits the numeral-wise additivity of Roman numerals and our procedure for Roman addition.

Suppose that a and b are positive integers, that $A_1, \ldots, A_m$ are the simple numerals (including repetitions, if any) in the Roman numeral for a, that $B_1, \ldots, B_n$ are the simple numerals in the Roman numeral for b, that each A_i denotes the integer a_i, and that each B_j denotes the integer b_j. Then by numeral-wise additivity,

$$a \cdot b = \left(\sum_{i=1}^{m} a_i\right) \cdot \left(\sum_{j=1}^{n} b_j\right).$$

And by the distributive law,

$$a \cdot b = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i \cdot b_j).$$

Since each a_i and b_j is denoted by a simple Roman numeral that occurs in the Roman numerals for a and b, the right sides of the above equalities can be expressed using only the signs for addition and multiplication, parentheses, and the simple numerals occurring in the Roman numerals for a and b. With this multiplication table:

	$(\mathrm{I})_m$	$(\mathrm{IV})_m$	$(\mathrm{V})_m$	$(\mathrm{IX})_m$
$(\mathrm{I})_n$	$(\mathrm{I})_{m+n}$	......	......	......
$(\mathrm{IV})_n$	$(\mathrm{IV})_{m+n}$	$(\mathrm{XVI})_{m+n}$	......	......
$(\mathrm{V})_n$	$(\mathrm{V})_{m+n}$	$(\mathrm{XX})_{m+n}$	$(\mathrm{XXV})_{m+n}$	......
$(\mathrm{IX})_n$	$(\mathrm{IX})_{m+n}$	$(\mathrm{XXXVI})_{m+n}$	$(\mathrm{XLV})_{m+n}$	$(\mathrm{LXXXI})_{m+n}$

the Roman numeral for $a \cdot b$ can be computed from the Roman numerals for a and b using our addition procedure. The multiplication table gives the Roman numerals for the products of the numbers denoted by all possible pairs of simple Roman numerals.[9] For example, the table gives 'MM' as the Roman numeral for IV · D: 'IV' is $(\mathrm{IV})_0$ and 'D' is $(\mathrm{V})_2$; according to the table the Roman numeral for the product of the numbers they denote is $(\mathrm{XX})_{2+0}$; this is $(\mathrm{XX})_2$, *i.e.*, 'MM'. Multiplication with Roman numerals can thus be accomplished by successively applying numeral-wise additivity, the distributive law, the multiplication table, and our addition procedure.

Although this procedure can be cumbersome for numbers denoted by long Roman numerals, it works quite well for numbers denoted by short ones. Suppose,

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with which most of us are familiar, it is false for Roman type systems generally. Indeed, one can make do with but four distinct symbols: 'I', 'V', 'X', and a bar. Using the symbols 'I', 'IV', 'V', and 'IX' with n bars above them to denote, respectively, $1 \cdot 10^n$, $4 \cdot 10^n$, $5 \cdot 10^n$, and $9 \cdot 10^n$, one can perspicaciously and uniquely mention any positive integer with but four distinct symbols.

9 Within the range of simple Roman numerals at a calculator's command. For the system described in the previous note, the table would be complete.

Note that, if one were to disallow subtractive numerals, this multiplication table would be even more compact: the second and fourth rows and columns could be eliminated.

for example, we wish to compute the Roman numeral for the product of XLVI and XIV. Then:

Numeral-wise additivity yields	(XL + V + I) · (X + IV)
The distributive law yields	(XL · X) + (XL · IV) + (V · X) + (V · IV) + (I · X) + (I · IV)
The multiplication table yields	CD + CLX + L + XX + X + IV
Step A (addition) yields	CD*CLX*L*XX*X*IV
Step B yields	CDCLLXXXXIV
Step C6 yields	$\underline{\mathrm{D}}\mathrm{LLXXXXIV}$
Step C4 yields	$\mathrm{D}\underline{\mathrm{C}}\mathrm{XXXXIV}$
Step C2 yields	$\mathrm{DC}\underline{\mathrm{XL}}\mathrm{IV}$

And this is the Roman numeral for the product desired. As with the addition procedure, obvious shortcuts (with training, one can often go directly to step B of the addition procedure) make this multiplication procedure much quicker in practice than it might appear.

III. Roman Numerals and the Abacus

The intimate connection between Roman numerals and the abacus has often been noted.[10] It may be described by saying that the Roman numeral for a number describes the state of an abacus when it represents that number, the number of occurrences of each simple numeral corresponding to the number of counters at play in the appropriate columns of an abacus when it represents that number.[11]

In light of this, it should come as no surprise that our addition and multiplication procedures parallel quite closely the procedures for addition and multiplication on an abacus. Indeed, our procedures are a kind of pencil and paper abacus: the main difference between our procedures and the corresponding abacus procedures is that an abacus operator concatenates step by step, applying transformations when appropriate, while we concatenate all at once and apply transformations to the result. The suggested shortcuts in our procedures would bring the two sets of techniques in line.

IV. Numbers and Numerals

In view of the relative simplicity of our computational procedures and their connection with computational techniques on the abacus, the common conviction

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[10] *Cf.* MENNINGER, *op. cit.*, p. 298, and TAISBAK, *op. cit.*, pp. 158–160. TAISBAK also cites A. NAGL's article on the abacus in PAULY-WISSOWA, *Real-Encyclopaedie*, suppl. III, col. 11.

[11] The number of occurrences of 'I' in a Roman numeral would correspond to the number of counters at play in the lower portion of the right-most column of an abacus when it represents the same number; the number of occurrences of 'V', to the number at play in the upper portion of that column; and so forth. For the purpose of this point, one must think of subtractive numerals like 'IV' as shorthand for an appropriate number of appropriate non-subtractive numerals, but TAISBAK (*op. cit.*, pp. 158–160) observes that subtractive numerals may have arisen from an abacus maneuver: an abacus operator adds four by dropping one counter in the lower portion of the right-most abacus column and raising one counter in the upper portion, if these operations are possible.

that computation with Roman numerals is difficult if not impossible is in need of explanation. An obvious (but uninteresting) explanation would be that no one thought of techniques that would work. A more interesting explanation, though, is suggested by a consideration of what seems to be the only argument for the conviction; that argument is due to MENNINGER. Discussing the importance of the introduction of Arabic numerals into Europe (an importance we would not, of course, deny), he writes:

> But though at first glance one merely notices the greater brevity brought about by the new numerals, a second glance lets one see a little deeper: with the new digits we can now for the first time make computations!
> With this remark we finally realize that writing numerals and making computations are two entirely different things; up to now we have generally had nothing to say about computations, although we have thoroughly discussed spoken numbers and written numerals. But didn't people make calculations with Roman numerals? No, they did not! The fairly simple multiplication:

325 · 47	in Roman numerals	CCCXXV · XLVII
2275		MMCCLXXV
13000		KMMM
15275		KWCCLXXV

> looks clearly impossible to the uninitiated reader.[12]

MENNINGER's claim that persons did not in fact make calculations with Roman numerals may well be true. (If so, the existence of our procedures shows that the fault lay in the persons, not in their numeral system.) But MENNINGER also holds that computation with Roman numerals is itself not possible. Using a translation of an Arabic computation into Roman numerals, he argues from the "impossible look" of the translated computation to this conclusion. MENNINGER's argument, note, requires the assumption that computation with Roman numerals is possible only if Arabic and Roman computations mirror one another: without this idea, the impossibility of MENNINGER's translated computation would be irrelevant to the impossibility of Roman computation itself. As a comparison between our procedures for Roman computation and the standard procedures for Arabic computation will show, this assumption is false. But one could be led to think it true by a confusion of numerals with numbers. If (making this confusion) one thought that when one is computing with Arabic numerals one is really manipulating numbers, it would be natural for one to suppose that numbers must be manipulated in the same manner regardless of the numeral system one is using, and thus to suppose that the intermediate stages of Roman computations, if such computations are possible, will parallel the intermediate stages of Arabic computations.[13]

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[12] MENNINGER, *op. cit.*, p. 298. We have taken the liberty of changing MENNINGER's Roman numerals for five thousand and ten thousand to 'W' and 'K', respectively.

[13] Further evidence that MENNINGER may be confusing numerals with numbers is his use of the expression "spoken numbers" in the passage quoted above. See also TAISBAK, *op. cit.*, p. 148, where only a few lines after distinguishing between numerals and numbers he proceeds to confuse them.

Once one makes this supposition, an argument of the sort presented by MENNINGER will be persuasive, if not compelling.

Computation techniques, though, involve the manipulation of numerals, not of numbers. The standard procedures for Arabic computation and those presented here for Roman computation are both techniques for computing numerals for numbers from other designators of those numbers, giving a calculator the numerals he needs to write sums and products. Arabic computation, for example, would yield the numeral '714' from the designator '$102 \cdot 7$', while Roman computation would yield the numeral 'DCCXIV' from the designator 'CII · VII'. In light of this and of the significant differences between the Roman and Arabic numeral systems, there is no reason to suppose that Roman computation is possible only if Roman and Arabic computations mirror one another.[14]

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