MA0004 25.5.2023 Skupina A

1. $a_n = \frac{2n}{n+3} \cdot \sin\left(\frac{n\pi}{2}\right)$

$a_1 = \frac{2\cdot 1}{1+3} \cdot \sin\frac{\pi}{2} = \frac{1}{2}\cdot 1 = \frac{1}{2} \longrightarrow a_5 = \frac{2\cdot 5}{5+3}\cdot \sin\frac{5\pi}{2} = \frac{10}{8}\cdot\sin\frac{\pi}{2} = \frac{10}{8}\cdot 1 = \frac{10}{8} \to \dots$

$a_2 = \frac{2\cdot 2}{2+3}\cdot \sin\pi = \frac{4}{5}\cdot 0 = 0 \longrightarrow a_6 = \frac{2\cdot 6}{6+3}\cdot\sin\frac{6\pi}{2} = \frac{12}{9}\cdot\sin\pi = \frac{12}{9}\cdot 0 = 0 \to \dots$

$a_3 = \frac{2\cdot 3}{3+3}\cdot\sin\frac{3\pi}{2} = 1\cdot(-1) = -1 \to a_7 = \frac{2\cdot 7}{7+3}\cdot\sin\frac{7\pi}{2} = \frac{14}{10}\cdot\sin\frac{3\pi}{2} = \frac{14}{10}\cdot(-1) = -\frac{14}{10} \to \dots$

$a_4 = \frac{2\cdot 4}{4+3}\cdot\sin 2\pi = \frac{8}{7}\cdot 0 = 0 \to a_8 = \frac{2\cdot 8}{8+3}\cdot\sin\frac{8\pi}{2} = \frac{16}{11}\cdot\sin 2\pi = \frac{16}{11}\cdot 0 = 0 \to \dots$

Rozdělme na 3 podposloupnosti:

- $n = 2k, k\in\mathbb{N}\cup\{0\}$: $\lim\limits_{n\to\infty}\frac{2n}{n+3}\cdot\sin\frac{n\pi}{2} = \lim\limits_{k\to\infty}\frac{2\cdot 2k}{2k+3}\cdot\lim\limits_{k\to\infty}\sin\frac{\cancel{2}k\pi}{\cancel{2}} = \lim\limits_{k\to\infty}\frac{4k}{2k+3}\cdot\sin k\pi = 2\cdot 0 = 0 \Rightarrow \boxed{0\in H(a_n)}$
- $n = 4k+1, k\in\mathbb{N}\cup\{0\}$: $\lim\limits_{n\to\infty}\frac{2n}{n+3}\cdot\sin\frac{n\pi}{2} = \lim\limits_{k\to\infty}\frac{2(4k+1)}{4k+1+3}\cdot\lim\limits_{k\to\infty}\sin\underbrace{\frac{(4k+1)\pi}{2}}_{\frac{4k\pi}{2}+\frac{\pi}{2}} = \lim\limits_{k\to\infty}\frac{8k+2}{4k+4}\cdot\lim\limits_{k\to\infty}\left[\sin\frac{\pi}{2}\right] = 2\cdot 1 = 2 \Rightarrow \boxed{2\in H(a_n)}$ $2k\pi$
- $n = 4k+3, k\in\mathbb{N}\cup\{0\}$: $\lim\limits_{n\to\infty}\frac{2n}{n+3}\cdot\sin\frac{n\pi}{2} = \lim\limits_{k\to\infty}\frac{2\cdot(4k+3)}{4k+3+3}\cdot\lim\limits_{k\to\infty}\left[\sin\frac{(4k+3)\pi}{2}\right] = \lim\limits_{k\to\infty}\frac{8k+6}{4k+6}\cdot\lim\limits_{k\to\infty}\left[\sin\left(\frac{4k\pi}{2}+\frac{3\pi}{2}\right)\right] = 2\cdot(-1) = -2$ $\boxed{-2\in H(a_n)}$

$H(a_n) = \{-2, 0, 2\} \Rightarrow \liminf a_n = -2$
$\limsup a_n = 2$

2. $f(x) = \sin(x) + 3,\ T[0,?]$

$f(0) = \sin(0) + 3 = 3 \Rightarrow T[0;3]$

$f'(x) = \cos x \Rightarrow f'(0) = \cos(0) = 1 \Rightarrow k = 1$

tečna t: $y = kx + q \Rightarrow y = x + q$

Dosazením $T[0,3]$: $3 = 0 + q \Rightarrow \boxed{t: y = x+3}$

normála n: $y = -\frac{1}{k}\cdot x + b \Rightarrow y = -x + b$

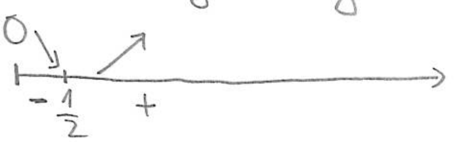
Dosazením $T[0,3]$: $3 = 0 + b \Rightarrow \boxed{n: y = -x+3}$

3. $f(x) = 2x^2 - \ln x$

$D(f) = \mathbb{R}^+ = (0,\infty)$

$f'(x) = 4x - \frac{1}{x} = \frac{4x^2-1}{x} \Rightarrow f'(x) = 0 \Leftrightarrow 4x^2 - 1 = 0 \Leftrightarrow x^2 = \frac{1}{4} \Leftrightarrow x = \pm\frac{1}{2}$

Nulové body či body, v nichž $f'(x)$ neex.: $x = \frac{1}{2}$ (s ohledem na $D(f)$)



$f(x)$ je klesající pro $x\in(0,\frac{1}{2})$
je rostoucí pro $x\in(\frac{1}{2},\infty)$

$f(-\frac{1}{2}) = 2\cdot(+\frac{1}{2})^2 - \ln(\frac{1}{2}) = \frac{1}{2} - \ln\frac{1}{2} \Rightarrow$ lokální minimum v bodě $[\frac{1}{2}, \frac{1}{2} - \ln\frac{1}{2}]$

4. $f(x) = \frac{-2x^2+x}{x+2}$

Asymptota se směrnicí: $y = ax + b$

Pro $x\to\pm\infty$ je asymptotou se směrnicí přímka $\boxed{y = -2x+5}$

$a = \lim\limits_{x\to\pm\infty}\frac{f(x)}{x} = \lim\limits_{x\to\pm\infty}\frac{\frac{-2x^2+x}{x+2}}{x} = \lim\limits_{x\to\pm\infty}\frac{-2x^2+x}{x\cdot(x+2)}$

$= \lim\limits_{x\to\pm\infty}\frac{-2x^2+x}{x^2+2x} = -2$

$b = \lim\limits_{x\to\pm\infty}(f(x) - ax) = \lim\limits_{x\to\pm\infty}\frac{-2x^2+x}{x+2} + 2x$

$= \lim\limits_{x\to\pm\infty}\frac{-2x^2+x+2x\cdot(x+2)}{x+2} = \lim\limits_{x\to\pm\infty}\frac{\cancel{-2x^2}+x+\cancel{2x^2}+4x}{x+2} = 5$

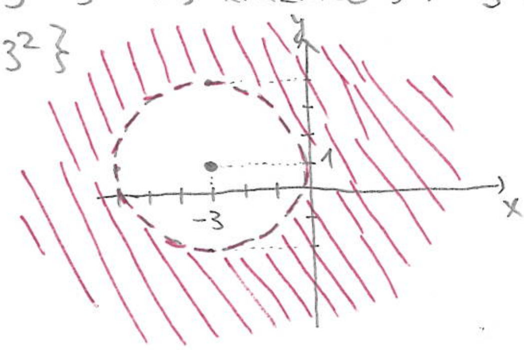
5. $z = \ln(1 + 6x - 2y + x^2 + y^2)$

Podmínka pro $D(z)$: $1 + 6x - 2y + x^2 + y^2 > 0$

$$\cancel{1} + (x^2 + 6x + 9) - 9 + (y^2 - 2y + 1) - \cancel{1} > 0$$

$$(x+3)^2 + (y-1)^2 > 9 = 3^2 \Rightarrow \text{kružnice s } r = 3 \text{ o středu } [-3,1]$$

$D(z) = \{[x,y] \in \mathbb{R}^2 \mid (x+3)^2 + (y-1)^2 > 3^2\}$



6. $f(x,y) = 6xy - x^3 - y^2 + 2$

$D(f) = \mathbb{R}^2$

$$\left.\begin{array}{l} f'_x = 6y - 3x^2 \\ f'_y = 6x - 2y \end{array}\right\} \quad \begin{array}{l} 6y - 3x^2 = 0 \\ 6x - 2y = 0 \Rightarrow 2y = 6x \Rightarrow y = 3x \end{array}$$

$6 \cdot 3x - 3x^2 = 0 \;/ :3$

$6x - x^2 = 0$

$x \cdot (6 - x) = 0$

$\nearrow x_1 = 0 \Rightarrow y_1 = 3 \cdot x_1 = 0 \Rightarrow S_1[0,0]$

$\searrow x_2 = 6 \Rightarrow y_2 = 3\,x_2 = 18 \Rightarrow S_2[6,18]$

$$\left.\begin{array}{ll} f''_{xx} = -6x, & f''_{xy} = 6 \\ f''_{yx} = 6, & f''_{yy} = -2 \end{array}\right\} \quad H(x,y) = \begin{vmatrix} -6x & 6 \\ 6 & -2 \end{vmatrix} = (-6x)\cdot(-2) - 6\cdot 6 = 12x - 36$$

$S_1[0,0]$: $H(0,0) = 12 \cdot 0 - 36 < 0 \Rightarrow S_1$ sedlový bod

$S_2[6,18]$: $H(6,18) = 12 \cdot 6 - 36 > 0 \Rightarrow S_2$ lok. extrém

$f''_{xx}(6,18) = -6 \cdot 6 = -36 < 0 \Rightarrow S_2$ lok. maximum