1. $\lim\limits_{n\to\infty}\left(1+\frac{2}{3n}\right)^{-3n+2} = \lim\limits_{n\to\infty}\left(1+\frac{2}{3n}\right)^{-3n}\cdot\lim\limits_{n\to\infty}\left(\left(1+\frac{2}{3n}\right)^{2}\right) =$ $\Big|$ $\frac{1}{k}=\frac{2}{3n} \Rightarrow 3n = 2k$, $n=\frac{2}{3}\cdot k$

$\left(1+\frac{2}{3n}\right)^{2} \to 1$

$= \lim\limits_{k\to\infty}\left[\left(1+\frac{1}{k}\right)^{\frac{2}{\cancel{3}}k}\right]^{-\cancel{3}} = \lim\limits_{k\to\infty}\left(1+\frac{1}{k}\right)^{-2k} = \underline{\underline{e^{-2}}}$

2. $\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{\sin 3x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim\limits_{x\to 0}\frac{x+4-4}{(\sin 3x)\cdot(\sqrt{x+4}+2)} = \lim\limits_{x\to 0}\frac{x\cdot 3}{\sin 3x\cdot 3\cdot(\sqrt{x+4}+2)}$

$\stackrel{\text{L'Hosp.}}{=}\lim\limits_{x\to 0}\frac{\frac{1}{2}\cdot(x+4)^{-\frac{1}{2}}}{3\cdot\cos 3x} = \lim\limits_{x\to 0}\frac{\frac{1}{2\cdot\sqrt{x+4}}}{3\cos 3x} = \frac{1}{12}$

$= \lim\limits_{x\to 0}\left(\frac{3x}{\sin 3x}\right)\cdot\lim\limits_{x\to 0}\frac{1}{3\cdot(\sqrt{x+4}+2)} = +\frac{1}{12}$

$\frac{3x}{\sin 3x} \to 1^{-1} = 1$

3. $f(x) = x + e^{-x^2}$, $D(f)=\mathbb{R}$

$f'(x) = 1 + e^{-x^2}\cdot(-2x)$

$f''(x) = e^{-x^2}\cdot(-2x)\cdot(-2x) - 2\cdot e^{-x^2} = e^{-x^2}\cdot(4x^2-2) = 2\cdot \underbrace{e^{-x^2}}_{>0}\cdot(2x^2-1)$

$D(f'')=\mathbb{R}$, $f''(x)=0 \iff 2x^2-1=0 \iff x^2=\frac{1}{2} \iff x=\pm\frac{\sqrt{2}}{2}$

$\cup$ $\cap$ $\cup$

$+$ $-\frac{\sqrt{2}}{2}$ $-$ $\frac{\sqrt{2}}{2}$ $+$

Pro $x\in\left(-\infty,-\frac{\sqrt{2}}{2}\right)\cup\left(\frac{\sqrt{2}}{2},\infty\right)$ je $f(x)$ konvexní

Pro $x\in\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ je $f(x)$ konkávní

Inflexní body $x=\pm\frac{\sqrt{2}}{2}$ → $f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}+e^{-\frac{1}{2}} \doteq 1{,}31$

$f\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}+e^{-\frac{1}{2}} \doteq 0{,}1$

4. $f(x) = \ln(2x^2+4)$, $x_0=0$, $f\left(\frac{1}{2}\right)=?$

$f(x_0) = f(0) = \ln 4$

$f'(x) = \frac{1}{2x^2+4}\cdot 4x = \frac{2x}{x^2+2} \to f'(x_0) = f'(0) = 0$

$f''(x) = \frac{2\cdot(x^2+2)-2x\cdot 2x}{(x^2+2)^2} = \frac{4-2x^2}{(x^2+2)^2} \to f''(x_0) = f''(0) = 1$

$f(x) \approx T_2(x_0) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}\cdot(x-x_0)^2$

$= \cancel{4}\ \ln 4 + 0\cdot x + \frac{1}{2}\cdot x^2 = \frac{x^2}{2} + \ln 4$

$f\left(\frac{1}{2}\right) \approx \ln 4 + \frac{1}{8} \doteq 1{,}51$

~~5~~ 6. $\lim\limits_{(x,y)\to(1,-2)}\frac{x\cdot(y+2)}{2x+y} = \left[\frac{0}{0}\right]$

1. Postupné limity:

$L_1 = \lim\limits_{y\to -2}\left[\lim\limits_{x\to 1}\frac{x\cdot(y+2)}{2x+y}\right] = \lim\limits_{y\to -2}\frac{y+2}{2+y} = 1$

$L_2 = \lim\limits_{x\to 1}\left[\lim\limits_{y\to -2}\frac{x\,(y+2)}{2x+y}\right] = \lim\limits_{x\to 1}\frac{0}{2x-2} = 0$

$L_1 \neq L_2 \Rightarrow$ limita neexistuje

2. Přibližování po přímkách:

$y = k\cdot(x-1)-2$

$\lim\limits_{(x,y)\to(1,-2)}\frac{x\cdot(y+2)}{2x+y} \stackrel{\text{subst.}}{=} \lim\limits_{x\to 1}\frac{x\cdot[k\cdot(x-1)\cancel{-2}\cancel{+2}]}{2x+k\,(x-1)-2}$

$= \lim\limits_{x\to 1}\frac{x\cdot k\cdot(x-1)}{2\cdot(x-1)+k\cdot(x-1)} = \lim\limits_{x\to 1}\frac{kx\cdot\cancel{(x-1)}}{\cancel{(x-1)}\cdot(k+2)} = \frac{k}{k+2}$

Limita závisí na param. $k \Rightarrow$ neexistuje

3. Přibližování po parabolách:

$y = k\cdot(x-1)^2-2$

$\lim\limits_{(x,y)\to(1,-2)}\frac{x\cdot(y+2)}{2x+y} \stackrel{\text{subst.}}{=} \lim\limits_{x\to 1}\frac{x\cdot[k\cdot(x-1)^2-2+2]}{2x+k\cdot(x-1)^2-2}$

$= \lim\limits_{x\to 1}\frac{k\cdot x\cdot(x-1)^2}{2\cdot(x-1)+k\,(x-1)^2} = \lim\limits_{x\to 1}\frac{k\cdot x\cdot(x-1)^{\cancel{2}}}{\cancel{(x-1)}\cdot[2+k\,(x-1)]}$

$= \frac{0}{2} = 0 \Rightarrow$ o existenci limity nelze rozhod

4. Polárními souřadnicemi:

$x = 1+\rho\cdot\cos\varphi$

$y = -2+\rho\cdot\sin\varphi$

$\lim\limits_{(x,y)\to(1,-2)}\frac{x\cdot(y+2)}{2x+y} \stackrel{\text{subst.}}{=} \lim\limits_{\rho\to 0}\frac{(1+\rho\cos\varphi)\cdot(\cancel{-2}+\rho\sin\varphi\cancel{+2})}{2\cdot(1+\rho\cdot\cos\varphi)+\rho\sin\varphi-2}$

$\lim\limits_{\rho\to 0}\frac{\cancel{\rho}\sin\varphi\cdot(1+\underbrace{\rho}_{\to 0}\cos\varphi)}{2\cancel{\rho}\cos\varphi+\cancel{\rho}\sin\varphi} = \lim\limits_{\rho\to 0}\frac{\sin\varphi}{2\cos\varphi+\sin\varphi}$

Limita závisí na param. $\varphi \Rightarrow$ neexistuje

5. $f(x,y) = x^2 - 4xy + y^3 + 4y$

$f'_x = 2x - 4y = 0 \Rightarrow 2x = 4y \Rightarrow x = 2y$

$f'_y = -4x + 3y^2 + 4 = 0$

$-4\cdot(2y) + 3y^2 + 4 = 0$

$3y^2 - 8y + 4 = 0$

$d = 64 - 4\cdot 3 \cdot 4 = 16$

$y_{1,2} = \frac{8 \pm 4}{6} = \begin{cases} 2 & \xrightarrow{x=4} x = 2y \\ \frac{2}{3} & \xrightarrow{x=\frac{4}{3}} x = 2y \end{cases}$

$S_1[4,2]$

$S_2[\frac{4}{3},\frac{2}{3}]$

$\left.\begin{array}{l} f''_{xx} = 2, \quad f''_{xy} = -4 \\ f''_{yx} = -4, \quad f''_{yy} = 6y \end{array}\right\} \quad H(x,y) = \begin{vmatrix} 2 & -4 \\ -4 & 6y \end{vmatrix} = 12y - 16$

$H(4,2) = 12\cdot 2 - 16 = 8 > 0$, $S_1[4,2]$ lok. extrém. Protože $f''_{xx}(4,2) = 2 > 0$
$S_1[4,2]$ lok minimum

$H(\frac{4}{3},\frac{2}{3}) = \cancel{12}^{4} \cdot \frac{2}{\cancel{3}_{1}} - 16 = -8 < 0$, $S_2[\frac{4}{3},\frac{2}{3}]$ je sedlový bod