MA0004 15.5.2023 $\nearrow 1$ Skupina B.

1. $\lim\limits_{n\to\infty}\left(1-\frac{3}{2n}\right)^{-4n+2} = \lim\limits_{n\to\infty}\left(1-\frac{3}{2n}\right)^{-4n}\cdot\lim\limits_{n\to\infty}\left(1-\frac{3}{2n}\right)^{2} = \left|\begin{array}{l}\text{subst.}\\ \frac{1}{k}=-\frac{3}{2n} \Rightarrow 2n=-3k \\ \qquad\qquad\quad n=-\frac{3}{2}\cdot k\end{array}\right.$

$= \lim\limits_{k\to\infty}\left(1+\frac{1}{k}\right)^{-\cancel{4}^{2}\cdot\left(-\frac{3}{\cancel{2}_{1}}k\right)} = \lim\limits_{k\to\infty}\left[\left(1+\frac{1}{k}\right)^{k}\right]^{6} = \underline{\underline{e^{6}}}$

2. $\lim\limits_{x\to 0}\frac{\sin 3x}{\sqrt{x+2}-\sqrt{2}}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}} = \lim\limits_{x\to 0}\frac{\sin 3x\cdot(\sqrt{x+2}+\sqrt{2})}{(x+2-2)} = \lim\limits_{x\to 0}\frac{\sin 3x}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{1}\cdot\frac{3}{3}$

$= \lim\limits_{x\to 0}\frac{\sin 3x}{3x}\cdot 3\cdot(\sqrt{x+2}+\sqrt{2}) = 1\cdot 3\cdot 2\sqrt{2} = \underline{\underline{6\sqrt{2}}}$

$\overset{\text{L'Hosp.}}{=} \lim\limits_{x\to 0}\frac{3\cdot\cos 3x}{\frac{1}{2}\cdot(x+2)^{-\frac{1}{2}}} = \lim\limits_{x\to 0}\frac{3\cdot\cos 3x}{\frac{1}{2\sqrt{x+2}}}$

$= 3\cdot 2\cdot\sqrt{2} = 6\sqrt{2}$

3. $f(x)=\ln(2x^2+4)$, $D(f)=\mathbb{R}$, protože $2x^2+4>0$ vždy

$f'(x) = \frac{1}{2x^2+4}\cdot 4x = \frac{2x}{x^2+2}$

$f''(x) = \frac{2(x^2+2)-2x\cdot 2x}{(x^2+2)^2} = \frac{4-2x^2}{(x^2+2)^2}$, $D(f'')=\mathbb{R}$, $f''(x)=0 \Leftrightarrow 4-2x^2=0$

$4=2x^2$

$2=x^2 \Leftrightarrow x=\pm\sqrt{2}$

$\cap$		$\cup$		$\cap$
$-$	$-\sqrt{2}$	$+$	$\sqrt{2}$	$-$

Pro $x\in(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$ je $f(x)$ konkávní. Pro $x\in(-\sqrt{2},\sqrt{2})$ je $f(x)$ konvexní.

Inflexní body: $x=\pm\sqrt{2} \Rightarrow f(-\sqrt{2})=\ln 8 \doteq 2{,}08$

$f(\sqrt{2})=\ln 8 \doteq 2{,}08$

4. $f(x)=x+e^{-x^2}$, $x_0=0$, $f(\frac{1}{2})=?$

$f(x_0)=f(0)=0+e^0=1$

$f'(x)=1+e^{-x^2}\cdot(-2x) \rightarrow f'(x_0)=f'(0)=1+0=1$

$f''(x)=e^{-x^2}\cdot(-2x)\cdot(-2x)-2\cdot e^{-x^2}=e^{-x^2}\cdot(4x^2-2) \rightarrow f''(x_0)=f''(0)=-2$

$f(x)\approx T_2(x_0) = f(x_0)+f'(x_0)\cdot(x-x_0)+\frac{f''(x_0)}{2!}\cdot(x-x_0)^2$

$= 1+1\cdot x-\frac{2}{2}\cdot x^2 = 1+x-x^2$

$f(\frac{1}{2})\approx 1+\frac{1}{2}-\frac{1}{4}=\frac{5}{4}=1{,}25$

6. ~~5.~~ $\lim\limits_{(x,y)\to(1,2)}\frac{y\cdot(x-1)}{2x-y} = \left[\frac{0}{0}\right]$

1. Postupné limity:

$L_1=\lim\limits_{y\to 2}\left[\lim\limits_{x\to 1}\frac{y\cdot(x-1)}{2x-y}\right]=\lim\limits_{y\to 2}\frac{0}{2-y}=0$

$L_2=\lim\limits_{x\to 1}\left[\lim\limits_{y\to 2}\frac{y\cdot(x-1)}{2x-y}\right]=\lim\limits_{x\to 1}\frac{2\cdot(x-1)}{2x-2}=\lim\frac{2(x-1)}{2(x-1)}=1$

$L_1\neq L_2 \Rightarrow$ limita neexistuje

2. Přibližování po přímkách:

$y=k\cdot(x-1)+2$

$\lim\limits_{(x,y)\to(1,2)}\frac{y\,(x-1)}{2x-y} \overset{\text{subst.}}{=} \lim\limits_{x\to 1}\frac{[k\cdot(x-1)+2]\cdot(x-1)}{2x-[k\cdot(x-1)+2]}$

$=\lim\limits_{x\to 1}\frac{[k\cdot(x-1)+2]\cdot(x-1)}{2x-2-k\cdot(x-1)}=\lim\limits_{x\to 1}\frac{[k\cdot(x-1)+2]\cdot(x-1)}{2\cdot(x-1)-k\cdot(x-1)}$

$=\lim\limits_{x\to 1}\frac{[k\cdot(x-1)+2]\cdot\cancel{(x-1)}}{\cancel{(x-1)}\cdot(2-k)}=\lim\limits_{x\to 1}\frac{k\cdot(x-1)+2}{2-k}=\frac{2}{2-k}$

Limita závisí na param. k $\Rightarrow$ neexistuje

3. přibližování po parabolách:

$y=k(x-1)^2+2$

$\lim\limits_{(x,y)\to(1,2)}\frac{y\cdot(x-1)}{2x-y} \overset{\text{subst.}}{=} \lim\limits_{x\to 1}\frac{[k\cdot(x-1)^2+2]\,(x-1)}{2x-[k\cdot(x-1)^2+2]}$

$=\lim\limits_{x\to 1}\frac{[k\cdot(x-1)^2+2]\cdot(x-1)}{2x-2-k\cdot(x-1)^2}=\lim\limits_{x\to 1}\frac{[k(x-1)^2+2]\cdot\cancel{(x-1)}}{\cancel{(x-1)}\cdot[2-k(x-1)]}$

$=\lim\limits_{x\to 1}\frac{k(x-1)^2+2}{2-k\cdot(x-1)}=1 \Rightarrow$ o existenci limity nelze rozhodnout

4. Polárními souřadnicemi:

$x=1+\rho\cdot\cos\varphi$, $y=2+\rho\sin\varphi$

$\lim\limits_{(x,y)\to(1,2)}\frac{y\cdot(x-1)}{2x-y} \overset{\text{subst.}}{=} \lim\limits_{\rho\to 0}\frac{(2+\rho\cdot\sin\varphi)\cdot[\cancel{1}+\rho\cos\varphi\cancel{-1}]}{2\cdot(\cancel{1}+\rho\cos\varphi)\cancel{-2}-\rho\sin\varphi}$

$=\lim\limits_{\rho\to 0}\frac{(2+\rho\sin\varphi)\cdot\cancel{\rho}\cos\varphi}{2\cancel{\rho}\cdot\cos\varphi-\cancel{\rho}\sin\varphi}=\lim\limits_{\rho\to 0}\frac{(2+\rho\sin\varphi)\cdot\cos\varphi}{2\cos\varphi-\sin\varphi}$

$=\frac{2\cos\varphi}{2\cos\varphi-\sin\varphi}$

Limita závisí na param. φ $\Rightarrow$ neexistuje

5. $f(x,y) = x^3 - 4xy + y^2 + 4x + 1$

$f'_x = 3x^2 - 4y + 4 = 0$

$f'_y = -4x + 2y = 0 \Rightarrow 2y = 4x \Rightarrow y = 2x$

$3x^2 - 4\cdot(2x) + 4 = 0$

$3x^2 - 8x + 4 = 0$

$d = 64 - 4\cdot 3\cdot 4 = 16$

$x_{1,2} = \frac{8 \pm 4}{6} = \begin{cases} 2 & y = 4 \\ \frac{2}{3} & y = \frac{4}{3} \end{cases}$

$S_1[2,4]$

$S_2[\frac{2}{3}, \frac{4}{3}]$

$\left. \begin{array}{ll} f''_{xx} = 6x, & f''_{xy} = -4 \\ f''_{yx} = -4, & f''_{yy} = 2 \end{array} \right\} \quad H(x,y) = \begin{vmatrix} 6x & -4 \\ -4 & 2 \end{vmatrix} = 12x - 16$

$H(2,4) = 12\cdot 2 - 16 = 8 > 0$, $S_1[2,4]$ lok. extrém. Protože $f''_{xx}(2,4) = 6\cdot 2 = 12 > 0$, $S_1[2,4]$ lok. minimum

$H(\frac{2}{3}, \frac{4}{3}) = \cancel{12}^{4} \cdot \frac{2}{\cancel{3}_{1}} - 16 = -8 < 0$, $S_2[\frac{2}{3}, \frac{4}{3}]$ je sedlový bod