Part 8

Projects

Applications projects using symbolic computing

42

CONTENTS

42.1 Symbolic computation

There have been a number of significant advances in symbolic computation and computer algebra manipulation in recent years. These are systems which bring together symbolic, numerical, and graphical operations in one software package. The mathematical methods introduced in this book are particularly appropriate contexts in which to have a first look at such systems.

The software Mathematica† has been used extensively in the production of the drawings of curves and surfaces, and in the checking of examples and problems, in this text. At an elementary level, Mathematica is particularly helpful, for example, with operations such as differentiation (including partial derivatives), the construction of Taylor series, elementary algebraic operations involving matrices and linear equations, elementary integration (including repeated integrals), and difference equations; but most topics in this book can be approached to some extent using Mathematica. It is also useful in curve sketching in that a quick view of the general feature of a curve can be obtained, which can then be revised and edited to produce detailed graphs as required.

It is not the purpose of this book to provide an introduction to Mathematica. There are a number of texts which do, including the handbook that comes with the system. There are other software packages including MAPLE+ which can also be used in mathematics. Apart from this chapter, *Mathematical Techniques* is software-free.

Useful information about Mathematica and its applications can be found in the following texts by Abell and Braselton (1992), Blackman (1992), Skeel and Keeper (1993) and Wolfram (1996).

† Mathematica is a registered trade mark of Wolfram Research Inc.
+ MAPLE is a registered trademark of Waterloo Maple Software.

42.2 Projects

The following projects are listed by chapter. They are selected samples of problems and do not cover every topic in the book. The intention is that they can be approached using mainly built-in Mathematica commands: very few problems require programming in Mathematica. It is generally inadvisable to attempt these problems by hand, since many could involve a great deal of manipulation, although some projects are prompted by examples and problems in the relevant chapters.

It is worth emphasizing that computer algebra systems usually generate outputs or answers without explanation of how the results are arrived at, unless the programming within them is investigated. Outputs can go wrong for many mathematical reasons. For example, a curve can oscillate too frequently for the built-in point spacing to detect, which can result in a false graph. This can be corrected by increasing the number of plot points, but the potential difficulty has to be recognized at the formulation stage. Symbolic computation is not a substitute for understanding mathematical techniques.

Mathematica notebooks for each project are available on the web at:

www.oxfordtextbooks.co.uk/orc/jordan_smith4e

Any comments should be sent to the authors at:

School of Computing and Mathematics, Keele University, Keele, Staffordshire ST5 5BG, UK. (Email: p.smith@maths.keele.ac.uk)

Chapter 1

1. Draw the graphs of $y=x^3$, $y=(x-1)^3$, $y-1=x^3$, $y-1=(x-1)^3$ for $-1.5 \leqslant x \leqslant 2.5$. How do they differ?

2. (a) Plot the points (n, n^2+1) for $n=1, 2, 3, 4, 5$.
   (b) Plot the points in (a) but with successive points joined by straight lines.
   (c) Plot $y=x^2$ between $x=0$ and $x=5$.
   (d) Show the curves from (b) and (c) on the same graph.

3. Plot curves defined by the following relations between x and y.
   (a) $x^2+3y^2=4$; $-2 \leqslant x \leqslant 2$;
   (b) $x^2+2y^2-xy+2y=4$; $-3 \leqslant x \leqslant 3$;
   (c) $x^4+2y^2-xy-2x^2y=4$; $-2 \leqslant x \leqslant 3$.

4. Define the function $f(x)=x(1-x^2)$. Plot the graphs
   (a) $y=f(x)$;
   (b) $y=f(1-x)$;
   (c) $y=f(-x)$;
   (d) $y=f(|x|)$; all for $-2 \leqslant x \leqslant 2$.

5. Define the Heaviside function $\mathrm{H}(t)$ and the signum function $\operatorname{sgn} t$. Plot graphs of the following functions on $-4 \leqslant t \leqslant 4$:
   (a) $\mathrm{H}(t)$;
   (b) $\operatorname{sgn} t$;
   (c) $\mathrm{H}(t)+\mathrm{H}(-t)$;
   (d) $\operatorname{sgn}(\sin t)$.

6. Plot the graphs of the curves defined by the following polar equations:
   (a) $r=\frac{1}{2}(1-\cos\theta)$ for $0 \leqslant \theta \leqslant \pi$ (cardioid).
   (b) $r=(4\sin^2\theta-1)\cos\theta$ for $0 \leqslant \theta < 2\pi$ (folium).

7. Express

$$\frac{1}{(x-1)(x-2)(x-3)(x-4)(x-5)}$$

in partial fractions.

Chapter 2

1. Define the function

$$f(x) = \frac{x \sin x - 1 + \cos x}{\sin 2x + 2 - 2\,e^x}.$$

Find $\lim_{x\to 0} f(x)$. Plot the function for $-0.5 \leqslant x \leqslant -0.001$ and for $0.001 \leqslant x \leqslant 0.5$, and check graphically that this agrees with the limit.

2. Find the derivative of

$$f(x) = 7x^2 + 8x^3 + 9x^4 + 10x^5 + 11x^6 + 12x^7$$

and its values $f'(0.2)$ and $f'(0.4)$.

3. Find the derivative of

$$f(x) = x^4 + 2x^3 - 3x^2 - 2x + 4.$$

Find the approximate values of x where $f'(x) = 0$, using a numerical solution routine. Plot graphs of $y = f(x)$ and $y = f'(x)$ on the same axes and compare the zeros of $f'(x)$ with the zero slopes on $y = f(x)$.

4. Find the equation of the tangent to the curve

$$y = x \sin 2x$$

at $x = 0.7$. Plot the graphs of the curve and its tangent.

5. Find the first three derivatives of

$$f(x) = x \sin^2 x + x^2 \sin(x^2),$$

and confirm that the first nonzero higher derivative at $x = 0$ is $f^{(3)}(0) = 6$.

6. Plot the graphs of $y = f(x)$, $y = f'(x)$, and $y = f''(x)$ for

$$f(x) = x^2(x^2 - 3)$$

in the interval $-2 \leqslant x \leqslant 2.5$. (This should confirm the results from Problem 2.19.)

Chapter 3

1. Display rules for the derivatives of the following general forms:
   (a) $f(x)g(x)$;
   (b) $f(x)/g(x)$;
   (c) $f(g(x))$;
   (d) $f(x)g(x)h(x)$;
   (e) $f(x)g(x)/h(x)$;
   (f) $f(h(x))/h(x)$.

2. Find the first derivatives of

$$f(x) = e^{\sin x \cos^2 x} \sin x.$$

The function is periodic. What is its minimum period? Plot its graph and the graph of $f'(x)$ over one cycle. Estimate where $f(x)$ is stationary and then find each of the roots of $f'(x) = 0$ to 5 decimal places using a root-finding routine.

3. If

$$x^2 + 2y^2 - xy - 2yx^2 = 4,$$

find dy/dx as a function of x and y.

Chapter 4

1. Display rules for the first and second derivatives with respect to x of the following general forms:
   (a) $f(x^2)$;
   (b) $f(\sin x)$;
   (c) $f(\sin(x^2))$.

2. Find the first and second derivatives of

$$f(x) = 0.1x^5 - 0.5x^4 + 0.2x^3 + x^2 - 0.7x + 2.2.$$

Estimate the roots of $f'(x) = 0$ from a graph of $y = f(x)$. Then find the roots to 5 decimal places by a root-finding routine. Calculate $f''(x)$ at each stationary point, and confirm the second-derivative test for stationary points. Points of inflection are given by $f''(x) = 0$. Find their locations on the original graph of $y = f(x)$.

3. Plot the graph of

$$y = \frac{x^2 - 1}{2x + 1},$$

and its asymptotes $y = \frac{1}{2}x - \frac{1}{4}$ and $x = -\frac{1}{2}$ (see Fig. 4.13).

4. Plot the graph of $y = f(x) = x^5 - 2x^3 + x^2 - 3x + 1$ in the interval $-1 \leqslant x \leqslant 3$, and estimate the roots of $f(x) = 0$ in this interval. Set up a Newton routine

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

for calculating the roots of $f(x) = 0$, and find, starting at $x = 0.5$ and 1.6, the roots to 10 significant figures. What is the smallest number of iterations required in each case to calculate the roots to 10 significant figures?

5. Plot the graph of $y = x + \sin 5x$ in the interval $0 \leqslant x \leqslant 25$ using
   (a) the default plotting routine,
   (b) plotting with 20 plot points,
   (c) plotting with 50 plot points.
   Explain why the graphs are different for this type of function.

Chapter 5

1. Obtain formulae for the Taylor polynomials for the following functions centred at $x = a$ as far as $(x - a)^3$:
   (a) $f(x)$;
   (b) $[f(x)]^2$;
   (c) $f(x)g(x)$;
   (d) $e^{f(x)}$.
   State the coefficient of $(x - a)^2$ in each case.

2. Find Taylor expansions about $x = 0$ up to and including x^5 for each of the following functions:
(a) e^x; (b) $(x+1)\cos x$;
(c) $\ln(1+\sin x)$; (d) $\exp(\sin(e^x - 1))$.

3. Find the Taylor polynomials for $(\sin^2 x)/x^2$ up to and including x^N for $N = 2, 4, 6$. Plot the graphs of the function and its Taylor polynomials for $0.001 \leqslant x \leqslant 2$, and compare them. At approximately what values of x do the Taylor polynomials visibly part company from the exact function?

4. Find the Taylor polynomials for $\ln x$ about $x = 1$ for $N = 6$. Construct an error function which is the difference of $\ln x$ and its Taylor polynomial. Show that, at 2.159 approximately, this error starts to exceed 0.2 as x increases. Plot this error function against x for $1 \leqslant x \leqslant 2.2$.

Chapter 6

1. Solve, for the complex number a, the equation $z = 0$ where

$$z = \frac{(2+3i)^4}{(1-5i)^3} + \frac{(a-2i)}{(1+5i)^4}.$$

2. If $z = x + iy$, find the real and imaginary parts of $z\,e^z \cos z$.

3. Find the 13 roots of $z^{13} = 1 + i$, and plot the roots on the Argand diagram.

4. Let $z_1 = 1 - 2i$, $z_2 = 3 + i$. Plot the following points on the Argand diagram:

$$z_1 + z_2,\quad \bar{z}_1 + \bar{z}_2,\quad z_1 - z_2,\quad \bar{z}_1 + z_2,\quad z_1 z_2,\quad z_1/z_2.$$

5. Find $|z|$ and Arg z, where

$$z = \frac{(1+2i)^4}{(1+3i)} + \frac{2(3-4i)^3}{1+4i}.$$

Chapter 7

1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & -4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & -1 & 2 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -2 & 1 & 2 \\ -3 & 1 & -3 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & 1 & 2 & 1 \\ p & p & 1 & 2 \\ 1 & -2 & -3 & 2 \\ 2 & 1 & 0 & -1 \end{bmatrix}.$$

Find and compare
(a) AB and BA; (b) $A(BC)$ and $(AB)C$;
(c) $(A+B)^T$ and $A^T + B^T$; (d) $(AB)^T$ and $B^T A^T$.

2. Find the inverse of

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

(see Problem 7.18). Find the equation of the parabola of the form $y = a + bx + cx^2$ through the points $(-1, -2)$, $(\frac{1}{2}, -1)$, and $(\frac{5}{2}, 2)$.

3. Let

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}.$$

Find A^2, A^4, A^8, A^{16}. How do you expect A^n to behave as $n \to \infty$?

Chapter 8

1. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 3 & 1 & 0 & -3 \\ 2 & -1 & 3 & -1 \\ 2 & -1 & 2 & 4 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 4 & -3 & 1 \\ 0 & -1 & 4 & 3 \\ -2 & -2 & 3 & 1 \\ -2 & 5 & 6 & -5 \end{bmatrix}.$$

Find det A, det B, det A^{-1}, and det AB. Confirm that

$$\det A^{-1} = 1/\det A, \qquad \det A \det B = \det AB.$$

2. Factorize the following determinants:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$;

(c) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$.

3. Find the values of a for which

$$\begin{vmatrix} 5 & a & -1 & 1 \\ 2 & 1 & a & 2 \\ 3 & a & 1 & 4 \\ -1 & 0 & a & 2 \end{vmatrix}$$

is zero.

Chapter 9

1. Plot the curve which has the position vector

$$r = (2\cos t)\hat{\imath} + (2\sin t)\hat{\jmath} + 0.3t\hat{k}$$

from $t = 0$ to $t = 20$. What is the curve called? The position vector represents a particle moving along the curve. Find the velocity vector $\dot{r}$ and the acceleration vector $\ddot{r}$ of the particle. Show that $\dot{r}\cdot\ddot{r} = 0$.

2. Plot the *trefoil knot* given parametrically by

$$r = (1 + a\cos 3t)(\cos 2t\,\hat{\imath} + \sin 2t\,\hat{\jmath}) + a\sin 3t\,\hat{k}$$

with $a \sim 0.25$ and $0 < t \leqslant 2\pi$.

Chapter 10

1. Show that

$$\begin{bmatrix} -2^{-3/2} & 2^{-1/2} & 2^{-3/2}3^{1/2} \\ -2^{-3/2} & -2^{-1/2} & 2^{-3/2}3^{1/2} \\ 2^{-1}3^{1/2} & 0 & 2^{-1} \end{bmatrix}$$

defines a rotation of axes. If each row defines the direction of the X, Y, Z axes in the x, y, z frame, find the equation of the plane $x + 2y - 2z = 1$ in the new axes.

Chapter 11

1. The area of a triangle whose vertices are the points with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is given by the formula

$$\tfrac{1}{2}|\boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{c}\times\boldsymbol{a} + \boldsymbol{a}\times\boldsymbol{b}|.$$

Devise a program based on this formula to determine the area for general vertices. What is the area if $\boldsymbol{a} = (1, 0, 1)$, $\boldsymbol{b} = (2, -1, 1)$, and $\boldsymbol{c} = (1, 1, 2)$? Plot a diagram showing the triangle.

2. A tetrahedron has vertices with position vectors

$$\boldsymbol{a} = (1, -1, 2), \quad \boldsymbol{b} = (-1, 2, 3),$$
$$\boldsymbol{c} = (2, -1, 3), \quad \boldsymbol{d} = (1, 3, -2).$$

Find its surface area. Draw a three-dimensional plot showing the tetrahedron viewed from the point with position vector (2.1, −2.4, 1.5).

Chapter 12

1. Use a row-reduction routine to solve the linear equations

$$\begin{aligned} x + 2y - 3z &= q, \\ 2x + py + z &= -1, \\ x - 2y - z &= 4, \end{aligned}$$

where p and q are two parameters. Determine for what values of p and q the equations have (a) a unique solution, (b) no solution, (c) an infinite set of solutions.

2. Use a row-reduction method to solve the linear equations

$$\begin{aligned} x + 2y + pz &= 5, \\ 3x + 2y + z &= q, \\ 2x - y + 4z &= 7, \end{aligned}$$

where p and q are two parameters. Confirm that

$$z = \frac{63 - 5q}{11 + 7p} \quad (p \neq -\tfrac{11}{7}),$$

and discuss the nature of solutions for all values of p and q.

3. Using a row-reduction instruction, show that

$$\begin{aligned} x_1 + 3x_3 &= 5, \\ -x_1 + x_2 - x_3 + x_4 &= -1, \\ x_1 + 2x_2 + 11x_3 &= 4, \\ -x_1 + 2x_2 + 3x_3 + x_4 &= 3 \end{aligned}$$

is an inconsistent set of equations.

Chapter 13

1. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -6 & 1 & 2 & 0 \\ 1 & 0 & -3 & -1 \\ 2 & 1 & -6 & 0 \\ -2 & 2 & 0 & -3 \end{bmatrix}.$$

How many linearly independent eigenvectors does A have?

Find the eigenvalues of the following matrices:
(a) A^{-1}; (b) A^2; (c) $A + kI$.

2. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Construct a matrix C of eigenvectors and confirm that

$$A = CDC^{-1},$$

where D is a diagonal matrix of eigenvalues. Obtain the general formula for

$$A^n = CD^nC^{-1}.$$

3. Find the inverse and transpose of

$$A = \tfrac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix},$$

and verify that A is an orthogonal matrix. Find the eigenvalues of A. What expected property do they have?

4. Find the eigenvalues of

$$A = \begin{bmatrix} 5 & 5 & -6 & 2 \\ -3 & 13 & -6 & 2 \\ -3 & 7 & 0 & 2 \\ 3 & -15 & 12 & 2 \end{bmatrix}.$$

Find the expression $\det(A - \lambda I_4)$, and demonstrate the Cayley–Hamilton theorem of Problem 13.21.

Chapter 14

1. Plot the graphs of the derivative $dy/dx = \sin 2x$ and the equation of the curve through $(\pi, -1)$ of which this is the derivative (see Example 14.7).

2. Plot the graph of

$$\frac{dy}{dx} = x\,e^{-x} + \sin x - x^2 \cos 2x,$$

for $0 \leqslant x \leqslant 10$. Show that an antiderivative which is zero when $x = 0$ is

$$y = 2 + \tfrac{1}{4}[-4(1+x)\,e^{-x} - 4\cos x - 2x\cos 2x + \sin 2x - 2x^2 \sin 2x].$$

Plot the graph of the signed area between $x = 0$ and $x = 10$.

Chapter 15

1. Set up a program to compute the area under the curve $y = f(x)$ between $x = a$ and $x = b$ using the approximation

$$h\sum_{n=0}^{N-1} f(x_n),$$

where $h = (b - a)/N$ and $x_n = a + nh$. Apply the method to the following functions, limits, and subdivision numbers:
(a) $f(x) = x^2$, $1 \leqslant x \leqslant 3$, $N = 200$;
(b) $f(x) = x\,e^{-x}$, $0 \leqslant x \leqslant 3$, $N = 20$;
(c) $f(x) = x^3 \sin x$, $0 \leqslant x \leqslant \pi$, $N = 30$;
(d) $f(x) = \cos(e^{-x})$, $0 \leqslant x \leqslant 1$, $N = 25$.
In cases (a), (b), and (c), compare the numerical result with the areas obtained by integration. In these cases, how many subdivisions are required to obtain a numerical result correct to 3 decimal places? In (a), show that over 10 000 steps are required. Why is this?

2. Use a symbolic integration program to obtain the following indefinite integrals:

(a) $\int (\ln x)^3\,dx$; (b) $\int \sin^5 x \cos^3 x\,dx$;

(c) $\int x^2 e^x \sin x\,dx$; (d) $\int \sqrt{(1 - x^2)}\,dx$;

(e) $\int \frac{dx}{x(x+1)(x+2)(x+3)}$;

(f) $\int \frac{dx}{(1 - x^3)}$.

Check each answer by recovering the integrands by differentiation.

3. Evaluate the following definite integrals:

(a) $\int_1^2 x(\ln x)^3\,dx$; (b) $\int_0^1 \frac{x\,dx}{\sqrt{(5 + 4x - 4x^2)}}$;

(c) $\int_0^{\frac{1}{2}} \frac{x^3\,dx}{(1 - x^2)^{\frac{5}{2}}}$; (d) $\int_0^1 \sum_{n=0}^{100} \frac{x^n}{x!}\,dx$.

4. Find

$$I(a) = \int_1^a (\ln x)^3\,dx.$$

Find the limit

$$\lim_{b\to 0} \frac{bI(1/b)}{(-\ln b)^3}.$$

How does $I(a)$ behave as $a \to \infty$? Does

$$\int_1^\infty (\ln x)^3\,dx$$

exist?

5. A cylindrical hole of circular cross-section and radius b is drilled through a sphere of radius $a > b$, the axis of the hole passing through the centre of the sphere. Find the volume of the remaining object. Display a diagram of the object for some values of a and b.

Chapter 16

1. Plot the graph of the polar equation $r = \sin 5\theta$ for $0 \leqslant \theta \leqslant 2\pi$. Find the area enclosed by the five 'petals' of the curve.
Show that the area of the $2n + 1$ petals of $r = \sin(2n+1)\theta$ $(n \geqslant 1)$ is independent of n.

2. Devise a program to generate the trapezium rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{N}[\tfrac{1}{2}f(a) + (f(x_1) + f(x_2) + \cdots + (f(x_{N-1})) + \tfrac{1}{2}f(b)].$$

Apply the program to the integral

$$\int_0^2 e^{-2x} \sin^2 x\,dx,$$

and compare the result with the exact value of the integral. Investigate how many steps are required to obtain a result accurate to 3 decimal places.

Apply the program also to Problem 16.20.

3. A thin plane metal plate consists of an isosceles triangle of height h and base length $2a$ with a semicircle of radius a attached symmetrically by its diameter to the base of the triangle. Find the location of its centroid on its axis of symmetry.

4. Set up a program to generate Simpson's rule

$$\int_a^b f(x)\,dx \approx$$

$$\frac{b-a}{3N}\left(f(a)+f(b)+4\sum_{k=1}^{\frac{1}{2}N} f(x_{2k-1})+2\sum_{k=1}^{\frac{1}{2}N-1} f(x_{2k})\right),$$

where N is an even number. Apply the method to $f(x)=e^{-x^2}$, with $b=1$, $a=0$. Compare results with the trapezium rule above.

Chapter 17

1. Illustrate the substitution method in integration by writing a program to integrate

$$\int \frac{x-2}{\sqrt{(5+4x-x^2)}}\,dx,$$

using the substitutions $x=u+2$, $u=3\sin t$. Integrate directly and through the substitutions.

2. Integrate the following, and compare your answers with computer-integrated ones:

(a) $\int \frac{x\,dx}{4x^2+1}$; (b) $\int \tan x\,dx$;

(c) $\int \cos^4 x\,dx$; (d) $\int \frac{x\,dx}{\sqrt{(x-1)}}$;

(e) $\int \frac{\sin^3 x}{\cos x}\,dx$.

3. Computer-integrate the infinite integrals

$$I_{10}=\int_0^\infty t^{10}e^{-t}\,dt,\quad I_{11}=\int_0^\infty t^{11}e^{-t}\,dt,$$

and confirm that $I_{11}/I_{10}=11$.

4. Computer-integrate the following infinite integrals:

(a) $\int_0^\infty e^{-x}\sin x\,dx$; (b) $\int_1^\infty \frac{\ln x}{x^{10}}\,dx$;

(c) $\int_1^\infty x^3 e^{-ax^2}\,dx$.

5. Evaluate the integral

$$f(a)=\int_1^a \frac{(\ln x)^6}{x^2}\,dx$$

for $a>1$. Find $f(10)$, $f(20)$, and $f(\infty)$. The results indicate that $f(a)$ tends to a limit very slowly as $a\to\infty$. Find where

$$g(x)=\frac{(\ln x)^6}{x^2}$$

has a maximum value, and plot the graph $y=g(x)$ for $1\leqslant x\leqslant 100$.

Chapter 18

1. Solve the differential equation $\dot{x}+x=0$, for the initial conditions (a) $x(0)=0$, (b) $x(0)=1$, (c) $x(0)=2$, and plot the solutions on the same axes for $0\leqslant t\leqslant 2$.

2. Solve the differential equations

(a) $2\ddot{x}+3\dot{x}+x=0$, (b) $\ddot{x}+2\dot{x}+2x=0$,
(c) $\ddot{x}+2\dot{x}+x=0$,

each for the six sets of initial conditions:

(i) $x(0)=0, \dot{x}(0)=1$;
(ii) $x(0)=0, \dot{x}(0)=2$;
(iii) $x(0)=0, \dot{x}(0)=3$;
(iv) $\dot{x}(0)=0, x(0)=1$;
(v) $\dot{x}(0)=0, x(0)=2$;
(vi) $\dot{x}(0)=0, x(0)=3$.

Plot all solutions on the same axes for each differential equation, for $0\leqslant t\leqslant 5$.

Chapter 19

1. Solve the differential equation $2\ddot{x}+3\dot{x}+x=\cos t$ subject to $\dot{x}(0)=0$, $x(0)=1$. Plot the solution for $0\leqslant t\leqslant 50$.

2. Solve the differential equation $\ddot{x}+x=\cos t$ subject to $x(0)=0$, $\dot{x}(0)=0$. Plot the solution for $0\leqslant t\leqslant 20$.

Chapter 20

1. Solve the differential equation $\ddot{x}+x=0$ subject to the initial conditions $x(0)=1$, $\dot{x}(0)=0$. Also solve $\ddot{x}+\sin x=0$, by a built-in numerical solution method for $0\leqslant t\leqslant 10$ subject to the same initial conditions. Plot both solutions for $0\leqslant t\leqslant 10$. Comparison of the plotted solutions will indicate by how much the period decreases when the linear approximation is used. Rerun the programs for different amplitudes $x(0)$.

Chapter 21

1. Draw the phasor diagram of the sum of the three phasors of

$$u(t) = 2\cos 10t, \quad v(t) = \cos(10t - \tfrac{1}{2}\pi),$$
$$w(t) = 3\cos(10t + \tfrac{1}{4}\pi)$$

(see Example 21.6).

Chapter 22

1. Draw the lineal-element diagram of $\mathrm{d}y/\mathrm{d}x = xy$, produced by a standard package in the square $\{0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ (see Section 22.1). Compare this with the exact solutions (see Section 22.1) drawn through the points (0, 0.2), (0, 0.4), and (0, 0.6).

2. Repeat the above process for the differential equation $\mathrm{d}y/\mathrm{d}x = x - y$ of Example 22.1.

3. Design a program for Euler's method (Section 22.2) for the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2, \qquad y(0) = 1$$

(see Example 22.4) with step length $h = 0.2$ and five steps. Run the program for the cases $h = 0.1$ and $h = 0.01$ and compare the results.

4. Plot numerical solutions for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y - x}{3x - y}$$

(Example 22.14 and Fig. 22.11) using built-in routines. As with many equations of this type it is often easier to solve the equivalent simultaneous equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3y - x,$$

numerically for various initial values of $x(0)$ and $y(0)$.

Chapter 23

1. By splitting the differential equation $\ddot{x} + 2x^3 = 0$ into the system

$$\dot{x} = y, \qquad \dot{y} = -2x^3,$$

and plotting four phase paths respectively through the four points

$$(x(0), y(0)) = (0.3, 0), (0.6, 0), (0.9, 0), (1.2, 0)$$

over the interval $-1.5 \leqslant x \leqslant 1.5$, show that the solutions appear to be periodic.

2. Plot phase paths for the van der Pol equation

$$\ddot{x} + 10(x^2 - 1)\dot{x} + x = 0$$

showing the limit cycle. Also show the corresponding (t, x) graph of the periodic solution (the periodic solution has an initial value close to $x(0) = 2, \dot{x}(0) = 0$).

Chapter 24

1. Computer algebra systems are quite efficient at finding Laplace transforms of complicated expressions involving standard functions. Test the system with the following transforms:
   (a) $\mathcal{L}\{t^8 \mathrm{e}^{-t}\}$;
   (b) $\mathcal{L}\{t^2 \mathrm{e}^{-t} \cos t\}$;
   (c) $\mathcal{L}\left\{\dfrac{\mathrm{d}^3 x}{\mathrm{d}t^3}\right\}$;
   (d) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant c \\ 0 & \text{if } t > c; \end{cases}$
   (e) $\mathcal{L}\{\mathrm{e}^t/t^{\frac{1}{2}}\}$;
   (f) $\mathcal{L}\{\cosh at\}$.

2. Solve

$$\dot{x} + 2x = \mathrm{e}^{-t}, \qquad x(0) = 3,$$

using a Laplace-transform package, and compare the answer with that of Example 24.12. Plot the input e^{-t} and the output against t for $0 \leqslant t \leqslant 3$.

3. Using a Laplace-transform package, solve the system

$$\ddot{x} + 2\dot{x} + x = a\cos \omega t, \qquad x(0) = 0, \dot{x}(0) = 0.$$

Plot the input and output functions for $a = 1$, $\omega = 1$, and $0 \leqslant t \leqslant 30$. Estimate the eventual amplitude of the periodic output.

4. Find the functions whose Laplace transforms are:

(a) $\dfrac{1}{s(s+1)(s+2)(s+3)}$; (b) $\dfrac{\mathrm{e}^{-s}}{(s^2+4)(s+1)}$.

Plot the functions in each case.

5. Consider the function $f(t) = \ln t$. Show the Laplace-transform package produces the transform

$$\frac{1}{s}(\gamma + \ln s),$$

where γ is *Euler's constant* given by

$$\gamma = \lim_{m \to \infty}\left(\sum_{k=1}^{m} \frac{1}{k} - \ln m\right).$$

Derive a program to calculate Euler's constant. It should give $\gamma = 0.577\,215\ldots$.

Chapter 25

1. Find the Laplace transform of the solution of

$$\ddot{x} + \omega^2 x = a\,\delta(t-1), \qquad x(0) = \dot{x}(0) = 0,$$

which has impulse input applied at time $t = 1$. Invert the transform and plot the output for $\omega = 4$, $a = 1$ (see Example 25.3).

2. Following the previous project, solve the more complicated problem with two impulses:

$$2\ddot{x} + 3\dot{x} + 2x = a\,\delta(t-\pi)\cos t + b\,\delta(t-2\pi),$$
$$x(0) = \dot{x}(0) = 0.$$

Plot the output for $a = b = 1$.

3. Let $f(t) = t^3$, $g(t) = \cos t$. Find the convolution

$$\int_0^t f(t-u)g(u)\,du.$$

Then verify that

$$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\left\{\int_0^t f(t-u)g(u)\,du\right\}.$$

4. A transfer function with a parameter a is given by (Section 25.10)

$$\mathcal{G}(z) = \frac{4z^3 - 8z^2 - 2z + 4}{6z^4 - 6z^3 - 2a^2z^2 + 3z^2 + 2a^2z - 2a^2}.$$

Find the locations of the poles of $\mathcal{G}(z)$. For what values of a do all poles lie within the unit circle (indicating transient stability)? Plot the poles on an Argand diagram for $a = 2$.

Chapter 26

1. Consider the period-2 sawtooth function defined over its fundamental interval $-1 < t \leqslant 1$ by $f(t) = t$. Find its general Fourier coefficient and output its first four terms. Plot and compare the graphs of this truncated series and the sawtooth for $-3 < t \leqslant 3$.

2. Repeat the previous problem but with the function

$$f(t) = \begin{cases} 1 & (0 \leqslant t < 1), \\ -1 & (-1 \leqslant t < 0). \end{cases}$$

Plot the graphs of $f(t)$ and the first 12 terms of its Fourier series. The graph should show the **Gibbs' phenomenon**, in which the Fourier series approximation overshoots the function at discontinuities. You can try it with (say) 20 terms or more, but you should include more interpolating points in these cases.

3. Find the Fourier coefficients of the 2π-periodic function defined by

$$f(x) = x^6 - 5\pi^2x^4 + 7\pi^4x^2$$

on the interval $-\pi \leqslant x < \pi$. What is the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}?$$

Chapter 27

1. Find the Fourier transforms of the following functions:
   (a) the top-hat function $\Pi(t)$;
   (b) the one-sided exponential $e^{-t}H(t)$;
   (c) $e^{-|t|}$;
   (d) $e^{-|t-1|}$;
   (e) $1/(1+t^2)$.
   Plot the graph of the transform in (e).

2. Find the functions whose Fourier transforms are
   (a) e^{-f^2};
   (b) $1/(4+f^2)$;
   (c) 2;
   (d) $2\cos(f-a)$.

Chapter 28

1. Plot the saddle surface $z = x^2 - y^2$ in the cylinder $x^2 + y^2 \leqslant 1$, using a three-dimensional parametric plot routine with parameters r and u where

$$(x, y, z) = (r\cos u, \quad r\sin u, \quad r^2\cos 2u).$$

Also draw a contour plot of the surface in the (x, y) plane on the square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$.

2. Plot the surface $z = xy(x^2 - y^2)$ in the cylinder $x^2 + y^2 \leqslant 1$ using the same routine as in Project 28.1 above, but with the parametric equations

$$(x, y, z) = (r\cos u, r\sin u, \tfrac{1}{4}r^3\sin 4u).$$

How would you describe this saddle? Draw its contour plot in the square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$.

3. For the function

$$f(x, y) = e^{x^2y}\sin(xy) + x\ln(x^2 + y^3),$$

verify that

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}.$$

4. Plot the surface given by $z = \cos xy$ over $-\pi \leqslant x \leqslant \pi$, $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$. Find the partial derivatives at $(\frac{1}{4}\pi, 1)$ and construct the equation of the tangent plane there. Finally plot the surface and its tangent plane.

5. Find the stationary points of

$$f(x, y) = 0.3x^3 + 0.2y^2 - x^2y - xy + 2y$$

numerically by solving

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

Plot the contours on the (x, y) plane for $-3 \leqslant x \leqslant 3$, $-9 \leqslant y \leqslant 3$.

Find the values of the second derivatives at each stationary point and check the second derivative tests (28.9) at each point.

6. Find the least-squares straight line fit to the points

$(0, 1.1),\quad (1, 2),\quad (2, 2.9),\quad (3, 3.9),$
$(4, 4.5),\quad (5, 5.1),$

in the (x, y) plane. Plot the data and the least-squares straight line fit. If you are using a built-in routine, check your results against that given by (28.10).

Chapter 29

1. Find the family of curves orthogonal to that of

$$\frac{dy}{dx} = y\,e^{-x}.$$

Plot both families of curves for $|x| \leqslant 2, |y| \leqslant 2$.

Chapter 30

1. Find where the function

$$f(x, y) = x^3 - 2xy - x + 3y^2$$

is stationary subject to the condition $x^2 + 2y^2 = 1$. Devise a program which uses the Lagrange-multiplier method (30.4): here is a suggested line of approach. First plot the contours of $z = f(x, y)$ and the curve $x^2 + 2y^2 = 1$. Locate the approximate coordinates of any point of tangency. Then use a built-in root-finding scheme to locate the stationary values. There should be four.

Chapter 31

1. Find the equation of the tangent plane to the surface

$$x^3y + zx + xy^2z = -3$$

at $(1, 2, -1)$.

2. Show graphically the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ (Example 31.9).

3. Find the envelope of the family of curves

$$y(a^2 - 1 + ax) = x$$

with parameter a. Plot the envelope and a sample of touching curves in $-3 \leqslant x \leqslant 3$.

Chapter 32

1. By repeated integration, evaluate the integral

$$\int_{-1}^{1}\int_{0}^{1} (x + y\,e^{-xy} + xy)\,dx\,dy,$$

using a symbolic routine. Plot the surface

$$z = x + y\,e^{-xy} + xy$$

over $0 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1$. Interpret the integral as the volume under the surface. Does the integral contain 'negative' volumes under the surface? Plot the positive part of the surface over the same rectangle.

2. Evaluate the repeated integral

$$\int_{0}^{a}\int_{-\sqrt{(a^2-y^2)}/a}^{\sqrt{(a^2-y^2)}/a} x^2y\,dx\,dy.$$

Plot the region of integration in the (x, y) plane, and then check that the integral has the same value with the order of the integration reversed.

Chapter 33

1. Let

$$\boldsymbol{f}(x, y, z) = xy\hat{\boldsymbol{\imath}} + yz\hat{\boldsymbol{\jmath}} + (z - y)x\hat{\boldsymbol{k}}.$$

Find $\boldsymbol{f}$ as a function of t on the line $x = t, y = t, z = t$. Evaluate the line integral

$$\int \boldsymbol{f}\cdot d\boldsymbol{r}$$

on this line between $(0, 0, 0)$ and $(1, 1, 1)$.
Repeat the process with the curve $x = t^2$, $y = t^3, z = t^4$, and the same end-points. Plot both paths of integration.

Chapter 34

1. Plot the surfaces defined parametrically by the following position vectors:
   (a) $\boldsymbol{r} = (3 + \cos v)\cos u\,\hat{\boldsymbol{\imath}} + (3 + \cos v)\sin u\,\hat{\boldsymbol{\jmath}} + \sin v\,\hat{\boldsymbol{k}}$ (see Section 34.3);
   (b) $\boldsymbol{r} = (1 + a\sin(bu))\cos v\,\hat{\boldsymbol{\imath}} + (1 + a\sin(bu))\sin v\,\hat{\boldsymbol{\jmath}} + u\hat{\boldsymbol{k}}$, where $a = 0.3$ and $b = 3.5$ (see Section 34.3).

2. Given that

$$\boldsymbol{f}(x, y, z) = e^{xyz}\hat{\boldsymbol{\imath}} + z\cos(xy)\hat{\boldsymbol{\jmath}} + (x^2 + y^2)\hat{\boldsymbol{k}},$$

find
   (a) div $\boldsymbol{f}$;
   (b) **curl** $\boldsymbol{f}$;
   (c) div **curl** $\boldsymbol{f}$;
   (d) **curl curl** $\boldsymbol{f}$ at the point $(1, 0, -1)$.

3. Using symbolic computation test the validity of the following identities:
   (a) $(\boldsymbol{F}\cdot\mathbf{grad})\boldsymbol{F} = \frac{1}{2}\mathbf{grad}(\boldsymbol{F}\cdot\boldsymbol{F}) - \boldsymbol{F}\times\mathbf{curl}\,\boldsymbol{F}$;
   (b) $\mathrm{div}(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot\mathbf{curl}\,\boldsymbol{F} - \boldsymbol{F}\cdot\mathbf{curl}\,\boldsymbol{G}$;
   (c) $\mathbf{curl}(\boldsymbol{F}\times\boldsymbol{G}) = (\boldsymbol{G}\cdot\mathbf{grad})\boldsymbol{F} - (\boldsymbol{F}\cdot\mathbf{grad})\boldsymbol{G} - \boldsymbol{G}\,\mathrm{div}\,\boldsymbol{F} + \boldsymbol{F}\,\mathrm{div}\,\boldsymbol{G}$;
   (d) $\mathrm{div}(U\,\mathbf{grad}\,V - V\,\mathbf{grad}\,U) = U\nabla^2V - V\nabla^2U$;
   (e) $\mathbf{curl\,curl}\,\boldsymbol{F} = \mathbf{grad}\,\mathrm{div}\,\boldsymbol{F} - \nabla^2\boldsymbol{F}$.

Chapter 35

1. A and B are the sets of integers defined by
$$A = \{2n + 5(-1)^n \mid n \in \mathbb{N}^+, 1 \leqslant n \leqslant 100\},$$
$$B = \{n^2 - n + 1 \mid n \in \mathbb{N}^+, 1 \leqslant n \leqslant 10\}.$$
Produce lists of the elements in $A \cup B$ and $A \cap B$. How many elements do each of these sets have?

2. Let A, B, and C be the following sets:
$$A = \{n(n-1) \mid n \in \mathbb{N}^+, 2 \leqslant n \leqslant 100\},$$
$$B = \{|n^2 - 100n| \mid n \in \mathbb{N}^+, 1 \leqslant n \leqslant 160\},$$
$$C = \{4n \mid n \in \mathbb{N}^+, 1 \leqslant n \leqslant 2200\}.$$
Verify the first distributive law
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
How many elements are there in the set $A \cap (B \cup A)$?

Chapter 36

1. Design programs to generate the truth tables for the OR gate, the AND gate, the NOT gate, the NAND gate, and the NOR gate.

2. Design a program to simulate the truth table in Example 36.3 which has the output
$$f = \overline{a * b} \oplus b \oplus c$$
for inputs a, b, and c.

Chapter 37

1. In the cutset method applied to the circuit in Fig. 37.23, the currents i_1, i_2, i_3, i_4, i_5 and the voltages v_a, v_b, v_c, v_d satisfy the nine equations
$$i_1 - i_3 + i_2 = 0, \qquad i_X - i_3 + i_2 = 0,$$
$$-i_Y + i_5 - i_3 + i_2 = 0,$$
$$-i_Y + i_4 + i_2 = 0,$$
$$i_1 = (v_a - v_b)/R_1,$$
$$i_2 = (v_c - v_b)/R_2,$$
$$i_3 = (v_b - v_d)/R_3,$$
$$i_4 = (v_c - v_d)/R_4,$$
$$i_5 = v_d/R_5,$$
where $i_X = 2$ A, $i_Y = 2$ A, and $R_1 = \frac{1}{2}\Omega$, $R_2 = 3\ \Omega$, $R_3 = 1\ \Omega$, $R_4 = 2\ \Omega$, $R_5 = 2\ \Omega$. Solve this set of linear equations for the currents and voltages.

2. Draw the labelled drawings of the bipartite graphs $K_{5,6}$ and $K_{6,6}$. Answer the following for each graph by the built-in diagnostic test.
   (a) How many edges has each graph?
   (b) Is the graph eulerian? If it is, list an eulerian walk.
   (c) Is it hamiltonian? If it is, list a hamiltonian cycle.

3. Check the complete graphs K_n, $2 \leqslant n \leqslant 7$, and the bipartite graphs $K_{i,j}$ $(2 \leqslant i \leqslant 5; i \leqslant j \leqslant 6)$ for planarity, using a built-in diagnostic test.

Chapter 38

1. Rework Example 38.2 using a symbolic package for solving difference equations. Solve the mortgage difference equation
$$Q_m - (1 + I)Q_{m-1} = -A,$$
with $I = 0.08$ and $Q_0 = P = 50\,000$ (in £). Given that $Q_{25} = 0$, find A. List the outstanding debt Q_m each year m to the nearest £. Plot (a) the outstanding debt against years and (b) the annual interest repayments $A - IQ_m$ against years.

2. Solve the following homogeneous difference equations:
   (a) $u_{n+2} - u_{n+1} - 12u_n = 0$;
   (b) $u_{n+2} + 2u_{n+1} + 2u_n = 0$;
   (c) $u_{n+2} + 4u_{n+1} + 4u_n = 0$;
   (d) $u_{n+3} + 3u_{n+2} + 3u_{n+1} + u_n = 0$, $u_0 = 0$, $u_1 = 1$, $u_2 = -1$.

3. Solve the following inhomogeneous difference equations:
   (a) $u_{n+2} - u_{n+1} - 12u_n = 2 + n + n^2$;
   (b) $u_{n+2} - u_{n+1} + 4u_n = 2^n$;
   (c) $u_{n+3} + 3u_{n+2} + 3u_{n+1} + u_n = n^2$, $u_0 = 0$, $u_1 = 1$, $u_2 = -1$.

4. Devise a program to generate cobweb plots for the first-order difference equation
$$u_{n+1} = -ku_n + k$$
for (a) $k = \frac{1}{2}$, (b) $k = \frac{3}{2}$, (c) $k = 1$, with initial value $u_0 = \frac{3}{4}$ in each case (see Example 38.3).

5. Display cobweb plots for the logistic difference equation
$$u_{n+1} = \alpha u_n(1 - u_n)$$
for selected values of α. Some suggested values are:
   (a) $\alpha = 2.8$ to show a stable fixed point;
   (b) $\alpha = 3.4$: find the period-2 solution;
   (c) $\alpha = 3.5$: find the period-4 solution;
   (d) $\alpha = 3.7$: chaotic output;
   (e) $\alpha = 3.83$: should be able to locate a stable period-3 solution.

6. Design a program to generate the period-doubling display shown in Fig. 38.11 for the logistic equation $u_{n+1} = \alpha u_n(1 - u_n)$ for α increasing from $\alpha = 2.8$ to $\alpha = 4$.

Chapter 39

1. (See Example 39.8.) A box contains 40 balls of which 7 are red, 12 are white, and 21 are black.

In each of the cases $n = 2, 3, 4, 5, 6, 7$, n balls are drawn at random from the box without replacement. What is the total number of n-ball selections which can be made? What is the probability that there are n ($n = 2, 3, 4, 5, 6, 7$) balls of the same colour? Show the probabilities graphically in a bar chart.

Chapter 40

1. List the probabilities of the binomial distribution for $n = 12$ and $p = 0.7$. Check that their sum is 1. Plot this discrete distribution as a bar chart.
2. Plot graphs of the probability density function (pdf) and the cumulative distribution function (cdf) for the standardized normal distribution $N(0, 1)$.
3. Model a sequence of n Bernoulli trials with success/failure equally likely, in which the number of successes is recorded. You could try $n = 50$ run 500 times and count the number of successes i for $i = 0, 1, 2, \ldots, n$. This should approximate to the binomial distribution ${}_nC_i p^i q^{n-i}$. Plot this distribution and compare it with the simulation.

Chapter 41

1. Devise a program to draw comparative box plots for the examination data given in Problem 41.2.
2. Produce a histogram and frequency polygon for the pipe length data given in the table accompanying Problem 41.4.
3. Some randomized points (x_i, y_i) are generated by the Mathematica command

```
Table[{x+0.2*Random[],x+2+
1.2*Random[]},{x,0,6,0.5}].
```

   Find the regression lines of y on x, and of x on y, for the data. Plot the data and both regression lines. Also find the *mass centre* of the data, and add this point to the graph. Where does the mass centre lie in relation to the regression lines?
4. Two dice are rolled and the average scores recorded. Compute the probabilities of the possible average scores, and plot them in a bar chart. Repeat the program for four and six dice. Plot bar charts in each case to illustrate the development normal distribution predicted by the central limit theorem.

Self-tests: Selected answers

Chapter 1

1.1 $2 < x \leqslant 3$.

1.2 $AB = BC = \sqrt{26}$, $AC = \sqrt{52}$; $\widehat{ABC}$ is a right angle.

1.3 The circles intersect at the points $(0, 1)$ and $(2, -1)$.

1.4 The graph is

$$x = \begin{cases} t & |t| \leqslant 1 \\ 0 & |t| > 1. \end{cases}$$

1.5 $\cos \frac{1}{12}\pi = \frac{1+\sqrt{3}}{2\sqrt{2}}$, $\quad \sin \frac{1}{12}\pi = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

1.6 $\sin(2\arctan x) = 2x/(1+x^2)$.

1.7 $r = 1 + \cos\theta$, which is a cardioid.

1.8 $y = 1/(1 - x^2 e^{-x})$.

1.9 Time $T = (\ln 10)/k$.

1.11 (b) $f(x) = \dfrac{a}{(a-b)^2(x-a)} - \dfrac{b}{(a-b)(x-b)^2} - \dfrac{a}{(a-b)^2(x-b)}$;

(c) $f(x) = \dfrac{a}{(x-a)^3} + \dfrac{1}{(x-a)^2}$.

1.12 Sum to infinity is

$$\frac{3}{(1-x)^2} - \frac{2}{1-x}.$$

1.13 (a) 5040; (b) 9990.

1.14 $2[1 + {}_{2n}C_2 x^2 + {}_{2n}C_4 x^4 + \cdots + {}_{2n}C_{2n} x^{2n}]$.

Chapter 2

2.1 Tangent: $y = -2x + 2$; normal: $y = -4x + \frac{19}{8}$; intersection point $(\frac{3}{16}, \frac{13}{8})$.

2.2 $\dfrac{dV}{dr} = 4\pi r^2$.

2.3 $\dfrac{dy}{dx} = 70(x^6 + x^9)$.

2.4 (a) 2; (b) 2; (c) 3.

2.5 $d(\cosh x)/dx = \sinh x$: $d(\sinh x)/dx = \cosh x$.

2.6 $(2r)!/r!$.

Chapter 3

3.1 $dy/dx = e^x(\sin x + \cos x)$.

3.2 $\dfrac{dy}{dx} = \dfrac{1 + x^2 - 2x^2 \ln x}{x(1+x^2)^2}$.

3.3 $dy/dx = 1728e^{12x}(1 + 12e^{12x})^{12}$.

3.4 $dy/dx = ka^x \ln a$.

3.5 $dy/dx = 2xe^{x^2}\cos(e^{x^2})$.

3.6 $dy/dx = \frac{1}{2}x^{-\frac{1}{2}}\cos x[(2 + \ln x) - 4x \sin x \ln x]$.

3.7 $dy/dx = (x-3)/[3x(1+2y^2)]$. At $(1, 1)$, $dy/dx = -\frac{2}{9}$.

3.8 $dy/dx = 2/(1 - \tanh^2 x)$.

3.9 $dy/dx = -(b/a)\cot t$. Tangents with slope (-1) occur at $(a^2, b^2)/\sqrt{(a^2+b^2)}$ and $(-a^2, -b^2)/\sqrt{(a^2+b^2)}$.

Chapter 4

4.1 (a) $f'(x) = e^x[\cos(x^2) - 2x\sin(x^2)]$;
(b) $f'(x^2) = e^{x^2}[\cos(x^4) - 2x^2\sin(x^4)]$;
(c) $df(x^2)/dx = 2xe^{x^2}[\cos(x^4) - 2x^2\sin(x^4)]$.

4.2 $x = 1$ is a maximum, and $x = 2$ is a minimum.

4.3 $x = 0$ is a minimum, and $x = 1$ is a point of inflection (using a slope test).

4.4 The area change is $\delta A = 8\pi r\delta r = 5.027$: the exact change is 5.152.

4.5 Solution is $x = 0.7686$ to four decimal places, requiring three steps.

Chapter 5

5.1 $1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5$.

5.2 Required accuracy needs terms as far as x^5.

5.3 $1 - \frac{1}{2} - \frac{1}{8}x^2 + \frac{13}{48}x^3$.

5.4 $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n!}$.

5.5 -2.

Chapter 6

6.1 (a) $4 + 3i$; (b) i; (c) $2i$.

6.2 $z = 1 + i$, $\bar{z} = 1 - i$, $z^2 = 2i$, $\bar{z}^2 = -2i$, $2z = 2 + 2i$, $2z = 2 + 2i$, $z\bar{z} = 2$.

6.3 $z = 2(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$, $\bar{z} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, $2z = 4(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$, $z^2 = 2\sqrt{7}(\cos\theta + i\sin\theta)$, where $\cos\theta = 2/\sqrt{7}$, $\sin\theta = -\sqrt{(3/7)}$.

6.4 $z^{10} = 32i$.

6.5 $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

6.6 $z = 2n\pi i$, $z = \ln(2 \pm \sqrt{3}) + 2n\pi i$, $(n = 0, \pm 1, \pm 2, \dots)$.

6.7 $S(\theta) = \cos(\cos\theta)\cosh(\sin\theta)$.

Chapter 7

7.1 In full the matrix is

$$\begin{bmatrix} -1 & 1 & 1 \\ -2 & 4 & -8 \\ -3 & 9 & -27 \end{bmatrix}.$$

7.2 $AB = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$, $BA = \begin{bmatrix} 4 & -1 & -1 \\ -1 & -2 & 7 \\ -2 & -1 & 5 \end{bmatrix}$.

7.3 $2A + 3B$, A^2, $AB + BA$ are symmetric: AB and BA are not symmetric.

7.4 $A^4 = abcdI_4$, so that $A^{-1} = A^3/(abcd)$.

Chapter 8

8.1 $\det A = 2(k-1)^2$; $k = 1$.

8.2 $D_n = (x-a)^{n-1}(x + na)$.

8.3 The adjoint and inverse are given by

$$\operatorname{adj} A = \begin{bmatrix} -3 & -k-2 & -2k+2 \\ -4 & 1 & 6 \\ -1 & k+1 & -2k-1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4k+5}\begin{bmatrix} 3 & k+2 & 2k-2 \\ 4 & -1 & -6 \\ 1 & -k-1 & 2k+1 \end{bmatrix}$$

The matrix is singular if $k = -\frac{5}{4}$. The product $A\operatorname{adj}(A)$ will always be zero for a singular matrix.

Chapter 9

9.1 $\overline{AD} = (29, 35)$; direction is 0.878… rads to x direction.

9.3 Relative speed = 86.02 lm/hr; direction is 35.5° E of S.

9.4 Plane is $x - 1 = \lambda - 2\mu$, $y + 1 = \lambda$, $z - 2 = 3\lambda + \mu$.

9.5 The point of intersection is $(-1, -2p/(1-p), (1+p)/(1-p))$; the locus is the straight line $x = 0$, $y + z = 1$.

9.6 $\ddot{r} = -\omega^2 \mathrm{r}$.

Chapter 10

10.4 (b) −45°.

10.5 (b) $(1/\sqrt{2}, 1/\sqrt{2}, 0)$.

10.6 Angle is $\arccos(-\frac{1}{6})$.

10.7 (b) Perpendicular distance are $1/\sqrt{14}$, $4/\sqrt{14}$.

10.8 (b) Line is $\dfrac{x - \frac{1}{5}}{\frac{1}{5}} = \dfrac{y + \frac{1}{5}}{-\frac{1}{5}} = \dfrac{z + \frac{1}{5}}{-\frac{1}{5}}$.

Chapter 11

11.1 $|\mathbf{c}| = \sqrt{26}$.

11.4 (a) $-7\hat{\imath} + 6\hat{\jmath} + \hat{k}$.

Chapter 12

12.1 Solution is $x_1 = 2$, $x_2 = -2$, $x_3 = -3$.

12.2 $A^{-1} = \begin{bmatrix} -2 & -1 & -5 & -2 \\ 5 & 2 & 9 & 4 \\ 7 & 3 & 13 & 6 \\ -8 & -3 & -15 & -7 \end{bmatrix}$.

12.3

(a) If $a \neq -3/2$, the system has the unique solution

$x = (a+b)/(3+2a)$,
$y = (-3+2b)/(3+2a)$,
$z = (a+b)/(3+2a)$.

(b) If $a = -3/2$ and $b \neq 3/2$, the system has no solutions.

(c) If $a = -3/2$ and $b = 3/2$, then the system has the set of solutions $x = \lambda$, $y = -1 + 2\lambda$, $z = \lambda$.

Chapter 13

13.1 Eigenvalues: $-1, 1-\sqrt{2}, 1+\sqrt{2}$. Eigenvectors $(-1, 2, 2)^T$, $(-1 + (1/\sqrt{2}), 1/\sqrt{2}, 1)^T$, $-1 - (1/\sqrt{2}), -1/\sqrt{2}, 1)^T$.

13.2 $k \neq \pm 1$.

13.3 Eigenvalues are $-2, 1, 3$. The corresponding eigenvectors are

$$(-2, 1, 2)^T, \quad (2, -1, 1)^T, \quad (3, 1, 2)^T.$$

A possible matrix **C** is given by

$$C = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

13.4

$$A^n = \frac{1}{15}\begin{bmatrix} 8 & -8 & -12 \\ 20 & -20 & -30 \\ -18 & 18 & 27 \end{bmatrix} + \frac{(-1)^n}{15}\begin{bmatrix} 8 & 4 & 8 \\ -10 & -5 & 10 \\ 12 & 6 & 12 \end{bmatrix} + \frac{2^n}{15}\begin{bmatrix} -15 & 6 & 10 \\ -15 & 15 & 10 \\ 15 & -6 & 15 \end{bmatrix}$$

13.5 The eigenvalues are −8, 3, 5, and the eigenvectors are $(-1, -3, 1)^T$, $(-3, 2, 3)^T$, $(1, 0, 1)^T$.

Chapter 14

14.1 $x = \frac{2}{3} - \frac{2}{3}\cos 3t + 4t$.

14.2 (a) $\frac{1}{3}e^{3x} - \cos 2x + C$; (b) $-3x^{-1} + C$; (c) $4 \ln |x| + C$.

14.3 (a) $-\frac{1}{2}e^{-x^2}$; (b) $\cos x\, e^{\sin x}$.

14.4 Signed area $= e - e^{-1} - 2$; geometrical area $= e + e^{-1} - 2$.

Chapter 15

15.1 Approximate area $= \frac{19}{25}$; exact area $= \frac{2}{3}$.

15.2 (a) $\frac{1}{3}(b^3 - a^3)$; (b) $(e^{x^3} - e^{x^2})/x$.

15.3 $\frac{1}{18}\sin^{18}x + C$.

15.4 $\text{rms}[f(t)] = a/\sqrt{2}$.

15.5 $\frac{1}{2}$.

15.6 $1/(1 + b^2)$.

15.7 $\sinh^3 x \cosh^2 x$ is an odd function; $\cos^3 t$ is odd about $t = \frac{1}{2}\pi$.

15.8 $I(x) = 2xe^{x^2}\cos(x^2) - e^x \cos x$.

Chapter 16

16.1 Volume $= \frac{29}{15}\pi$.

16.2 Area $= \frac{1}{6}\pi$.

16.3 $\bar{y} = 2\int_0^1 y\sqrt{(1-y)}\,dy / \int_{-1}^1 (1 - x^2)\,dx$.

Chapter 17

17.1 $\frac{1}{2}x - \frac{1}{12}\sin(6x + 8) + C$.

17.2 $-\frac{1}{2}\cos(x^2) + C$.

17.3 $I_1 = -\frac{1}{4}\cos^4 x + C$, $I_2 = -\frac{1}{12}\cos^4(3x + 2) + C$.

17.4 $I_1 = \ln 2$, $I_2 = \frac{1}{2}(\ln 2)^2$.

17.5 (a) $4 - 2\ln 3$; (b) π.

17.6 $\frac{9}{2}(x + 3)^{-1} + \frac{1}{4}\ln|x + 1| + \frac{3}{4}\ln|x + 3| + C$.

17.7 $\dfrac{x^{n+1}}{n+1}\left(\ln x - \dfrac{1}{n+1}\right)$, $(n \neq -1)$; $\frac{1}{2}(\ln x)^2$, $(n = -1)$.

17.8 $\dfrac{x^{\alpha+1}}{(\alpha+1)^3}[2 - 2(\alpha + 1)\ln x + (\alpha + 1)^2(\ln x)^2]$.

Chapter 18

18.1 $x = e^{10t-20}$.

18.2 (a) $x = Ae^t + Be^{at}$; (b) $x = (A + Bt)e^t$.

18.3 $x = e^{2t}(A\cos 3t + B\sin 3t)$.

Chapter 19

19.1 (a) $x = \frac{2}{9}e^{-2t}$; (b) $x = \frac{2}{5}\cos 2t - \frac{1}{5}\sin 5t$; (c) $x = t^3 + t^2 - \frac{4}{3}t - \frac{10}{9}$.

19.2 The complex solution is $x = -e^{-1+it}/(2 + i)$.
(a) $x = e^{-t}(-\frac{2}{5}\cos t - \frac{1}{5}\sin t)$;
(b) $x = e^{-t}(\frac{1}{5}\cos t - \frac{2}{5}\sin t)$.

19.3 A particular solution is $x = -\frac{1}{2}t\, e^{-t}$.

19.4 $x = \frac{3}{8}(-e^{-t} + e^{3t}) - \frac{1}{2}t\, e^{-t}$.

19.5 $x = -\frac{1}{2}(\cos t - \sin t)e^{-t-\cos t} + C\, e^{-\cos t}$.

Chapter 20

20.1 $x(t) = 2\cos(3t + \frac{1}{6}\pi)$.

20.2 The amplitude of the superimposed waves is $C\cos\frac{1}{2}(\phi_1 - \phi_2)$. Cancellation occurs if $\phi_1 - \phi_2 = \pi$.

20.3 $x = (A + Bt)\, e^{-kt}$.

20.4 The resonant phase occurs at the polar angle given by $(2k^2, -2k\sqrt{(\omega_0^2 - 2k^2)})$.

20.5 Nodes occur at $z = [(2n + 1)\pi + (\phi_1 - \phi_2)]/2k$.

Chapter 21

21.1 $X = 2\, e^{\frac{1}{2}\pi i}$.

21.2 $X = -0.1319 - 0.00141i$.

21.3 $p(t) = \sqrt{(14 + 4\sqrt{2})}\cos(5t + \phi)$ where ϕ is the polar angle of $(2\sqrt{2}, 2 + \sqrt{2})$.

Chapter 22

22.1 The isoclines are given by the hyperbolas $x^2 - y^2 =$ constant.

22.2 General solution is $x^2(x^2 - 2y^2) =$ constant.

22.3 General solution is $x^2y + xy^2 + \sin xy = \text{constant}$.

22.4 General solution is $xy^2 = C(y - x)^2$, where C is a constant.

Chapter 23

23.1 The origin is the only equilibrium point. The equation of the phase paths is $y^2 = \frac{1}{2}x^4 + C$, where C is a constant.

23.2 For $c < 0$, the origin is a saddle; for $0 < c < \frac{1}{4}$, the origin is a stable node; and for $c > \frac{1}{4}$ the origin is a spiral.

23.3 Equilibrium points are at $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$. Solutions are $x = \pm 1$, $y = \pm 1$.

23.4 The origin is a centre, the points $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$ are all saddle points.

23.5 Since $\dot{r} > 0$ for $r \neq 1$, the limit cycle is stable.

Chapter 24

24.1 (a) $2/(s^2 + 4)$.

24.2 (b) $2(1 - e^{-\frac{1}{2}}e^{-s})/(2s + 1)$.

24.3 $\mathcal{L}\{t^3 e^{-kt}\} = 6/(s + k)^4$.

24.5 $(s^2 - 2s - 6)X(s) - 2s - 3$.

24.6 $x(t) = -e^{2t} + 2e^{3t}$.

24.8 $\mathcal{L}\{(e^{-t} - 1)/t\} = \ln[s/(s + 1)]$.

Chapter 25

25.1 $i(t) = (K/L)\cos(t/\sqrt{LC})$.

25.2 $x(t) = -e^{t} + 2e^{2t}$.

Chapter 26

26.1 The Fourier coefficients are $a_0 = 8\pi^2/3$, $a_n = 4/n^2$, $b_n = -4\pi/n$, $(n = 1, 2, \ldots)$.

26.2 $\frac{1}{8}\pi^2$

26.3 Sine series is $\displaystyle\sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2 - 1)} \sin 2nt$.

Chapter 27

27.1 $2/(1 + 4\pi^2 f^2)$.

Chapter 28

28.1 (a) $\partial f/\partial x = -2y\cos(xy)\sin(xy)$, $\partial f/\partial y = -2x\cos(xy)\sin(xy)$;
(b) $\partial f/\partial x = -2x\sin(x^2 - y^2)$, $\partial f/\partial y = 2y\sin(x^2 - y^2)$;
(c) $\partial f/\partial x = (xy)^x[1 + \ln(xy)]$, $\partial f/\partial y = x^2(xy)^{x-1}$;

28.3 Tangent planes are given by $\pm x \pm y - z = 2$. The tangent planes intersect the x, y plane in a square.

28.4 For maximum volume $a = 2\sqrt{[A/(3\sqrt{3})]}$.

28.5 $a = \dfrac{6[2\sum ny_n - (N + 1)\sum y_n]}{N(N^2 - 1)}$,

$b = \dfrac{2[-3\sum ny_n + (2N + 1)\sum y_n]}{N(N - 1)}$,

where all summations are from 1 to N.

28.6 $K(\alpha) = 3\pi/(16\alpha^5)$.

Chapter 29

29.1 At $(3, 4)$, $\delta z = \frac{3}{5}\delta x + \frac{4}{5}\delta y$. The approximate change is -0.02.

29.2 Percentage increase in volume is approximately 9%.

29.3 In terms of x, the rate can be expressed as

$$\frac{dz}{ds} = \sqrt{2}(x - 2)e^{-x^2 - 2(1-x)^2}$$

for $0 \leq x \leq 1$.

29.4 $dy/dx = -(x + y)/(x + 4y)$. The maximum occurs at $(-2/\sqrt{3}, 2/\sqrt{3})$ and the minimum at $(2/\sqrt{3}, -2/\sqrt{3})$.

29.5 The direction of the normal is $(\frac{3}{2} + \frac{1}{2}\sqrt{13}, \frac{5}{4} + \frac{3}{4}\sqrt{13})$.

29.6 $df/ds = 2\sqrt{5}$.

Chapter 30

30.1 $dz/dt = -3\sin t(\sin^2 t - 3\cos^2 t)$. Stationary at $t = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi, \frac{5}{3}\pi$.

30.2 Stationary points are at $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$.

30.3 The families curves are confocal ellipses and hyperbolas.

Chapter 31

31.1 Maximum error = 0.261 units for an area $A = 1.5$ units.

31.3 The point is $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$.

31.4 The tangent plane is $5x + 2y + 3z = 10$.

31.5 The directional derivative $(-2, -2, 1)$.

31.6 Restricted stationary values occur at $(1, 1, 2)$, $(-1, -1, -2)$, $(1, -2, -1)$, $(-2, 1, -1)$.

31.7 The envelope is the parabola $y^2 = x + \frac{1}{4}$.

Chapter 32

32.1 $I = J = 28$.

32.2 $I = \frac{1}{2}\pi$.

32.3 Volume $= 152/3$.

32.4 $I = e^{-1}$.

32.5 The moment of inertia is $\frac{1}{3}M(a^2 + b^2)$, where M is the mass of the plate.

32.6 The volume is $V = \frac{1}{2}\pi$.

32.7 Both areas $= \frac{1}{12}$.

Chapter 33

33.2 9/10.

Chapter 34

34.1 The field lines are ellipses being the intersection of circular cylinders and inclined planes.

34.3 Surface area is $\frac{1}{6}(5\sqrt{5} - 1)$.

34.4 Volume $= \frac{1}{3}Ah$; volume of tetrahedron $= \frac{1}{12}a^3$; volume of octahedron $= \frac{1}{3}a^3\sqrt{2}$.

34.5 $\mathbf{curl}\ \mathbf{F} = (x - 2yz)\hat{\imath} - yz\hat{\jmath} - x\hat{k}$; $\mathbf{curl}\ \mathbf{G} = (2y - 1)\hat{\imath} - 2x\hat{\jmath} - \hat{k}$.

Chapter 35

35.1 (a) $S_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$; (b) $S_2 = \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1, \frac{3}{2}\}$.

35.2 $A \cup B = \{x | x \in \mathbf{R} \text{ and } -1 \leqslant x \leqslant 2, x = 3\ x = 4\}$; $A \cap B = \{1, 2\}$.

35.3 (a) Same as Fig. 35.9b; (b) same as Fig. 35.9d; (c) elements which are not *only* in A or B or C.

Chapter 36

36.1 The output is $\overline{a * b}$ which has the truth table:

a	b	$\overline{a * b}$
0	0	1
0	1	1
0	0	1
1	1	0

36.2 $f = \overline{(a * b) \oplus \bar{c}}$. The truth table is

a	b	c	f
0	0	0	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

36.3 $f = (\bar{a} * \bar{b}) \oplus (a * \bar{b}) \oplus (\bar{a} * b)$: this problem and Self-test 36.1 have the same truth table.

Chapter 37

37.1 (a) $\{1, 2, 2, 2, 3\}$; (b) $\{2, 2, 3, 3\}$; (c) $\{1, 1, 1, 2, 3\}$; (d) $\{3, 3, 3, 3, 3, 3\}$; (e) $\{4, 4, 4, 4, 4\}$.

37.2 21 are connected of which three are regular with degrees 0, 2 and 4.

37.3 (b)(ii) A spanning tree could be the graph with edges $\{ba, bf, bg, bc, ge, cd\}$.

37.4 $ad(b + c) + eh(g + f)$.

37.5 By Euler's theorem: (a) the dodecahedron has 20 vertices; (b) the icosahedron has 30 edges.

Chapter 38

38.1 At 6.5% the repayment is £8198.15; at 7% the repayment is £8526.64.

38.2 The fixed point is $(\frac{1}{2}(\sqrt{3} - 1), \frac{1}{2}(\sqrt{3} - 1))$. The iteration gives $u_2 = 0.460$, $u_3 = 0.288$, $u_4 = 0.417$, $u_5 = 0.326$ to 3 decimal places, which indicates stability.

38.3 $u_n = (A + Bn + \frac{1}{8}n^2)2^n$.

Chapter 39

39.1 $P(A_1) = \frac{1}{6}$; (i) $P(A_2) = \frac{5}{9}$; (ii) $P(A_3) = \frac{17}{18}$.

39.2 $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B) + P(A \cap B \cap C)$.

39.3 The probability of six red cards is 0.0113.

39.4 Component is faulty with probability 0.942.

39.5 (a) 0.000125; (b) 0.1354; (c) 0.1426; (d) 0.1425.

39.6 $\frac{8}{21}$, the same as the probability for the second drawn component.

Chapter 40

40.1 The probability that the sum 7 occurs at throw i is $p_i = \frac{1}{6}(\frac{5}{6})^{i-1}$, $(i = 1, 2, \ldots)$.

40.2 The probability p_i that i individuals are over-height is given by

$$p_i = \left(\frac{13}{20}\right)^i \left(\frac{7}{20}\right)^{8-i} {}_8C_i, \; (i = 0, 1, 2, \ldots, 8).$$

40.3 Expected value is 2.8, and the variance is 1.82.

40.5 With $n = 20$, $\lambda = 2$ the distributions are compared in the following table:

i	0	1	2	3	4	5	6
Binomial	0.122	0.270	0.285	0.190	0.90	0.032	0.009
Poisson	0.135	0.271	0.271	0.180	0.090	0.036	0.012

40.6 $\gamma = \alpha/(\alpha\beta + 1)$; the probabilities are (a) γt_0;

(b) $\dfrac{\alpha}{\alpha\beta+1}\left[\beta + \dfrac{1}{\alpha} e^{-\alpha(t_0-\beta)}\right]$.

Chapter 41

41.1 The medians and quartiles are as follows:

	1st quartile	median	3rd quartile	mean
Paper 1	41.5	47	56	47.7
Paper 2	47.5	58	63.5	54.9
Paper 3	43.5	50	59.5	52.5
Paper 4	45	49	65	54.3

41.2 $m_1 \approx 1966$; $m_2 \approx 2034$. $m_2 - m_1$ behaves like $\sqrt{n}$ as $n \to \infty$.

Answers to selected problems

Full solutions of these end-of-chapter problems can be found at the website: www.oxfordtxtbooks.co.uk/orc/jordan_smith4e

Chapter 1

1.2 (a) $y = -2x + 3$; (b) $y = 1$; (c) $y = \frac{2}{3}x - \frac{1}{3}$. Intersections are $A : (2, 1)$, $B : (\frac{5}{4}, \frac{1}{2})$, $C : (1, 1)$.

$AB = \frac{1}{4}\sqrt{13}$, $AC = 1$, $BC = \frac{1}{4}\sqrt{5}$.

1.3 (b) Slope $= \frac{1}{3}$. Intersection with axes at $(2, 0)$, $(0, -\frac{2}{3})$.

1.4 (b) $(y + 2)/(x + 1) = -2$, so $y = -2x - 4$. (d) $(y - 2)/(x - 1) = 3$, so $y = 3x - 1$.

1.7 (b) Centre $(1, 0)$, radius 2. (d) Centre $(\frac{1}{2}, -\frac{1}{2})$, radius $\frac{1}{2}\sqrt{11}$.

1.9 (b) $x = -\frac{3}{5} \pm \frac{1}{5}\sqrt{14}$, $y = -\frac{1}{5} \pm \frac{1}{5}\sqrt{14}$.

1.14 (b) 1. (d) $-1/\sqrt{2}$. (f) $-\sqrt{3}/2$.

1.16 (b) $\cos x$; (d) $-\cos x$.

1.17 (b) $2\cos\frac{1}{2}(x + y)\sin\frac{1}{2}(x - y)$.

1.18 In the following, n represents any integer: (b) $\frac{1}{2}\pi + n\pi$; (d) $\frac{1}{6} + \frac{1}{3}n$; (f) $2n$.

1.19 (b) amp. = 1.5; ang. freq. = 0.2; period = 31.41; phase = −0.48.

1.20 (b) $\frac{1}{2}x - \frac{3}{2}$; (d) $\arcsin\frac{1}{2}x$, $0 \leqslant x \leqslant 2$. (f) $\arccos(\arcsin x)$, $0 \leqslant x \leqslant \sin 1$. (h) $-\frac{1}{2} + (1 + 4x)^{\frac{1}{2}}$, $x \geqslant -\frac{1}{4}$.

1.22 (b) $\frac{1}{2}e^2$; (d) $\frac{1}{3}\ln\frac{1}{3}$, or $-\frac{1}{3}\ln 3$; (f) 2; (h) $\pm\sqrt{2}$; (l) Hint: write $\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x})$ and obtain a quadratic equation for e^{2x}. $x = \frac{1}{2}\ln(4 + \sqrt{17})$.

1.26 Hint: $x = \tanh y = (e^y - e^{-y})/(e^y + e^{-y})$. Form an equation for e^y and solve it.

1.28 $5\cos(\omega t - 0.927)$.

1.29 $C = 2$, $\alpha = 1.386$, $f(2) = 1/8$.

1.30 Tidal period = 12.57 h. It floats for 9.20 h. Hint: it floats when $\sin 0.5t \geqslant -0.666$. Sketch $y = \sin 0.5t$ and $y = 0.666$ and find the intersections.

1.33 The vertex is $(-4, 7)$.

1.36 (b) $2/(x + 2) - 1/(x + 1)$. (d) $1/2x - 1/(x + 1) + 1/2(x + 2)$. (f) $1/4x - 1/4(x + 2) - 1/2(x + 2)^2$. (h) $1/2(x - 3) + 1/2(x + 1)$.

1.37 (b) $1/[2(x - 1)] + 1/[2(x^2 + 1)] - x/[2(x^2 + 1)]$.

1.38 (b) $x - 3 - 1/(x + 1) + 8/(x + 2)$.

1.39 (b) $1 + 1/2 + 1/5 + 1/10 + 1/17$.

1.40 (b) $\sum_{n=2}^{6}(\frac{1}{3})^n = (\frac{1}{3})^2 + (\frac{1}{3})^3 + \cdots + (\frac{1}{3})^6$

$= (\frac{1}{3})^2[1 + \frac{1}{3} + \cdots + (\frac{1}{3})^4]$.

Now (1.31) gives the sum in the brackets. Finally we obtain 121/729. (e) −341/1024.

1.44 (c) 1/99; (e) 30/11.

1.45 (b) 10/9; (d) 2/3.

1.47 (b)(i) $2^m m!$.

1.49 (c) 256; (d) 20; (f) 59.

1.50 (a) 72; (b) 360.

1.51 (b) 24; (d) 164.

1.54 (a) 2880; (b) 720.

1.55 (a) 120; (b) 720; (c) 220; (d) 1000.

Chapter 2

2.1 (b) 0.5; (e) 2; (g) 1.

2.2 (c) 6; (e) $-\frac{1}{4}$; (g) −4.

2.3 (c) $-1/x^2$; (f) $4x$.

2.4 (c) −8.

2.5 (c) 32, −32.

2.8 (c) $dE/dT = 4kT^3$.

2.9 (b) $7x^6 - 18x^5 + 1$.

2.11 Use the formula for $\tan(A - B)$ in Appendix B(b).

2.12 (b) $\frac{1}{2}$; (d) 1; (g) 2; (i) $\pi/180 = 0.0175$.

2.15 (a) $2\cos x + 3\sin x$.

2.16 (b) $y = 24x - 39$; (d) $y = e^{-1}x$.

2.17 (b) $6x - 2$, 6, 0.

2.20 $y = (-x + x_0 + 2a^2x_0^3)/(2ax_0)$.

Chapter 3

3.1 (b) $x \cos x + \sin x$; (f) $2x \ln x + x$.

3.2 (b) $1/(1+x)^2$; (f) $(x^2 - 2x \sin x \cos x)/x^4 \cos^2 x$. (m) nx^{n-1}.

3.3 (d) $f\dfrac{dg}{dx} + g\dfrac{df}{dx}$, $g\dfrac{d^2f}{dx^2} + 2\dfrac{df}{dx}\dfrac{dg}{dx} + f\dfrac{d^2g}{dx^2}$,

$$g\frac{d^3f}{dx^3} + 3\frac{d^2f}{dx^2}\frac{dg}{dx} + 3\frac{df}{dx}\frac{d^2g}{dx^2} + f\frac{d^3g}{dx^3}.$$

3.4 (b) $-2 \cos x \sin x$; (e) $2 \sin x/\cos^3 x$; (j) $12x^2(x^3+1)^3$; (n) $-3\,e^{-3x}$.

3.5 (f) $\frac{1}{2}x^{-\frac{1}{2}}$; (i) $-\frac{1}{2}x^{-\frac{3}{2}}$.

3.6 (f) $e^{-t}(\cos t - \sin t)$; (k) $2 \sin x(\cos x - \sin x)/x^3$.

3.9 (c) $(-2x \sin x^2)/\cos x^2$. The original function only has a meaning when $\cos x^2 > 0$.

3.10 (b) $e^t(\cos t + t \cos t - t \sin t)$.

3.11 (b) $dy/dx = -y^{\frac{1}{2}}/x^{\frac{1}{2}}$. This can be written in other ways; for example, put $y^{\frac{1}{2}} = 1 - x^{\frac{1}{2}}$ from the equation of the curve.

3.15 (b) -5.

3.16 (b) $dy/dx = \pm x/[2\sqrt{(1-(x/2)^2)}]$.

Chapter 4

4.1 (b) $2t^2$; (c) $4t^3$.

4.2 (c) $x = e^{-1}$ (min); (g) $x = 0$ (min); (i) $x = -1/\sqrt{3}$ (min), $x = 1/\sqrt{3}$ (max); (t) Points of inflection at $x = n\pi$; maxima at $x = (2n+\frac{1}{2})\pi$; minima at $(2n-\frac{1}{2})\pi$.

4.5 If base $= x$ and rectangle height $= y$, then $A = xy + \frac{1}{8}\pi x^2$ (constant), and $P = (1+\frac{1}{2}\pi)x + 2y$. Substitute for y from the formula for A to express P in terms of x only. The minimum of P is reached when $x = [2A/(1+\frac{1}{4}\pi)]^{\frac{1}{2}}$.

4.10 (b) $\delta y \approx -0.2$ (exact value $-0.227\ldots$). (d) $\delta y \approx -0.4$ (exact value -0.5).

4.11 (a) $\delta v \approx -0.11$; (d) $\delta A \approx -0.08$.

Chapter 5

5.1 (b) $(1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$. For 2 decimal places, we need $|\frac{1}{16}x^3| < 0.005$, or $-0.43 < x < 0.43$. (d) To four terms,

$$\sin 2x \approx 2x - 1.333x^3 + 0.267x^5 - 0.025x^7,$$

where (for this context) the coefficients are rounded to 3 decimal places. For two-decimal accuracy, we need $-0.79 < x < 0.79$.

5.3 (b) The terms in the expansion of $\sin x$ are of size $|x|^{2n-1}/(2n-1)!$ with $n = 1, 2, \ldots$. We need to choose n so that this is less than 0.000 05 when $x = \pm 2$. The first value within the limits is $n = 7$. The polynomial is

$$x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \tfrac{1}{9!}x^9 - \tfrac{1}{11!}x^{11} + \tfrac{1}{13!}x^{13}.$$

5.4 (b) $\frac{1}{2}\pi - x$.

5.5 (b) $\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \cdots$, $-2 < x < 2$. (h) $1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \cdots$, valid for all x.

5.6 (b) $1 + \frac{1}{2}x - \frac{1}{8}x^2$.

5.7 (b) $\tan x \approx (x - \frac{1}{6}x^3 + \frac{1}{120}x^5)(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4)^{-1}$
$\approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5$.

5.8 (d) $\ln(1+x+x^2) = \ln[x^2(1+1/x+1/x^2)]$
$= 2\ln x + \ln(1+1/x+1/x^2)$.

Then treat $1/x + 1/x^2$ as the small variable.

5.11 (b) Suppose that the first nonzero derivative is the N th : $f^{(N)}(c) \neq 0$. Consider whether N is even or odd, and whether $f^{(N)}(c)$ is positive or negative.

5.17 (c) $\frac{1}{2}(e^x + e^{-x})$.

5.18 (c) $\frac{4}{3}$.

Chapter 6

6.1 (b) $3 \pm i$.

6.3 (b) $3 - 5i$; (d) $9 + 3i$; (f) $1 + 6i$.

6.5 (d) $-\frac{13}{25} - \frac{9}{25}i$.

6.6 (a) $-4i$; (c) $-\frac{1}{5} + \frac{8}{5}i$.

6.7 (a) $1 - i$; (c) $-2i$.

6.8 (b) $16.233 - 0.167i$; (d) 88.669.

6.9 (b) $|z_2| = 8$; Arg $z_2 = -\frac{1}{3}\pi$. (d) $|z_4| = 3$; Arg $z_4 = \pi$.

6.10 (b) $y = 2$; (d) the parabola, $y^2 = 4x$; (f) $y = x$ $(x \geqslant 0)$.

6.11 (a) $\sqrt{2}\,e^{\frac{3}{4}\pi i}$; (d) $14\,e^{-\frac{1}{3}\pi i}$; (g) $e^2\,e^i$; (j) $\sqrt{2}\,e^{\frac{3}{4}\pi i}$.

6.16 (a) $2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$; (c) $(2n+1)\pi i$.

6.18 (a) $\cos(\ln 2) + i \sin(\ln 2)$.

6.23 (a) $x^2 - y^2 + 2xyi$. (d) $\cos x \cosh y - i \sin x \sinh y$.

6.28 $2+i, 2-i, -1-i, -1+i$.

6.29 (b) $e^{2\cos\theta}\cos(2\sin\theta)$.

Chapter 7

7.2 $x = -2, y = 1$.

7.6 $BA = \begin{bmatrix} -10 & -5 \\ 20 & 10 \end{bmatrix}$.

7.7 $A^2 + C^2 = \begin{bmatrix} -5 & 6 & 16 \\ -8 & 11 & 2 \\ -6 & -6 & -7 \end{bmatrix}$.

7.11 A^{2n-1}.

7.16 $x = -17, y = -2, z = 8$.

Chapter 8

8.1 (c) 1; (e) −1.

8.4 (b) 1728; (d) −8132.

8.6 $(b - c)(c - a)(a - b)(a + b + c)$.

8.14 $x = a, b, c, -a - b - c$.

8.16 $\det(AB) = -36, A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 1 & -5 \\ -2 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Chapter 9

9.1 (a) $\overline{PQ} = (5, -3), \overline{QP} = (-5, 3)$.

9.2 (f) Length = 5, $\theta = 126.9°$.

9.3 (b) $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

9.4 $\overline{BE} = (0, -4)$; $BE = 4$; bearing south.

9.5 (c) $\sqrt{6}$.

9.7 (b) $2\boldsymbol{a} = (6, 4, 6), 3\boldsymbol{b} = (3, 3, 6), 2\boldsymbol{a} - 3\boldsymbol{b} = (3, 1, 0)$.

9.10 (a) $(3, 3, -6)$. (b) $(X + 2)^2 + (Y - 1)^2 + (Z + 3)^2 = 1$.

9.16 Speed $10\sqrt{2}$; direction towards north east. (Hint: use $\boldsymbol{v}_{we} = \boldsymbol{v}_w - \boldsymbol{v}_c$ in components, with $\boldsymbol{v}_N = (u, v)$.)

9.22 (b) $\frac{3}{4}\boldsymbol{a} + \frac{1}{4}\boldsymbol{b}$; (c) $\frac{3}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$.

9.23 (a) $(\boldsymbol{a} + \lambda\boldsymbol{b})/(1 - \lambda)$. (b) $(\boldsymbol{a} - \lambda\boldsymbol{b})/(1 + \lambda)$. (c) The point is on the extension of AB in the direction of $\overline{AB}$.

9.26 (a) $y + z = 1$. (b) $3x - 2y - z = 0$.

9.27 $\sqrt{2}$.

9.28 (a) $\pm(3/\sqrt{34}, 4/\sqrt{34}, 3\sqrt{34})$. (b) $\pm(\frac{2}{7}, \frac{3}{7}, \frac{6}{7})$.

9.29 (a) $-3\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$. Length $\sqrt{29}$.

9.36 $\boldsymbol{r} = \frac{1}{2}\left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|} + \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)$. (Hint: draw a diagram involving $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$.)

9.37 The minimum separation occurs when $t = 12\frac{1}{2}$ s.

Chapter 10

10.1 (a) 10. (e) zero.

10.3 If your diagram is a parallelogram $ABCD$, the theorem obtained is $AC^2 + BD^2 = 2(AB^2 + AD^2)$. If you use the triangle rule the result gives the median of a triangle in terms of the sides.

10.5 (a) 6. (b) −5.

10.6 (a) 35.3°.

10.8 54.7°.

10.9 $33x^2 + 13y^2 - 95z^2 + 48xy - 144yz + 96zx = 0$.

10.10 32.5°, 78.9°, 68.6°.

10.12 $F = -\frac{13}{11}\mathbf{a} + \frac{30}{11}\mathbf{b} + \frac{57}{11}\mathbf{c}$.

10.16 $\alpha = -\frac{2}{5}, \beta = \frac{7}{5}, \gamma = \frac{2}{15}$.

10.17 $x = 0, y = 0, z = 1$.

10.18 (a) $(2\sqrt{2}, 0)$. (b) $\left(X - \frac{1}{\sqrt{2}}\right)^2 + \left(Y + \frac{1}{\sqrt{2}}\right)^2 = 1$.

10.19 (c) $(l, m, n) = (\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3})$.

10.21 (a) $\pm(\frac{3}{13}, \frac{4}{13}, \frac{12}{13})$.

10.26 (a) 19.1°.

10.30 (a) P_1 is $-y + z = 4$, P_2 is $2x - 2y + z = 5$. (b) 45°. (c) $2\sqrt{2}$. (d) and (e). The line L is given by $\boldsymbol{r} = \lambda(1, 4, -4)$. Show that intersection with P_1 and P_2 occurs when $\lambda = -\frac{1}{2}$.

10.34 Begin by finding any two points on the line of intersection. (The resulting form is not unique.)

Chapter 11

11.1 (a) $(4, 7, 5)$. (d) −9. (h) $(-24, 3, 15)$.

11.9 Hint: the determinant is equal to $\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c})$, where $\overline{OA} = \boldsymbol{a}$, etc.

11.12 $X = -\frac{1}{6}, Y = -\frac{2}{3}, Z = -\frac{5}{6}$.

11.13 (c) $\lambda = \mu = -\frac{1}{2}, v = -\frac{3}{2}$. L_3 meets L_1 at $(-1, 0, -\frac{1}{2})$ and L_2 at $(\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$.

11.15 (a) $(\frac{24}{3}, \frac{16}{3}, -\frac{4}{3})$. (b) $(\frac{16}{3}, \frac{24}{3}, -\frac{16}{3})$. (c) $\frac{16}{3}\mathrm{i}$. (Note: the *unit* vector in the direction of $\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ is $\frac{1}{3}(\hat{\imath} - 2\hat{\jmath} - 2\hat{k})$.)

11.16 (a) −6. (b) 6. (c) 0. (d) 0. (e) $-2\sqrt{3}$.

Chapter 12

12.1 (c) $x_1 = 1, x_2 = -1, x_3 = -5$. (e) $x_1 = 2, x_2 = -1, x_3 = 2, x_4 = 2$.

12.7 $x_1 = 40, x_2 = 88, x_3 = -68, x_4 = -59.$

12.9 (b) $\frac{1}{25}\begin{bmatrix} 5 & 0 & -5 \\ -6 & 10 & 1 \\ 7 & 5 & 3 \end{bmatrix}.$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$

12.12 The shadow on the z plane has vertices at the points $(-1, 0, 0)$, $(-1, -2, 0)$, $(1, 0, 0)$.

12.16 Non-trivial solutions if $k = 1, -1, 4$.

12.18 Non-trivial solutions if $k = -6, -1, 3, 4$.

12.22 $x_1 = 1.398, x_2 = 1.090, x_3 = -0.2844, x_4 = -0.3697.$

Chapter 13

13.1 (b) Eigenvalues 4, 9. Eigenvectors $\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(e) Eigenvalues $3 - 4\sqrt{2}, 3 + 4\sqrt{2}$. Eigenvectors

$$\begin{bmatrix} -1 - 2\sqrt{2} \\ 7 \end{bmatrix}, \begin{bmatrix} -1 + 2\sqrt{2} \\ 7 \end{bmatrix}.$$

13.4 (c) Eigenvalues $(-2, 2, 3)$. Eigenvectors

$$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

13.7 $a = -2$ and $a = -\frac{7}{2}$.

13.12 The matrix C is given by

$$C = \begin{bmatrix} 7 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}.$$

13.16 $\lim_{n\to\infty} A^n = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$

13.22 Eigenvalues are 0, 4, 4, 12.

13.26 $A^{3n} = I_3, A^{3n+1} = A, A^{3n+2} = A^2.$

Chapter 14

14.1 (a) $\frac{1}{6}x^6 + C; \frac{3}{5}x^5 + C; \frac{1}{2}x^4 + C; \frac{1}{9}x^3 + C; 3x^2 + C; 3x + C; C.$

(g) $e^x + C; -e^{-x} + C;$
$\frac{5}{2}e^{2x} + C; -2e^{-\frac{1}{2}x} + C; -\frac{3}{2}e^{-2x} + C.$

(k) $x + \ln x + C$ $((x + 1)/x = 1 + x^{-1})$;
$2x - 2x^{\frac{1}{2}} + C; \ln|x| - 2x^{-1} - \frac{1}{2}x^{-2} + C.$

14.2 (b)
$-\frac{1}{5}(1 - x)^5 + C; -\frac{2}{3}(8 - 3x)^{-\frac{3}{2}} + C; \frac{3}{2}(1 - x)^{\frac{4}{3}} + C.$

14.3 (b) $-\ln|1 - x| + C; -\frac{1}{5}\ln|4 - 5x| + C.$

14.4 (c) $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

14.5 $x^2 e^x - 2x e^x + 2 e^x + C.$

14.6 (a) 2; (h) $-\ln 2$.

14.7 (c) $4 - x^2 \geqslant 0$ if $-1 \leqslant x \leqslant 2$, and $4 - x^2 \leqslant 0$ if $2 \leqslant x \leqslant 3$. The geometrical area is

$$|F(x)|_{-1}^{2} + |F(x)|_{2}^{3},$$

where $F(x) = 4x - \frac{1}{3}x^3$.

14.8 (a) $At + B$; (b) $\frac{1}{6}t^3 + At + B$.

Chapter 15

15.1 (b) $\lim_{\delta x\to 0} \sum_{x=-1}^{x=1} x^5\,\delta x = \int_{-1}^{1} x^5\,dx = [\frac{1}{6}x^6]_{-1}^{1} = 0.$

15.2 (b) $\int (x + 1)^{\frac{1}{2}}\,dx = \frac{2}{3}(x + 1)^{\frac{3}{2}} + C.$

15.3 (c) $\int_0^2 dx = [x]_0^2 = 2$; (i) $\frac{2}{3}(2^{\frac{3}{2}} - 1)$.

15.4 (b) $\int_{-1}^{1} (x^2 - 1)\,dx = [\frac{1}{3}x^3 - x]_{-1}^{1} = -\frac{4}{3}.$

15.5 (b) $\int_0^{\infty} e^{-\frac{1}{2}v}\,dv = -2\,[e^{-\frac{1}{2}v}]_0^{\infty} = -2(0 - 1) = 2.$

15.6 (c) $2/\pi$; (h) $\int_0^T (1 - e^{-t})\,dt = T + e^{-T} - 1.$

$$\frac{1}{T}(T + e^{-T} - 1) = 1 + T^{-1}e^{-T} - T^{-1} \to 1$$

as $T \to \infty$.

15.7 The integrands are (a) even; (b) odd; (c) odd; (d) odd.

15.9 (b) The exact result is $\sqrt{\pi}/2$.

15.10 (e) $\frac{1}{2}(x + 1)^{-\frac{1}{2}}\sin(x + 1) - \frac{1}{2}x^{-\frac{1}{2}}\sin x.$

15.11 (b) $x \leqslant -1 : \frac{1}{2}$ (constant); $-1 \leqslant x \leqslant 1 : \frac{1}{2}x^2$;

$x \geqslant 1 : \frac{1}{2}$ (constant)

15.14 6.

Chapter 16

16.1 $5.\dot{3}\times 10^{-3}$.

16.2 $\int_2^4 (20-10t)\,dt = -20,\ x(4) = -17.$

16.3 (b) $\frac{1}{2}\pi$; (g) π.

16.5 (a) $\frac{4}{3}\pi ab^2$.

16.6 $v = \int_1^2 \pi x^2\,dy = \int_1^2 \pi(2y)^2\,dy = 28\pi/3.$

16.7 Put $x=0$ at A; moment $= \int_0^L mx\,dx = \frac{1}{2}mL^2.$

16.8 1.18.

16.9 0.015 g.

16.12 A sketch shows that $x(x-1) \geqslant -x$ if $0 \leqslant x \leqslant 2$.

Therefore the area is

$$\lim_{\delta x\to 0} \sum_{x=0}^{x=2} [x(x-1)-(-x)]\,\delta x = \int_0^2 x^2\,dx = \tfrac{8}{3}.$$

16.13 $\frac{3}{2}$.

16.14 (b) π.

16.15 In a plane perpendicular to the end, y is downward and x is horizontal; the origin is at the top. Area elements are horizontal strips of width δy in the end face. Force $=\frac{1}{2}\rho gLH^2$. Moment $=\frac{1}{6}\rho gLH^3$.

16.16 Distance of centre of mass from vertex is $\frac{3}{4}H$.

16.17 $\frac{1}{12}\sigma a^3 b$ (σ = mass per unit area).

16.18 (a) $\frac{1}{4}\sigma BH^3$; (b) $\frac{1}{48}\sigma HB^3$, where σ is mass per unit area.

16.23 $8a$.

Chapter 17

17.1 (c) $-\frac{1}{3}e^{-3x}+C$; (f) $-\frac{1}{12}(3-2x)^6+C$; (j) $(2x-3)^{\frac{1}{2}}+C$; (n) $\frac{1}{2}\ln|2x+3|+C$; (o) $\ln|1-x|+1/(1-x)+C$.

17.2 (b) $-\frac{2}{3}\cos\frac{1}{2}(3t-1)+C$; (e) $-\frac{2}{3}(-t)^{\frac{3}{2}}+C$;

17.3 (d) $\frac{1}{2}\sin(x^2+3)+C$. (j) $\frac{1}{2}\ln(1+x^2)+C$.

17.4 (c) $\frac{1}{6}\sin^3 2x+C$. (g) Put $\cot 2x = \cos 2x/\sin 2x$, then $u=\sin 2x$, giving $\frac{1}{2}\ln|\sin 2x|+C$. (j) $\frac{1}{3}\cos^3 x-\cos x+C$.

17.5 (b) 205/32; (e) $-\ln 2$; (h) $\frac{1}{2}\ln 2$; (k) zero; (n) $(2/\omega)\cos\phi$.

17.6 (b) $\frac{1}{2}\pi$; (d) $\frac{1}{4}\pi+\frac{1}{2}$; (f) $\frac{3}{8}\pi$.

17.7 (e) $\tan x - x + C$; (f) $-x^{-1}-\arctan(x^{-1})+C$; (k) $\frac{1}{2}[\arcsin x + x(1-x^2)^{\frac{1}{2}}]+C$.

17.8 (b) $\frac{1}{2}\ln|x/(x+2)|+C$.
(d) $\ln|x+1|-\frac{1}{2}\ln|2x+1|+C$.
(f) $\ln|x|-\frac{1}{2}\ln(x^2+1)+C$.
(i) $\frac{1}{2}\ln[(1+\sin x)/(1-\sin x)]+C$.

17.9 (b) $\frac{1}{2}\ln(x^2-2x+3)+C$. (e) $\ln(e^x+e^{-x})+C$. (f) $2\ln(x^{\frac{1}{2}}+1)+C$.

17.10 (b) $\frac{1}{3}x e^{3x}-\frac{1}{9}e^{3x}+C$.
(f) $2x\sin\frac{1}{2}x+4\cos\frac{1}{2}x+C$.
(i) $\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$. (j) $x^{n+1}[\ln x-1/(n+1)]/(n+1)$.
(k) Hint: bring together the two terms $\int(\ln x/x)\,dx$.

17.11 (a) Hint: there are two stages required; see Example 15.20.

17.12 Hint: the same integral occurs on both sides but with a different factor.

17.13 (b) zero; (d) $\frac{1}{2}$; (h) π.

17.15 $F(0)=\frac{1}{2}\pi$, $F(1)=1$, $F(4)=\frac{3}{16}\pi$, $F(5)=\frac{8}{15}$.

17.16 (a) $2(\ln 2)^3-6(\ln 2)^2+12\ln 2-6$.
(b) $F(0)=2$, $F(1)=\pi$, $F(4)=\pi^4+12\pi^2+48$, $F(5)=\pi^5+20\pi^3+120\pi$.

17.23 (c) $(a/b)\arctan[(a\tan x)/b]+C$;
(d) $\ln(\tan\frac{1}{2}x)+C$;
(g) $\ln|\sec x+\tan x|+C$; (j) $\ln[(1+\sqrt{5})/2]$;
(k) $8(6\sqrt{3}+1)/15$.

17.25 Coordinates of centroid: $(\frac{3}{5}h, 0)$.

Chapter 18

18.2 (b) $x=A\,e^{\frac{1}{2}t}$; (e) $x=A\,e^{-\frac{4}{3}t}$; (i) $x=A\,e^{t}$.

18.3 (b) $x=e^{\frac{1}{3}(t-1)}$; (d) $x=10\,e^{-(t+1)}$.

18.4 $I(t)=I_0\,e^{-Rt/L}$. I reduces to a fraction $1/n$ of itself in any interval of length $(L/R)\ln n$.

18.5 (a) $A(t)=C\,e^{-kt}$ (C arbitrary).

(b) The half-life $T=\dfrac{1}{k}\ln 2$ years. The information implies that $e^{-20k}=1-0.175=0.825$, so $k=0.0096$. Therefore $T=72$ years.

18.6 If $N(t)$ is the number, then $\delta N\approx 20(\frac{1}{2}N)\,\delta t$ so the equation is $dN/dt=10N$. In the second experiment there is an average death-rate of 1 per rabbit per year, so $dN/dt=9N$.

18.7 (b) $A\,e^{t}+B\,e^{-2t}$. (e) $A\,e^{t/2\sqrt{3}}+B\,e^{-t/2\sqrt{3}}$.
(l) $A\,e^{-3t}+Bt\,e^{-3t}$.
(n) $A+Bt$ (this is an exception to (18.10)).

18.9 (b) $\frac{2}{3}(e^{t} - e^{-2t})$.
(d) The general solution is $A\,e^{-x} + Bx\,e^{-x}$, $y = e(x-1)e^{-x}$.

18.10 (b) $A\cos 3t + B\sin 3t$.
(d) $A\cos\omega_0 t + B\sin\omega_0 t$. (f) $e^{t}(A\cos t + B\sin t)$.
(i) $e^{-\frac{2}{3}t}(A\cos\frac{1}{3}\sqrt{2}t + B\sin\frac{1}{3}\sqrt{2}t)$.

18.11 (c) $a\cos\omega_0 t + (b/\omega_0)\sin\omega_0 t$.

18.12 $\theta = \alpha\cos(g/l)^{\frac{1}{2}}t$.

18.13 The initial angular velocity $d\theta/dt$ is v/l;

$$\theta = \frac{v}{(lg)^{\frac{1}{2}}}\sin\left(\frac{g}{l}\right)^{\frac{1}{2}}t.$$

18.14 $\theta = 0.0719\,e^{-0.033t}\sin 0.696t$.

18.18 $A = (Mg/P)\,e^{-\rho g(y-H)/P}$.

Chapter 19

19.1 (b) $-\frac{1}{3}t^3 - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{11}{27}$.
(d) $\frac{3}{5}e^{2t}$. (i) $-\frac{2}{15}\sin 3t$.
(k) $-\frac{3}{25}\cos 2t + \frac{4}{25}\sin 2t$.

19.2 (d) $\frac{1}{5}(-6\cos t - 3\sin t)$.
(f) $-\frac{2}{137}(4\cos 2t + 11\sin 2t)$.
(h) $\frac{3}{65}e^{t}(4\cos 2t + 7\sin 2t)$.

19.3 (b) $-\frac{3}{4}t\cos 2t$.

19.4 (b) $\frac{1}{2}t^2e^{t}$; (e) $\frac{1}{2}t\,e^{t}\sin t$.

19.5 (c) $A\,e^{\frac{1}{2}t} + B\,e^{-\frac{1}{2}t} - 1 - \frac{3}{17}\cos 2t$.
(i) $A\cos x + B\sin x + x^2 - 1 + \frac{1}{5}e^{3x}$.

19.6 (c) $-\frac{1}{2} + A\,e^{t^2}$.
(g) $(\sin x - \cos x - x\cos x + A)/(x+1)$.
(l) $(x+1)\ln|x+1| + 1 + A(x+1)$.

19.9 $11\frac{1}{2}$ minutes.

Chapter 20

20.1 (b) $3\cos(\omega t + \pi)$. (e) $3\cos(2t + \frac{1}{2}\pi)$.
(h) $5\cos(2t + \phi)$, $\phi = -\arctan\frac{4}{3}$.

20.2 (c) x leads y by π.

20.3 (b) (i) 0.318 cycles/s. (ii) 0.316 cycles/s.
(iii) About 3 cycles.

20.4 (b) $C = \sqrt{(4 - \sqrt{6})}$, $\phi = \arctan(1/(\sqrt{6} - 1))$, $(-\frac{1}{2}\pi < \phi < 0)$.

20.7 The solutions are of exponential type.

20.8 $x = e^{-4t} - 4\,e^{-6t}$.

20.9 $A\,e^{-kt} + Bt\,e^{-kt}$.

20.10 (a) Period $= 1.0508$.
(b) Amplitude $= 10/[(36-\omega^2)^2 + \omega^2]^{\frac{1}{2}}$, phase $= -\arctan[\omega/(36-\omega^2)]$.
(c) Resonance: $\omega = 5.958$.

Chapter 21

21.1 (b) $-2e^{\frac{1}{2}\pi i}$ ($2e^{-\frac{1}{2}\pi i}$ in standard form).

21.2 (d) $2e^{-\frac{3}{4}\pi i}$; $2\cos(\omega t - \frac{3}{4}\pi)$.
(i) $e^{1.97i}$; $\cos(\omega t + 1.97)$.

21.3 (b) $1 - e^{-\frac{1}{2}\pi i} = 1 + i = \sqrt{2}\,e^{\frac{1}{4}\pi i}$.

21.4 (b) $1 - 3e^{-\frac{1}{2}\pi i} + e^{\frac{1}{2}\pi i} = 1 + 4i = \sqrt{17}\,e^{i\phi}$, where $\phi = \arctan 4 = 1.33$.

21.6 (b) $R + \omega Li$. (d) $R/(1 + \omega RCi)$.
(i) $R + i\omega L/(1 - \omega^2LC)$.
(k) $i\omega RL/[R(1 - \omega^2LC) + i\omega L]$.

21.7 $V = \boldsymbol{Z}\boldsymbol{I}$ and $V = 2$.
(d) $\boldsymbol{I} = 2(1 + i\omega RC)/R$; $|\boldsymbol{I}| = 2(1 + \omega^2R^2C^2)^{\frac{1}{2}}/R$; $\arg\boldsymbol{I} = \arctan(\omega RC)$.

21.8 (b) $V_1/V_0 = \frac{2}{13}(3 - 2i)$; $V_0/I_1 = \frac{1}{2}(5 - i)$.

Chapter 22

22.4 (b) $2x^2 - y^2 = C$. (g) $y = x/(1 + Cx)$.
(k) $x = \pm 2^{-\frac{1}{2}}(C - t^3)^{-\frac{1}{2}}$ for $t^3 < C$.
(n) $\arctan y + \arctan x = \frac{1}{4}\pi$. Take the tangent of this expression and use the formula for $\tan(A + B)$; we find that $y = (x+1)/(x-1)$.

22.6 (b) $y = \frac{1}{16}(x^2 + C)^2$ for $x^2 + C > 0$. $y = 0$ is also a solution. (d) Those parts of the curves $y = \sin(\ln|x| + C)$ for which x and dy/dx have the same sign. Also $y = \pm 1$ are solutions.

22.7 (b) $y^3 - 3xy = C$. (d) $xy - y^2 - x^2 = C$.
(f) $y^3 + y - x^3 = C$. (h) $y + \cos y + \sin x = C$.
(j) $e^{x+y} + y - x = C$.

22.8 (b) $xy + y/x = C$; (d) $x/y + y - x = C$;
(e) $y/x - x/y - 1/x = C$; (f) $x^2/(2y^2) + 1/(xy) = C$.

22.12 (b) $x(1 + 2y^2/x^2)^{\frac{1}{4}} = C$.
(d) $x^2 - 4y^2 = Cy^3$.

Chapter 23

23.2 (b) $y = Cx$ (this is not covered by (23.22)).
(d) $xy = C$ (a saddle).

23.4 (b) Saddle (i.e. unstable). $m = \frac{1}{2}(-3 \pm \sqrt{13})$.
(f) Stable spiral; directions are clockwise round origin.

23.5 (b) Equilibrium points at (1, 1). (1, 1) is a stable spiral, anticlockwise about (1, 1). (d) Equilibrium points at $(-1, 0)$, $(0, 0)$, $(0, 1)$; $(0, 0)$ is a centre and $(-1, 0)$, $(1, 0)$ are saddle points.

Chapter 24

24.1 (b) $4/(s+1)$; (d) $6/s^3 - 1/s$; (g) $(3-s)/(s^2+1)$.

24.2 (b) $1/s - 2/(s+2)$; (e) $(3s-4)/(s^2+4)$; (g) $\frac{1}{2}[1/s - s/(s^2+4)]$.

24.3 (b) $1/(s+2)^2$; (d) $(s-2)/(s^2-4s+5)$; (i) $(s^2-9)/(s^2+9)^2$; (l) $24/(s+1)^5$.

24.5 (b) 1; (d) $\frac{1}{8}t^4$; (g) $\frac{3}{2}e^{\frac{1}{2}t}$; (k) $\frac{1}{2}e^t + \frac{1}{2}e^{-t}$; (o) $2\cos 2t - \frac{1}{2}\sin 2t$; (s) $\frac{1}{2}e^t t^2$; (u) $\frac{1}{3}(\cos t - \cos 2t)$.

24.6 (e) $(2s^2+3s-2)X(s) - 10s - 9$.

24.7 (b) $2e^t + e^{-2t}$; (e) $3e^{-t}\cos 2t$; (f) $y = \frac{1}{4}e^x + \frac{1}{4}e^{-x} + \frac{1}{2}\cos x$.

24.8 (b) $3 - 3\cos t + \sin t$.
(e) $-\frac{1}{8}e^{-t} + \frac{9}{8}e^t - \frac{1}{4}t\,e^t + \frac{1}{4}t^2 e^t$.
(i) $-\frac{7}{6}e^t - \frac{1}{2}e^{-t} + \frac{3}{4}e^{2t} - \frac{1}{12}e^{-2t}$.

24.9 (b) $x = \frac{3}{8} + \frac{5}{8}e^{4t} + \frac{1}{2}t\,e^{4t}$; $y = -\frac{3}{16} + \frac{3}{16}e^{4t} + \frac{1}{4}t\,e^{4t}$.

24.10 (b) $e^t(\frac{1}{2}A + \frac{1}{2}B + \frac{3}{2}) + e^{-t}(\frac{1}{2}A - \frac{1}{2}B + \frac{3}{2}) - 3$, where A and B are arbitrary. This is the same as $C\,e^t + D\,e^{-t} - 3$, where C and D are arbitrary.

24.13 $e^{-2}e^{-2s}[(s+1)^2-1]/[(s+1)^2+1]^2 = e^{-2}e^{-2s}\,s(s+2)/(s^2+2s+2)^2$.

24.14 (b) $H(t)\sin t - H(t-1)\cos(t-1)$.

24.15 (b) $(\frac{1}{8}e^{2t} + \frac{1}{8}e^{-2t} - \frac{1}{4})H(t)$, $-(\frac{1}{8}e^{2(t-1)} + \frac{1}{8}e^{-2(t-1)} - \frac{1}{4})H(t-1)$.
(d) $\frac{1}{2}H(t)t\sin t + \frac{1}{2}H(t-\pi)(t-\pi)\sin(t-\pi)$.

Chapter 25

25.3 Hint for working: $s^2 + 2ks + \omega^2$ has real factors when $k^2 > \omega^2$; so put $s^2 + 2ks + \omega^2 = (s-\alpha)(s-\beta)$, where $\alpha, \beta = -k \pm (k^2 - \omega^2)^{\frac{1}{2}}$. Then $x(t)$ is given by

$$(\alpha-\beta)^{-1}[(\alpha+\kappa)e^{\alpha t} - (\beta+\kappa)e^{\beta t}]H(t) + I(\alpha-\beta)^{-1}[e^{\alpha(t-t_0)} - e^{\beta(t-t_0)}]H(t-t_0),$$

where $\kappa = 1 + 2k$.

25.4 By proceeding as suggested, we obtain

$$u(x) = Ax + \tfrac{1}{6}Bx^3 + (Mg/6K)(x - \tfrac{1}{2}l)^3 H(x - \tfrac{1}{2}l).$$

The conditions at $x = l$ give $A = Mgl^2/16K$, $B = -Mg/2K$. This problem could be solved by integrating the equation four times, and linking the solutions over $[0, \frac{1}{2}l]$ and $[\frac{1}{2}l, l]$ by the condition that $u(x)$, $u'(x)$, $u''(x)$ are continuous at $x = \frac{1}{2}l$, but this is automatically secured in the Laplace-transform method.

25.5 (b) $2s/(6s^2+s+1)$.

25.6 (b) $V_2/V_1 = 3/(20s^2+12s+5)$; $V_2/I = 3/(4s^2+1)$.

25.7 (b) t; (f) $1 - \cos t$; (h) $\frac{1}{2}(-t\cos t + \sin t)$; (j) $n!m!t^{n+m+1}/(n+m+1)!$.

25.8 (b) $\dfrac{1}{2\omega}\displaystyle\int_0^t f(\tau)(e^{\omega(t-\tau)} - e^{-\omega(t-\tau)})\,d\tau$.

25.9 (b) $\cosh t$.

25.19 (a) $x(t) = \delta(t) + 2\delta(t-T) + \delta(t-2T)$, $X(s) = 1 + 2e^{-sT} + e^{-2sT}$.

25.21 (a) $z^{-1} + 2z^{-2} - z^{-3}$. (b) $1 - z^{-1} + z^{-2} - \cdots = z/(z+1)$. (c) $2z/(2z-1)$. (d) $z/(z^2-1)$.

25.22 (a) $Tz/(z-1)^2$.

25.23 (a) $(z-1)/(z+1)$, $g(t) = \{1, -2, 2, -2, \dots\}$.

25.27 (a) Unstable. Poles at $z = \pm 2$, giving growth $\frac{3}{4}2^n$ and $\frac{1}{4}(-1)^n 2^n$. (c) Stable. Poles at $z = \pm\frac{1}{2}i$, giving decay $\dfrac{1}{4}\dfrac{1}{2^n}\cos\dfrac{1}{2}\pi n$.

Chapter 26

26.1 (b) $a_n = 0$, $b_n = -2(-1)^n/n$.
(e) $a_n = 0$, $b_n = \dfrac{2}{\pi n}[1 + (-1)^n - 2\cos(\frac{1}{2}n\pi)]$.

26.2 (b) $b_n = 0$, $a_0 = \dfrac{2\pi^2}{3}$, $a_n = \dfrac{4}{n^2}(-1)^n (n = 1, 2, \dots)$.
(c) $b_n = 0$, $a_n = -\dfrac{4(-1)^n}{\pi(4n^2-1)}$.

26.3 (a) $a_0 = \frac{1}{2}\pi$, $a_{2n} = 0$, $a_{2n-1} = -\dfrac{2}{\pi n^2}$, $b_n = -\dfrac{(-1)^n}{n}$ $(n = 1, 2, \dots)$.

26.5 Series sum is $\frac{1}{4}\pi$.

26.8 $F = 2$.

26.10 $a_0 = 0$, $a_n = 0$, $b_n = \dfrac{4\beta}{\pi n^3}[1 - (-1)^n] (n = 1, 2, \dots)$.

26.16 (a) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin(2n-1)\pi t$.

26.18 $\dfrac{2}{\pi} - \displaystyle\sum_{n=1}^{\infty} \frac{4}{\pi(4n^2-1)}\cos 2n\omega t$.

26.23 (b) $R(t) = \dfrac{1}{2} + \dfrac{4}{\pi}\left[\cos\pi t + \dfrac{1}{3^2}\cos 3\pi t + \dfrac{1}{5^2}\cos 5\pi t + \cdots\right]$.

26.26 (b) $\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

26.30 $\frac{1}{2} + \frac{i}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n} e^{i2\pi nt/T}$.

Chapter 27

27.1 $X_s(f) = 4\pi f/(1+4\pi^2 f^2)$; $X_c(f) = 2/(1+4\pi^2 f^2)$.

27.8 $x(t) = 2\int_0^{\infty} X(f) \cos 2\pi ft \, dt$ where

$X(f) = 2\int_0^{\infty} x(t) \cos 2\pi ft \, dt$.

27.11 (c) $2c \operatorname{sinc} cf \cos 2\pi bcf$.

27.12 (a) $\frac{1}{2} \operatorname{sinc}^2 \frac{1}{2} f$. (b) $\frac{1}{2} \operatorname{sinc}^2 \frac{1}{2} f \, e^{-i3\pi f}$.

27.17 $\{1/[\alpha + i(2\pi f + \beta)] + 1/[\alpha + i(2\pi f - \beta)]\}$.

27.19 (b) $1/(1+i2\pi f)^2$.

27.20 (b) The Fourier transform is $\operatorname{sinc}^2(f)\, e^{-i2\pi(a+b)f} \leftrightarrow \Lambda[t-(a+b)]$.

Chapter 28

28.3 (c) $4x - 2y - 1; -6y - 2x - 1$.
(f) $y - 2; x - 1$. (i) $2y/(x+y)^2; -2x/(x+y)^2$.
(k) $x(x^2+y^2)^{-\frac{1}{2}}; y(x^2+y^2)^{-\frac{1}{2}}$.

28.4 (c) $\frac{\partial V}{\partial x} = g'(r)\cos\theta; \frac{\partial V}{\partial y} = g'(r)\sin\theta$.

28.8 $\partial^2 f/\partial x^2, \partial^2 f/\partial y^2$, and $\partial^2 f/\partial x\,\partial y = \partial^2 f/\partial y\,\partial x$ are given in order: (b) 2, 4, 3. (d) $2y/x^3, 0, -1/x^2$.
(h) $108(3x-4y)^2, 192(3x-4y)^2, -144(3x-4y)^2$.
(k) $-r^{-3} + 3x^2 r^{-5}, -r^{-3} + 3y^2 r^{-5}, 3xyr^{-5}$, where $r = (x^2+y^2)^{\frac{1}{2}}$.

28.10 (b) $2x + 2y - z = 4$; one normal is $(2, 2, -1)$.
(d) $3x + 4y + 8z = 29$; one normal is $(-\frac{3}{2}, -2, -1)$.

28.11 78.9° or 101.1°.

28.12 (b) $(1, -1)$, min; (d) $(n\pi, m\pi)$; min if n and m odd, max if n and m even, otherwise saddle;
(h) (0, 0) saddle; (1, 1) minimum; (k) (0, 0), saddle.

28.14 (a) $a = b = c = 7$; (b) $a = b = c = 4$.

28.15 The maximum is 9, attained at $(2, \pm 1)$.

28.16 Minimum distance $= \sqrt{2}$.

28.18 (b) Depth $= 2^{-\frac{2}{3}} V^{\frac{1}{3}}$; square base, side $2^{\frac{1}{3}} V^{\frac{1}{3}}$.

28.23 Lowest point is $z = \frac{3}{4}a$ at $(0, a)$ and (a, a).

Chapter 29

29.1 (b) $\delta z = 0.0718\ldots$ (exactly). The incremental approximation gives $\delta z \approx 0.0784$. Error $= 9.1\%$.

29.3 (b) $-\delta y(\delta n + \delta y)/(1 + \delta y)$.

29.6 -5.7%.

29.7 1.67% reduction, approximately.

29.9 (b) $-2\sqrt{2}$; (d) zero (it is the same in all directions).

29.10 (b) $-\frac{3}{4}$; (e) $-\frac{1}{2}$; (j) 1.

29.12 (b) $x_1 x/a^2 + y_1 y/b^2 = x_1^2/a^2 + y_1^2/b^2$.
(f) $ax_1 x + h(y_1 x + x_1 y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

29.16 (b) $x^{-1} - y^{-1} =$ constant; (d) $e^x + e^y =$ constant.

29.17 (b) $y^2 - x^2 = b^2 - a^2$.

29.19 (b) 49.8° or 130.2°.

(d) Hint: compare Problem 29.12f.

29.21 (b) $(0, \frac{1}{2})$; (d) $(-\frac{1}{4}, 1)$.

29.22 (b) $(2, 1)/\sqrt{5}$.

29.23 (b) $\phi = 0$.

Chapter 30

30.2 (b) $-4 \sin t \cos t$; (d) $2\sin(t^2) + 4t^2\cos(t^2)$.

30.3 It is easiest to start by expressing the distance D in terms of polar coordinates (r, θ), (R, ϕ) by using the cosine rule (Appendix B(f)). Then

$$\frac{dD}{dt} = -\frac{(Rv - rV)\sin(\phi - \theta)}{[R^2 + r^2 - 2Rr\cos(\phi-\theta)]^{\frac{1}{2}}},$$

where $\theta = vt/r$, $\phi = Vt/R$.

30.4 (b) $x = y = 3$. (e) The coordinates of the nearest point on the given line are $(\frac{3}{5}, \frac{1}{5})$. Distance $= 2/\sqrt{5}$.

30.5 (b) (0, 0), (2, 0). (A suitable parametrization is $x = 1 + \cos t$, $y = \sin t$.)
(d) $(\pm 6/\sqrt{5}, \pm 4/\sqrt{5})$. (A suitable parametrization would be $x = 2/\cos t$, $y = 2\tan t$.)

30.8 (b) $\ddot{x} = -2\dot{r}\dot{\theta}\sin\theta + \ddot{r}\cos\theta - \dot{\theta}^2 r\cos\theta - \ddot{\theta} r\sin\theta$,
$\ddot{y} = 2\dot{r}\dot{\theta}\cos\theta + \ddot{r}\sin\theta - \dot{\theta}^2 r\sin\theta + \ddot{\theta} r\cos\theta$.

30.9 (c) $\partial f/\partial u = -2v^2/u^3$, $\partial f/\partial v = 2v/u$.

30.10 (b) $\partial^2 f/\partial u^2 = 12u^2 - 2v^2$, $\partial^2 f/\partial u\,\partial v = -4uv$,
$\partial^2 f/\partial v^2 = -2u^2 + 12v^2$.

30.11 It is easiest to put $x^2 - y^2$ in terms of uv. Finally,
$\partial^2 f/\partial u^2 = 16v^2 g''(4uv)$, $\partial^2 f/\partial v^2 = 16u^2 g''(4uv)$,
$\partial^2 f/\partial u \partial v = 4g'(4uv) + 16uvg''(4uv)$.

Chapter 31

31.1 (b) $\delta f \approx -x(x^2+y^2)^{-\frac{3}{2}}e^{-t}\,\delta x - y(x^2+y^2)^{-\frac{3}{2}}e^{-t}\,\delta y - (x^2+y^2)^{-\frac{1}{2}}e^{-t}\,\delta t$.
(e) $\delta f \approx 2(x_1-x_2)\,\delta x_1 - 2(x_1-x_2)\,\delta x_2 + 2(y_1-y_2)\,\delta y_1 - 2(y_1-y_2)\,\delta y_2$.

31.2 −0.07.

31.3 It is easiest to write $\delta(1/R) \approx -\delta R/R^2$. We obtain $\delta R \approx 0.198\,\delta R_1 + 0.018\,\delta R_2 + 0.334\,\delta R_3$. The required δR_3 is −0.108.

31.4 Put $ax^3 - bx - c = f(a, b, c, x)$ and use (31.1).

31.5 (b) Hint: use logarithmic differentiation: $\delta w \approx -3\,\delta x + 3\,\delta z$ and $\delta w \approx 2(\pm 0.6)$. What is the significance of the absence of a term in δy?

31.8 (b) $2\delta x + 4\delta y - 6\delta z = 0$. For $\partial z/\partial x$, put $\delta y = 0$: $\partial z/\partial x = \frac{1}{3}$. Similarly $\partial z/\partial y = \frac{2}{3}$.

31.11 (b) $(2, -3, 5)$; (d) $(3x^2, 0, 9z^2)$; (f) $(-x/r^3, -y/r^3, -z/r^3)$, where $r = (x^2+y^2+z^2)^{\frac{1}{2}}$.

31.12 (b) $(0, 2y, 2z)$. Unit vector $= (0, y/(y^2+z^2)^{\frac{1}{2}}, z/(y^2+z^2)^{\frac{1}{2}})$.

31.13 (b) $\cos\phi = 11/(3\sqrt{14})$, so $\phi = 11.5°$ (i.e. the angle of intersection of smallest magnitude).

31.15 (b) $\hat{s}\cdot(2x, -2y, -3)$.

31.16 (b) (Check that $\hat{s}$ as given is a unit vector.)
$$\frac{df}{ds} = 7.51.$$

31.17 (b) $-2\hat{\imath} - 2\hat{k}$.

31.18 (b) $(\pm 1, 0, 0)$ and $(\pm 1, \frac{1}{4}, \frac{3}{16})$.
(d) $x = y = z$ is a line of stationary points (excluding the origin). (e) $x = y = z = \pm 1, \sqrt{3}, \lambda = \pm\frac{1}{2}\sqrt{3}$.

31.19 Stationary at (1, 0, 0), (−1, 0, 1), (−1, 0, −1).

31.21 (b) (3, 3, 3); (e) $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$; (g) $(\frac{1}{3}, \frac{7}{3}, 1)$.

31.26 (b) $4xy = 1$; (d) $x^2 + y^2 = 1$.

Chapter 32

32.1 (b) e − 2; (d) $(d-c)(b-a)$; (i) $-\frac{1}{3}$; (m) $\frac{1}{2}\ln 2$.

32.2 (b) Zero. Refer to the signed-volume analogy (30.2b). (f) ln(27/16).

32.4 $\frac{4}{3}$.

32.5 (b) $\displaystyle\int_0^1\int_0^x f(x, y)\,dy\,dx.$

(d) $\displaystyle\int_{-1}^1\int_0^{\sqrt{(1-x^2)}} f(x, y)\,dy\,dx.$

(g) $\displaystyle\int_{-1}^0\int_0^{1+y} f(x, y)\,dx\,dy + \int_0^1\int_0^{1-y} f(x, y)\,dx\,dy.$

32.6 (b) $\frac{2}{3}$; (d) $\frac{3}{2}$; (h) $\frac{1}{12}$.

32.7 (b) 1.

32.8 (b) $\frac{1}{4}\pi$; (d) $\frac{15}{8}$; (f) $\frac{\pi}{16}$.

32.9 $2a^3(4+3\pi)/9$.

32.10 (a) $2(u^2+v^2)$; (b) 2; (c) 1/5; (d) $-2\cosh v$.

32.11 The value of the integral is $2(257 - 129\sqrt{2})/5$.

32.12 Area = 1/12.

32.13 1/e.

32.14 Volume = 64/3.

32.15 1/4.

32.18 (a) $\sqrt{\pi}(|b| - |a|)$.

Chapter 33

33.1 (b) 1.

33.2 (b) $\frac{4}{3}$; (d) $\frac{2}{3}$; (f) 0.

33.3 (b) π; (d) $\frac{8}{3}\pi^3$.

33.5 (b) 2; (d) 0; (g) 0.

33.6 (b) 1; (d) −3; (f) $\frac{15}{2}$.

33.7 (b) 0.

33.8 (b) $-\frac{3}{2}$.

33.9 Zero.

33.11 Put $x = x(u, v)$ and $y = y(u, v)$, where u and v are the new coordinates. Then put
$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \text{ etc.}$$

33.14 $\frac{3}{8}\pi$.

33.16 (b) Non-conservative.

Chapter 34

34.1 $\pi[a^3 - (a-h)^3]/a$.

34.2 (a) 1/84; (b) 1/24; (c) 13/384.

34.5 $2\sqrt{6} + 2\sinh^{-1}(\sqrt{2})$.

34.6 Scalar potential is $e^{xyz} + \cos xy + zx + C$.

34.7 Scale factors are $h_1 = h_2 = \sqrt{(u^2+v^2)}$, $h_3 = uv$.

34.11 (b) div $\mathbf{F} = 2z$.

34.12 (b) curl $\mathbf{F} = 2x\hat{\imath} + (x - 2y)\hat{\jmath} + \hat{k}$.

34.14 (a) $5r^2$; (b) 0; (c) $3\mathbf{r}r$; (d) 0; (e) 0; (f) $12r$.

34.18 0.

Chapter 35

35.1 (c) −2, −1, 0, 1, 2, 3, 4; (f) 1, 4, 9.

35.3 (c) $A \cup B = \{-4, -3, -2, -1, 1, 2, 3, 4\}$.

35.4 (b) $A \cap B = \{x \mid x \in \mathbb{N}^+ \text{ and } -5 \leqslant x \leqslant 2\}$.
(d) $A \cap B = \{1\}$.

35.5 (b) $B \backslash (A \cup C)$; (d) $(B \cap D) \backslash A$.

35.6 (b) $S_1 \backslash (S \cup S_2 \cup \cdots \cup S_r)$.

35.7 (b) $[(A \backslash A_1) \backslash B_1] \cup B_2$.

35.10 $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

35.12 (b) 66.

Chapter 36

36.6 See table below.

36.10 (a) $(a*b)*(b \oplus c)$; (d) $\overline{(\bar{a}*b \oplus a*\bar{b})(c*d)}$.

36.15 (a) If a_1 represents the state of switch S_1, etc., then the switching function is

$(a_1 \oplus a_2) \oplus [(a_3 \oplus a_4)*a_5]$.

36.16 See the table below.

Solution for 36.6

a	b	c	$(a \oplus \bar{b})*(a \oplus \bar{c})$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Solution for 36.16

a_1	a_2	a_3	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Chapter 37

37.3 Twenty are planar.

37.4 Five are connected.

37.8 Six not including reversed order.

37.13 There are three different paths between a and e.

37.14 Five vertices.

37.15 $i_1 = -\frac{4}{21}i_0, i_2 = -\frac{1}{21}i_0, i_3 = -\frac{10}{21}i_0, i_4 = -\frac{11}{21}i_0$,
$i_5 = \frac{1}{21}i_0, i_6 = \frac{4}{21}i_0, i_7 = -\frac{3}{7}i_0$.

37.17 The transfer function is

$$Q = \frac{PG_1G_2G_3}{1 - G_2H_1 + G_1G_2G_3H_2}.$$

37.19 (a) The transfer function is

$$Q = \frac{PG_1G_2G_3}{1 + G_2H_1 - G_1G_2G_3H_2}.$$

37.20 (a) $\dfrac{G_1G_3}{(1 - G_1G_2H_2)(1 + G_3H_1)}$.

(d) $\dfrac{G_1G_2G_3G_4}{1 + G_2G_3H_2} + \dfrac{G_5G_6G_7}{1 - H_1}$.

37.24 $SAFT$, length 12.

37.25 2 ties.

37.27 (b) Framework is overbraced.

37.28 Two ties.

37.29 Waiting times are $13T/3$ and $4T$.

Chapter 38

38.1 £1790.85, 0.487%.

38.2 (b) 16.9 years.

38.3 (b) 0, $(-1 \pm \sqrt{13})/6$.

38.6 $f(n) = (\ln n)/\ln 2$.

38.8 (b) $u_n = A3^n + B(-3)^n$.
(c) $u_n = 3^n(A \cos \frac{1}{2}n\pi + B \sin \frac{1}{2}n\pi)$.

38.11 (a) (ii) $u_n = -\frac{3}{16}n + \frac{1}{8}n^2$.
(b) (ii) $u_n = \frac{1}{5}n + \frac{11}{25}$.
(c) (iii) $u_n = \frac{1}{2}n^2$.
(d) (iii) $u_n = \frac{1}{18}n^2 3^n$.

38.13 $D_n(1) = n + 1$.

38.16 $u_n = \frac{5}{4} - \frac{9}{4}(-\frac{1}{3})^n$.

38.17 $d_k = k(N - k)$.

38.19 $s_n = \frac{1}{4}n^2(1 + n)^2$.

38.22 $0 < \alpha < 1$.

38.23 Oscillates between 0.4953 and 0.8124.

38.24 The periodic values of the 2-cycle are 0.4 and 0.8.

Chapter 39

39.1 (d) 63.

39.2 The probability that the score is 7 or less is 7/12.

39.5 $n(A \cup B) = 10$.

39.6 (b) Ace of clubs or ace of spades drawn; (d) any ace or any heart or any black card drawn; (f) any heart except the ace of hearts; (h) ace of hearts or any black card.

39.7 (b) 1/221; (b) 0.004 166… .

39.9 (b) 5040; (d) 7.

39.11 (a) 27 216; (c) 3360.

39.12 (b) 156 849.

39.15 270 725; 0.010 56… .

39.17 (a) 9/209; (c) 16/665; (d) 683/1463.

39.18 (b) 0.872; (c) 0.4.

39.19 (b) 0.37; (c) 0.82.

39.20 With the same probability of failure 0.98, probability that circuit fails is 0.963.

39.21 (b) 1/495; (c) 4/99.

39.22 Overall probability is approximately 1/53.7.

39.23 Mean number of plays to the end of the game is $2^{n-1}/n$.

Chapter 40

40.1 $P(X \geqslant 1) = 0.833$.

40.2 $P(X \geqslant 6) = 1/32$.

40.3 Mean = 4; standard deviation = 1.633.

40.4 Mean = $(a + b)/2$; standard deviation = $(b - a)/(2\sqrt{3})$.

40.6 Mean number of non-faulty components to failure is 82.33; standard deviation of the number of components to failure is 82.83.

40.7 $1/2^9$.

40.9 Probability that a bottle fails the test is 0.000 67.

40.10 (a) 0.777; (c) 0.223.

40.11 (b) 0.528.

40.13 (b) $P(Z \leqslant 0.7) = 0.758$.

40.14 On average 30% of operations take longer than 40 seconds.

40.15 Standard deviation of 1 if $a = \sqrt{5}$ and $A = 3/(20\sqrt{5})$.

40.16 Maximum value of standard deviation is 121.6.

40.17 Probability that just two bulbs will be still working is 0.242.

Chapter 41

41.1 (b) Mean = 24.1; median = 24.5; interquartile range = 17.

41.3 Sample mean = 25.3; mode = 25.1; variance = 0.0644.

41.5 About 11 intervals.

41.6 Estimated variance of the sample is 1/12.

41.8 $k_1 = -1.1337$; $k_2 = 1.1337$.

41.9 For full data $\hat{a} = -0.0071$; $\hat{b} = 18.76$.

Appendices

A Some algebraical rules

(a) Index laws for real numbers

(i) $a^0 = 1$.

(ii) $a^p a^q = a^{p+q}$.

(iii) $a^{-p} = 1/a^p$.

(iv) $(a^p)^q$ or $(a^q)^p = a^{pq}$ (so $a^{p/q} = (a^p)^{1/q}$ or $(a^{1/q})^p$).

(v) $a^p b^p = (ab)^p$

Conventionally, $a^{\frac{1}{2}}$ and $\sqrt{a}$ represent the **positive root** when we are talking about real numbers (for complex numbers, see Chapter 6). For all the rules to hold in *all* cases, a must be positive so that $a^{p/q}$ is a always real number. For example, $(-8)^{\frac{1}{2}}$ or $\sqrt{(-8)}$ is not real: there is no real number whose square is equal to -8. But $(-8)^{\frac{1}{3}}$ or $\sqrt[3]{(-8)} = -2$.

(b) Quadratic equations

$ax^2 + bx + c = 0$ has the solutions

$$x_1, x_2 = [-b \pm \sqrt{(b^2 - 4ac)}]/2a.$$

(i) In terms of x_1 and x_2, the factors are

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

(ii) Sum and product of solutions:

$$x_1 + x_2 = -b/a, \qquad x_1 x_2 = c/a.$$

(c) Binomial theorem

(i) If n is a positive integer (or whole number)

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

where the binomial coefficients are denoted by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

There are $(n+1)$ terms in this sum, and it is symmetrical in a and b.

An important special case is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.$$

(ii) *Pascal's triangle*. Each entry (apart from the numerals 1) is the sum of two previous entries – that above, and that above and to the left – as illustrated by the underlined group:

$n=1$	1	1			
$n=2$	1	2	1		
$n=3$	1	$\underline{3}$	$\underline{3}$	1	
$n=4$	1	4	$\underline{6}$	4	1

and so on. Thus

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

(iii) *Permutations and combinations (see Section* 1.17*).*

$$_nP_r = \frac{n!}{(n-r)!}, \qquad _nC_r = \frac{n!}{(n-r)!\,r!}.$$

(d) Factorization

$a^2 - b^2 = (a+b)(a-b)$,
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$,
$a^3 + b^3 = (a+b)(a^2-ab+b^2)$.

(e) Constants

$\mathrm{e} = 2.718\,281\,82\ldots$,
$\pi = 3.141\,592\,65\ldots$,
1 radian $= 57.295\,78\ldots^\circ$,
$1^\circ = 0.017\,45\ldots$ radians,
$360^\circ = 2\pi$ radians.

(f) Sums of powers of integers

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

B Trigonometric formulae

(a) Relation between trigonometric functions

$\sin^2 A + \cos^2 A = 1$,
$\tan A = \sin A/\cos A$; $\sec A = 1/\cos A$; $\operatorname{cosec} A = 1/\sin A$.

(b) Addition formulae

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$,
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$,
$\tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$.

(c) Addition formulae: special cases

$\sin 2A = 2 \sin A \cos A$,
$\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1 = 1 - 2\sin^2 A$,
$\tan 2A = 2 \tan A/(1 - \tan^2 A)$,
$\sin 3A = 3 \sin A - 4 \sin^3 A$,
$\cos 3A = 4 \cos^3 A - 3 \cos A$.

(d) Product formulae

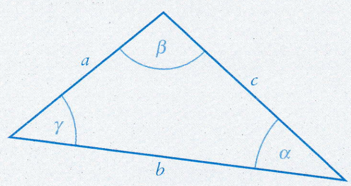
$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$,
$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$,
$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$.
$\sin C + \sin D = 2 \sin \frac{1}{2}(C + D) \cos \frac{1}{2}(C - D)$,
$\sin C - \sin D = 2 \sin \frac{1}{2}(C - D) \cos \frac{1}{2}(C + D)$,
$\cos C + \cos D = 2 \cos \frac{1}{2}(C + D) \cos \frac{1}{2}(C - D)$,
$\cos C - \cos D = -2 \sin \frac{1}{2}(C + D) \sin \frac{1}{2}(C - D)$.

(e) Product formulae: special cases

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$,
$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$,
$\sin^3 A = \frac{1}{4}(3 \sin A - \sin 3A)$,
$\cos^3 A = \frac{1}{4}(3 \cos A + \cos 3A)$.

(f) Triangle formulae

(i) $\alpha + \beta + \gamma = 180°$.
(ii) *Cosine rule*: $a^2 = b^2 + c^2 - 2bc \cos \alpha$.
(iii) *Sine rule*: $\dfrac{\sin \alpha}{a} = \dfrac{\sin \beta}{b} = \dfrac{\sin \gamma}{c}$.



(g) Trigonometric equations

In the following, n represents any integer (i.e. any whole number, positive or negative); x is in radians.

(i) $\sin x = 0$ and $\tan x = 0$ when $x = n\pi$; $\cos x = 0$ when $x = \frac{1}{2}\pi + n\pi$.
(ii) The following formulae show how to obtain all the solutions of certain equations when one solution has been obtained (e.g. a hand calculator or

a computer gives only one solution of $\sin x = -\frac{1}{2}$, namely $x = \arcsin(-\frac{1}{2}) = -0.5236\ldots$).

If $\sin\alpha = c$, then all the solutions of $\sin x = c$ are $x = n\pi + (-1)^n\alpha$.
If $\cos\beta = c$, then all the solutions of $\cos x = c$ are $x = 2n\pi \pm \beta$.
If $\tan\gamma = c$, then all the solutions of $\tan x = c$ are $x = n\pi + \gamma$.

(h) Hyperbolic functions

$\cosh x = \frac{1}{2}(e^x + e^{-x})$; $\sinh x = \frac{1}{2}(e^x - e^{-x})$; $\tanh x = \sinh x/\cosh x$;

$\text{sech}\, x = 1/\cosh x$; $\coth x = \cosh x/\sinh x$; $\text{cosech}\, x = 1/\sinh x$,

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$,

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$,

$\cosh^2 x - \sinh^2 x = 1$,

$\sinh 2x = 2 \sinh x \cosh x$,

$\cosh 2x = \cosh^2 x + \sinh^2 x$,

$\cosh ix = \cos x$; $\sinh ix = i \sin x$;

$\sinh^{-1} x = \ln[x + (x^2 + 1)^{\frac{1}{2}}]$,

$\cosh^{-1} x = \ln[x + (x^2 - 1)^{\frac{1}{2}}] \quad (x \geqslant 1)$,

$\tanh^{-1} x = \frac{1}{2}\ln[(1 + x)/(1 - x)] \quad (-1 < x < 1)$.

C Areas and volumes

(a) The area of a triangle is $\frac{1}{2}bh$, where b is the length of one side and h its height from that side.

(b) The circumference of a circle is $2\pi r$, where r is its radius.

(c) The area of a circle is πr^2, where r is its radius.

(d) The area of a circle sector is $\frac{1}{2}r^2\theta$, where r is its radius and θ the angle of the sector in radians.

(e) The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is its radius.

(f) The surface area of a sphere is $4\pi r^2$, where r is its radius.

(g) The volume of a cone is $\frac{1}{3}Ah$, where h is its height and A the cross-sectional area of its base.

(h) The area of an ellipse is πab, where a and b are the lengths of its semi-axes.

(i) The area of a regular n-sided polygon of side-length a is $\frac{1}{4}na^2 \cot(\pi/n)$.

D A table of derivatives

y	$\frac{dy}{dx}$
c (constant)	0
x^n (n any constant)	nx^{n-1}
e^{ax}	$a\,e^{ax}$
k^x ($k > 0$)	$k^x \ln k$
$\ln x$ ($x > 0$)	x^{-1}
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a/\cos^2 ax$
$\cot ax$	$-a/\sin^2 x$
$\sec ax$	$(a \sin ax)/\cos^2 ax$
$\operatorname{cosec} ax$	$-(a \cos ax)/\sin^2 ax$
$\arcsin ax$	$a/(1-a^2x^2)^{\frac{1}{2}}$
$\arccos ax$	$-a/(1-a^2x^2)^{\frac{1}{2}}$
$\arctan ax$	$a/(1+a^2x^2)$
$\sinh ax$	$a \cosh ax$
$\cosh ax$	$a \sinh ax$
$\tanh ax$	$a/\cosh^2 ax$
$\sinh^{-1} ax$	$a/(1+a^2x^2)^{\frac{1}{2}}$
$\cosh^{-1} ax$	$a/(a^2x^2-1)^{\frac{1}{2}}$
$\tanh^{-1} ax$	$a/(1-a^2x^2)$
$u(x)v(x)$	$u\frac{dv}{dx}+v\frac{du}{dx}$
$\frac{u(x)}{v(x)}$	$\frac{1}{v^2}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)$
$\frac{1}{v(x)}$	$-\frac{1}{v^2}\frac{dv}{dx}$
$y(u(x))$	$\frac{dy}{du}\frac{du}{dx}$
$y(v(u(x)))$	$\frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}$

E Tables of indefinite and definite integrals

$f(x)$	$\int f(x)\,dx$ (C is an arbitrary constant.)
x^m $(m \neq -1)$	$\dfrac{1}{m+1}x^{m+1} + C$
x^{-1}	$\ln\lvert x\rvert + C$, or $\ln\lvert Cx\rvert$
e^{ax}	$(1/a)\,e^{ax} + C$
k^x $(k > 0)$	$k^x/\ln k + C$
$\ln x$ $(x > 0)$	$x\ln x - x + C$
$\sin ax$	$-(1/a)\cos ax + C$
$\cos ax$	$(1/a)\sin ax + C$
$\tan ax$	$-(1/a)\ln\lvert\cos ax\rvert + C$ or $-(1/a)\ln\lvert C\cos ax\rvert$
$\cot ax$	$(1/a)\ln\lvert\sin ax\rvert + C$ or $(1/a)\ln\lvert C\sin ax\rvert$
$\sec ax$	$-(1/2a)\ln[(1-\sin ax)/(1+\sin ax)] + C$
$\operatorname{cosec} ax$	$(1/2a)\ln[(1-\cos ax)/(1+\cos ax)] + C$
$\arcsin ax$	$(1/a)(1-a^2x^2)^{\frac{1}{2}} + x\arcsin ax + C$
$\arccos ax$	$-(1/a)(1-a^2x^2)^{\frac{1}{2}} + x\arccos ax + C$
$\arctan ax$	$-(1/a)\ln(1-a^2x^2)^{\frac{1}{2}} + x\arctan ax + C$
$\sinh ax$	$(1/a)\cosh ax + C$
$\cosh ax$	$(1/a)\sinh ax + C$
$\tanh ax$	$(1/a)\ln\{\cosh ax\} + C$
$1/(x^2+a^2)$	$(1/a)\arctan(x/a) + C$
$1/(x^2-a^2)$	$(1/2a)\ln\lvert(x-a)/(x+a)\rvert + C$ or $(1/a)\tanh^{-1}(x/a) + C$
$1/(a^2-x^2)^{\frac{1}{2}}$	$\arcsin(x/a) + C$ (or $-\arccos(x/a) + C$)
$1/(a^2+x^2)^{\frac{1}{2}}$	$(1/a)\sinh^{-1}(x/a) + C$ or $\ln[x + (x^2+a^2)^{\frac{1}{2}}] + C$
$1/(x^2-a^2)^{\frac{1}{2}}$	$\ln[x + (x^2-a^2)^{\frac{1}{2}}] + C$
$x\,e^{ax}$	$(1/a^2)(ax-1)\,e^{ax} + C$
$x\cos ax$	$(1/a^2)(\cos ax + ax\sin ax) + C$
$x\sin ax$	$(1/a^2)(\sin ax - ax\cos ax) + C$
$x\ln x$	$\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$
$e^{ax}\cos bx$	$[1/(a^2+b^2)]\,e^{ax}(a\cos bx + b\sin bx) + C$
$e^{ax}\cos bx$	$[1/(a^2+b^2)]\,e^{ax}(-b\cos bx + a\sin bx) + C$

A table of definite integrals

$$\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}$$

$$\int_0^{\frac{1}{2}\pi} \sin x \, dx = \int_0^{\frac{1}{2}\pi} \cos x \, dx = 1$$

$$\int_0^\pi \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}\pi, & m = n \end{cases} \quad (m, n \text{ positive integers})$$

$$\int_0^\pi \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}\pi, & m = n \end{cases} \quad (m, n \text{ positive integers})$$

$$\int_0^\pi \sin mx \cos nx \, dx = \begin{cases} 0, & m+n \text{ even} \\ 2m/(m^2-n^2), & m+n \text{ odd} \end{cases} \quad (m, n \text{ positive integers})$$

$$\int_0^\infty x^n e^{-x} \, dx = n! \quad (n = 0, 1, 2, \ldots)$$

$$\int_0^\infty e^{-ax^2} \, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a > 0)$$

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2} \quad (a > 0)$$

$$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2} \quad (a > 0)$$

Gradshteyn and Ryzhik (1994) is a useful source of hundreds of indefinite and definite integrals.

F Laplace transforms, inverses, and rules

In the following tables, n and m represent a positive integer or zero. The constants k and c are arbitrary unless otherwise indicated.

Transforms

$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{kt}	$\dfrac{1}{s-k}$
$t^n e^{kt}$	$\dfrac{n!}{(s-k)^{n+1}}$
$\cos kt$	$\dfrac{s}{s^2+k^2}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
$t \cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
$t \sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
$H(t-c)$ $(c>0)$	e^{-cs}/s
$\delta(t-c)$ $(c>0)$	e^{-cs}

Inverses

$F(s)$	$f(t)$
$\dfrac{1}{s^m}$	$\dfrac{1}{(m-1)!} t^{m-1}$
$\dfrac{1}{s-k}$	e^{kt}
$\dfrac{1}{(s-k)^m}$	$\dfrac{1}{(m-1)!} t^{m-1} e^{kt}$
$\dfrac{s}{s^2+k^2}$	$\cos kt$
$\dfrac{1}{s^2+k^2}$	$\dfrac{1}{k} \sin kt$
$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	$t \cos kt$
$\dfrac{s}{(s^2+k^2)^2}$	$\dfrac{1}{2k} t \sin kt$
e^{-cs}/s $(c>0)$	$H(t-c)$
e^{-cs} $(c>0)$	$\delta(t-c)$

Summary of rules: In the following rules, $F(s) \leftrightarrow f(t)$.

Scale rule (24.5) $\quad f(kt) \leftrightarrow \dfrac{1}{k} F\left(\dfrac{s}{k}\right)$ and $F(ks) \leftrightarrow \dfrac{1}{k} f\left(\dfrac{t}{k}\right)$ $(k>0)$.

Shift rule, or multiplication by e^{kt} (24.7) $\quad$ If k is any constant, $e^{kt} f(t) \leftrightarrow F(s-k)$.

Powers of t (24.8) $\quad$ If n is a positive integer, then $t^n f(t) \leftrightarrow (-1)^n \dfrac{d^n F(s)}{ds^n}$.

Derivatives (24.12) $\quad \dfrac{df(t)}{dt} \leftrightarrow sF(s) - f(0), \quad \dfrac{d^2 f(t)}{dt^2} \leftrightarrow s^2 F(s) - sf(0) - f'(0)$.

Delay rule (24.15) $\quad$ If $c>0$, then $e^{-cs} F(s) \leftrightarrow f(t-c) H(t-c)$ (where H is the Heaviside unit function).

1/s as an integration operator (25.1) $\quad$ If $F(s) \leftrightarrow f(t)$, then $\dfrac{1}{s} F(s) \leftrightarrow \int_0^t f(\tau)\,d\tau$.

Convolution theorem (25.11) $\quad$ If $g(t) \leftrightarrow G(s)$ and $f(t) \leftrightarrow F(s)$, then

$$F(s)G(s) \leftrightarrow \int_0^t g(t-\tau) f(\tau)\,d\tau \left(= \int_0^t g(\tau) f(t-\tau)\,d\tau \right).$$

G Exponential Fourier transforms and rules

	Signal (time function)	Transform (frequency distribution)
Fourier transform pair	$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi ft}\, df$	$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi ft}\, dt$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(f) + BX_2(f)$
Time scaling	$x(At)$	$\lvert A\rvert^{-1}X(A^{-1}f)$
Time reversal	$x(-t)$	$X(-f)$
Time delay	$x(t-B)$	$X(f)\, e^{-i2\pi Bf}$
Frequency scaling	$\lvert C\rvert^{-1}x(C^{-1}t)$	$X(Cf)$
Frequency shift	$x(t)\, e^{i2\pi Dt}$	$X(f-D)$
Modualtion	$x(t)\cos 2\pi Kt$ $x(t)\sin 2\pi Kt$	$\frac{1}{2}[X(f+K) + X(f-K)]$ $\frac{1}{2}i[X(f+K) - X(f-K)]$
Differentiation	$dx(t)/dt$ $d^n x(t)/dt^n$	$(i2\pi f)X(f)$ $(i2\pi f)^n X(f)$
Duality	$X(t)$	$x(-f)$
Convolution	$\int_{-\infty}^{\infty} x_1(u)x_2(t-u)\, du = x_1(t) * x_2(t)$ $= \int_{-\infty}^{\infty} x_1(t-u)x_2(u)\, du$	$X_1(f)X_2(f)$
Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f-v)X_2(v)\, dv$ $= \int_{-\infty}^{\infty} X_2(v)X_2(f-v)\, dv$
Periodic function $x_P(t)$	$x_P(t)$ (period T)	$\sum_{n=-\infty}^{\infty} X_n\, \delta(f - nf_0)$, where $f_0 = 1/T$, $X_n = f_0 \int_{\text{Period}} x_P(t)\, e^{-2\pi i f_0 t}\, dt$

Short table of Fourier transforms

Signal	Transform	Signal	Transform
$\Pi(t) = H(t-\frac{1}{2}) - H(t+\frac{1}{2})$	$\operatorname{sinc} f$	$\frac{1}{1+t^2}$	$\pi\, e^{-2\pi\lvert f\rvert}$
$\operatorname{sinc} t$	$\Pi(f)$	$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\Lambda(t) = \begin{cases} 1+t, & -1<t<0 \\ 1-t, & 0<t<1 \\ 0, & \text{elsewhere} \end{cases}$	$\operatorname{sinc}^2 f$	$\delta(t)$	1
		1	$\delta(f)$
$\operatorname{sinc}^2 t$	$\Lambda(f)$	$\cos 2\pi f_0 t$	$\frac{1}{2}[\delta(f+f_0) + \delta(f-f_0)]$
$e^{-t}H(t)$	$1/(1+i2\pi f)$	$\sin 2\pi f_0 t$	$\frac{1}{2}i[\delta(f+f_0) - \delta(f-f_0)]$
$t\, e^{-t}H(t)$	$1/(1+i2\pi f)^2$	$III_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)\ (T>0)$	$f_0 III_{f_0}(f) \quad (f_0 = 1/T)$
$e^{-\lvert t\rvert}$	$2/(1+4\pi^2 f^2)$		

H Probability distributions and tables

(a) Distributions, means, and variances

(i) Discrete distributions

Distribution	Probability	Mean (μ)	Variance (σ^2)
Binomial	$\frac{n!p^r q^{n-r}}{(n-r)!r!}$	np	$np(1-p)$
Geometric	$(1-p)^{r-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$\frac{\lambda^n e^{-\lambda}}{n!}$	λ	λ
Pascal	${}_{r-1}C_{k-1}p^k(1-p)^{r-k}$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
Hypergeometric	$\frac{{}_wC_r\,{}_bC_{n-r}}{{}_{w+b}C_n}$	$\frac{nb}{w+b}$	$\frac{nwb(b+w+n)}{(w+b)^2(w+b-1)}$

(ii) Continuous distributions

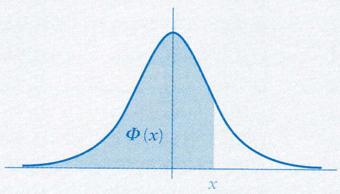
Distribution	Density	Mean (μ)	Variance (σ^2)
Exponential	$\begin{cases}\lambda e^{-\lambda x}, & x \geqslant 0\\ 0, & x < 0\end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Uniform	$\begin{cases}1/(b-a), & a < x < b\\ 0, & \text{elsewhere}\end{cases}$	$\frac{1}{2}(a+b)^2$	$\frac{1}{12}(b-a)^2$
Standardized normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$	0	1

(b) Cumulative normal distribution tables

Standardized cumulative normal distribution giving the values of

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt$$

for $0 \leqslant x \leqslant 3.0$ at 0.01 intervals. For $x < 0$, $\Phi(x)$ can be calculated from $\Phi(-x) = 1 - \Phi(x)$.



x	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.0633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9137	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table giving x for specified values of $\Phi(x)$ for $0.50 \leqslant \Phi(x) \leqslant 0.99$ at 0.01 intervals

$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x
0.50	0.0000	0.67	0.4399	0.84	0.9945
0.51	0.0251	0.68	0.4677	0.85	1.0364
0.52	0.0502	0.69	0.4959	0.86	1.0803
0.53	0.0753	0.70	0.5244	0.87	1.1264
0.54	0.1004	0.71	0.5534	0.88	1.1750
0.55	0.1257	0.72	0.5828	0.89	1.2265
0.56	0.1510	0.73	0.6138	0.90	1.2816
0.57	0.1764	0.74	0.6433	0.91	1.3408
0.58	0.2019	0.75	0.6745	0.92	1.4051
0.59	0.2275	0.76	0.7063	0.93	1.4758
0.60	0.2533	0.77	0.7388	0.94	1.5548
0.61	0.2793	0.78	0.7722	0.95	1.6449
0.62	0.3055	0.79	0.8064	0.96	1.7507
0.63	0.3319	0.80	0.8416	0.97	1.8808
0.64	0.3585	0.81	0.8779	0.98	2.0537
0.65	0.3853	0.82	0.9154	0.99	2.3263
0.66	0.4125	0.83	0.9542		

I Dimensions and units

Physical quantities of different types, such as acceleration, force, momentum, electrical potential, can be classified by expressing them as simple combinations of certain primary dimensions such as mass, length and time. These expressions determine how we can state the magnitude of a physical quantity – for example any velocity can be expressed in metres per second, but never in metres per kilogram. Five primary dimensions provide a basis sufficient for all common purposes. Their names, the algebraic symbols denoting their dimension, and appropriate units (the international (SI) system) are shown in the following table.

Basic quantity	Dimension symbol	SI unit	Unit symbol
length	**L**	metre	m
mass	**M**	kilogram	kg
time	**T**	second	s
electric current	**I**	ampere	A
absolute temperature	$\boldsymbol{\theta}$	Kelvin	K

We can now assign dimensions to any *derived* physical quantity, which classifies it without indicating its magnitude. For example, the velocity at any moment of any particle, substance, electromagnetic wave, etc., could in principle be measured as (*a distance travelled*)/(*time taken*). Symbolically we write:

$$[\text{velocity}] = \mathbf{LT^{-1}},$$

where the square brackets mean 'the dimension of'. The form $\mathbf{LT^{-1}}$ of the right-hand side indicates that an appropriate SI unit of measurement would be metres per second. The following table comprises mechanical and electromagnetic quantities, their dimensions, and conventional SI terms for certain special units of measurement. Notice how known dimensional forms may be multiplied and divided to obtain more complicated ones.

Derived quantity	Dimension	SI unit	Symbol	Name
angle	dimensionless	–	rad	radian
velocity (displacement/unit time)	$\mathbf{LT^{-1}}$	m s^{-1}	–	–
acceleration (velocity/unit time)	$\mathbf{LT^{-2}}$	m s^{-2}	–	–
force (mass × acceleration)	$\mathbf{MLT^{-2}}$	kg m s^{-2}	N	newton
momentum (mass × velocity)	$\mathbf{MLT^{-1}}$	kg m s^{-1}	–	–
moment of momentum	$\mathbf{ML^2T^{-1}}$	kg m^2 s^{-1}	–	–
pressure (force/unit area)	$\mathbf{ML^{-1}T^{-2}}$	N m^{-2}	Pa	pascal
work, energy (force × distance)	$\mathbf{ML^2T^{-2}}$	N m	J	joule
area	$\mathbf{L^2}$	m^2	–	–
volume	$\mathbf{L^3}$	m^3	–	–
power (work/unit time)	$\mathbf{ML^2T^{-3}}$	J s^{-1}	W	watt
angular frequency (radian/unit time)	$\mathbf{T^{-1}}$	s^{-1}	Hz	hertz
charge (current × time)	$\mathbf{IT}$	A s	C	coulomb
potential (work/unit charge)	$\mathbf{ML^2T^3I^{-1}}$	J C^{-1}	V	volt
resistance (potential/unit current)	$\mathbf{ML^2T^3I^{-2}}$	V A^{-1}	$\boldsymbol{\Omega}$	ohm
magnetic flux (work/unit current)	$\mathbf{ML^2T^{-2}I^{-1}}$	J A^{-1}	Wb	weber
inductance (magnetic flux/unit current)	$\mathbf{ML^2T^{-2}I^{-2}}$	Wb A^{-1}	H	henry

For comprehensive tables of units and constants, consult Kaye and Laby (1995).

If two physically meaningful expressions are equal, then both sides must obviously have the same physical dimensions. This often provides a useful check on a calculation. Also, in any expression containing the sum of two or more terms, the terms must all have the same dimensions if it is to make any physical sense. For example, expressions equivalent to the form (*energy* + *momentum*), or (*current* + *voltage*) can have no physical significance. However, in such cases the dimensions of any letters used as *constant factors* must not be overlooked: the expression (*momentum* + (k × *energy*)) could be meaningful provided $[k] = \mathbf{TL}^{-1}$.

The dimensions of quantities that appear as derivatives and integrals are treated in the following way. Suppose for example that t is time ($[t] = \mathbf{T}$) and $x(t)$ is a function representing displacement ($[x] = \mathbf{L}$). Then

$$\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = \mathbf{LT}^{-1}, \quad \left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = \mathbf{LT}^{-2},$$

and so on. Also

$$\left[\int_a^b x(t)\,\mathrm{d}t\right] = \mathbf{LT}, \quad \text{and} \quad \left[\int_a^d t\,\mathrm{d}t\right] = \mathbf{T}^2.$$

These follow from the definition of the integral as a sum.

Dimensional analysis is helpful in checking the validity of equations. For example, in the pendulum equation (see eqn 20.22)

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0$$

all terms should have the same dimensions, which is true since

$$\left[\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right] = \mathbf{T}^{-2} \quad \text{and} \quad \left[\frac{g}{l}\sin\theta\right] = \mathbf{LT}^{-2}\mathbf{L}^{-1} = \mathbf{T}^{-2},$$

where g is the acceleration due to gravity, l is the length of the pendulum, and the angle θ and $\sin\theta$ are dimensionless. Physically the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l^2}\sin\theta = 0,$$

with the same definition of symbols could not represent a general physical law because the dimensions of the two terms are different.

Dimensionless analysis indicates how equations can be simplified by making them dimensionless. In the pendulum equation above, let $\tau = t\sqrt{(g/l)}$. Then the dimensionless pendulum equation becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} + \sin\theta = 0$$

which includes pendulums of all lengths, in any uniform gravitational field.

Further reading

Abell, M.L. and Braselton, J.P. (1992) *Mathematica by Example*, Academic Press, San Diego.

Blachman, N. (1992) *Mathematica: A Practical Approach*, Academic Press, San Diego.

Boyce, W.E. and DiPrima, R.C. (1997) *Elementary Differential Equations and Boundary Value Problems* (6th edn), Wiley, New York.

Garnier, R. and Taylor, J. (1991) *Discrete Mathematics for New Technology*, Adam Hilger, Bristol.

Gradshteyn, I.S. and Ryzhik, I.M. (1994) *Table of Integrals, Series, and Products* (5th edn), Academic Press, San Diego.

Grimmett, G.R. and Stirzaker, D.R. (2001), *Probability and Random Processes* (3rd edn), Oxford University Press.

Jordan, D.W. and Smith, P. (2007a) *Nonlinear Ordinary Differential Equations* (4th edn), Oxford University Press.

Jordan, D.W. and Smith, P. (2007b) *Nonlinear Ordinary Differential Equations: Problems and Solutions*, Oxford University Press.

Kaye and Laby (1995) *Tables of Physical and Chemical Constants*, National Physical Laboratory (16th edn) (available online at www.kayelaby.npl.co.uk).

Montgomery, D.C. and Runger, G.C. (1994) *Applied Statistics and Probability for Engineers*, Wiley, New York.

Råde, L. and Westergren, B. (1995) *Mathematics Handbook for Physics and Engineering*, Studentlitteratur, Lund.

Riley, K.F., Hobson, M.P. and Bence, S.J. (1997) *Mathematical Methods for Physics and Engineering*, Cambridge University Press.

Roberts, G.E. and Kaufman, H. (1966) *Table of Laplace Transforms*, Saunders, Philadelphia.

Seggern, D.H. von (1990) *CRC Handbook of Mathematical Curves and Surfaces*, CRC Press, Baton Roca.

Skeel, R.D. and Keeper, J.B. (1993) *Elementary Numerical Computing with Mathematica*, McGraw-Hill, New York.

Whitelaw, T.A. (1983) *An Introduction to Linear Algebra*, Blackie, Glasgow.

Wilson, R.J. and Watkins, J.J. (1990) *Graphs: An Introductory Approach*, Wiley, New York.

Wolfram, S. (1996) *The Mathematica Book* (3rd edn), Wolfram Media/Cambridge University Press.

Zwillinger, D. (1992) *Handbook of Differentials Equations* (2nd edn), Academic Press, Boston.

Index

Pages of the main topics are given in heavy type for quick reference.

C

D

E

F

J

K

L

M

Q

R

T

U

V

W

Z

FEEDBACK ON THE PREVIOUS EDITION:

'This textbook offers an accessible and comprehensive grounding in many of the mathematical techniques required in the early stages of an engineering or science degree and also for the routine methods needed by first and second year mathematics students.'

ENGINEERING DESIGNER, 2003

'The authors do not attempt to dodge theoretical hurdles. They are careful to explain many of the less intuitive properties of functions and to highlight generalizations without becoming over abstract... There are significant changes in content in the opening chapter, where the foundation material has been expanded usefully.'

TIMES HIGHER EDUCATION SUPPLEMENT, NOVEMBER 2002

'Thoroughly recommended.'

ZENTRALBLATT MATH, 2002

***Mathematical Techniques* covers all the topics required by a science, engineering or mathematics student in the early stages of their degree. Assuming a minimum of prior knowledge, basic subjects are covered from scratch or topics are revised anew, giving a firm foundation for the topics that follow.**

With a huge array of end of chapter problems, and new self-check questions, the fourth edition of *Mathematical Techniques* provides extensive opportunities for students to build their confidence in the best way possible: by using the maths for themselves.

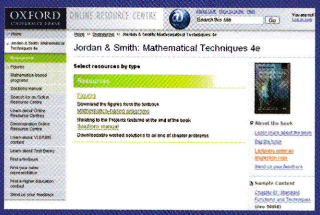
- Short, modular chapters, a large index, and careful cross-referencing make the book flexible enough to be used on a wide variety of courses.
- Over 500 fully-worked Examples show how the theory is applied and offer valuable guidance for the reader when tackling the problems.
- Self-check questions and over 2000 end of chapter problems provide extensive opportunities for students to practise using the concepts presented.
- Emphasis on techniques and applications helps readers to move through the subject without being hampered by formal analytic proofs.
- A series of Projects at the end of the book encourages students to use mathematical software to further exploit and illustrate the concepts covered.

NEW TO THIS EDITION:

- Each chapter opens with a new introduction, which explains the content and aim of the chapter, and places it in context for the student.
- The whole text has been reviewed, and colour introduced, in order to improve clarity and to facilitate navigation through the text.
- New self-check questions appear at the end of most Sections. These supplement the end of chapter problems, and provide the opportunity to check progress.
- A new Appendix covers Dimensional Analysis and Units.
- Topics revised in this edition include conic sections, linear dependence, integration, nonlinear differential equations, stationary values, vector fields and Stokes's theorem, and difference equations.

Cover illustration: Simon Witter

Online Resource Centre
www.oxfordtextbooks.co.uk/orc/jordan_smith4e



The Online Resource Centre features the following resources for all users of the book:

- Figures from the book in electronic format, ready to download.
- A downloadable solutions manual, featuring worked solutions to all end of chapter problems.
- Mathematica-based programs, relating to the Projects featured at the end of the book.

OXFORD
UNIVERSITY PRESS

www.oup.com

ISBN 978-0-19-928201-2

9 780199 282012