EXAMPLE 1 Geometric Interpretation. Existence and Uniqueness of Solutions

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

If we interpret x_1, x_2 as coordinates in the x_1x_2-plane, then each of the two equations represents a straight line, and (x_1, x_2) is a solution if and only if the point P with coordinates x_1, x_2 lies on both lines. Hence there are three possible cases:

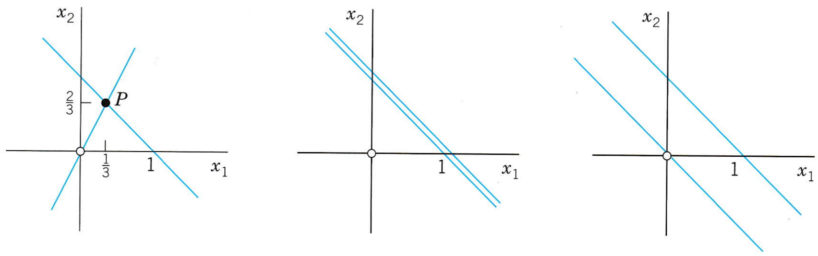
(a) Precisely one solution if the lines intersect.

(b) Infinitely many solutions if the lines coincide.

(c) No solution if the lines are parallel

For instance,

$x_1 + x_2 = 1$	$x_1 + x_2 = 1$	$x_1 + x_2 = 1$
$2x_1 - x_2 = 0$	$2x_1 + 2x_2 = 2$	$x_1 + x_2 = 0$
Case (a)	Case (b)	Case (c)



If the system is homogenous, Case (c) cannot happen, because then those two straight lines pass through the origin, whose coordinates 0, 0 constitute the trivial solution. If you wish, consider three equations in three unknowns as representations of three planes in space and discuss the various possible cases in a similar fashion. See Fig. 156. ■

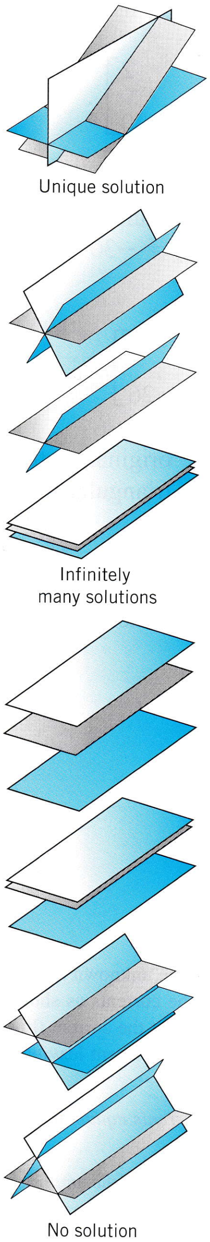


Fig. 156. Three equations in three unknowns interpreted as planes in space

Our simple example illustrates that a system (1) may perhaps have no solution. This poses the following problem. Does a given system (1) have a solution? Under what conditions does it have precisely one solution? If it has more than one solution, how can we characterize the set of all solutions? How can we actually obtain the solutions? Perhaps the last question is the most immediate one from a practical viewpoint. We shall answer it first and discuss the other questions in Sec. 7.5.

Gauss Elimination and Back Substitution

This is a standard elimination method for solving linear systems that proceeds systematically irrespective of particular features of the coefficients. It is a method of great practical importance and is reasonable with respect to computing time and storage demand (two aspects we shall consider in Sec. 20.1 in the chapter on numeric linear algebra). We begin by motivating the method. If a system is in **"triangular form,"** say,

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ 13x_2 &= -26 \end{aligned}$$

we can solve it by **"back substitution,"** that is, solve the last equation for the variable, $x_2 = -26/13 = -2$, and then work backward, substituting $x_2 = -2$ into the first equation