Příklad 1

Jsou dány dvě různé báze α, β vektorového prostoru $\mathbb{R}^3$. Najděte matice přechodu $P_{\beta,\alpha}, P_{\alpha,\beta}$ a určete souřadnice vektoru $\vec{u}_\alpha = (1,2,1)$ v bázi β a souřadnice vektoru $\vec{v}_\beta = (-1,0,3)$ v bázi α.

1. $\alpha = ((1,0,1);(2,1,1);(0,0,2))$,
   $\beta = ((0,1,1);(1,0,2);(2,0,2))$.

$$P_{\beta,\alpha}: \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 & 1 & 2 \end{array}\right) \sim \begin{array}{c} \\ \\ -r_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 2 & 1 & 1 & 2 \end{array}\right) \sim \begin{array}{c} \\ \\ -2r_2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 2 \end{array}\right) \sim$$

$$\sim \begin{array}{c} \\ +r_3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -4 & 2 \end{array}\right) \sim \begin{array}{c} \\ \\ :(-2) \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & -1 & -4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 1/2 & 2 & -1 \end{array}\right)$$

$$P_{\beta,\alpha} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 1/2 & 2 & -1 \end{pmatrix} \rightarrow \vec{u}_\beta = P_{\beta,\alpha} \cdot \vec{u}_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 1/2 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 7/2 \end{pmatrix}_\beta$$

$$P_{\alpha,\beta}: \begin{array}{c} \\ \\ -r_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 2 & 2 \end{array}\right) \sim \begin{array}{c} \\ \\ +r_2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \end{array}\right) \sim \begin{array}{c} -2r_2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 \end{array}\right) \sim$$

$$\begin{array}{c} \\ \\ :2 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & 0 \end{array}\right)$$

$$P_{\alpha,\beta} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1/2 & 0 \end{pmatrix} \rightarrow \vec{v}_\alpha = P_{\alpha,\beta} \cdot \vec{v}_\beta = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1/2 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}_\alpha$$

Příklad 2

Lineární zobrazení $\varphi: U \to V$ je zadáno maticí A_S ve standardních bázích U, V. Pro zadané báze α prostoru U a β prostoru V určete matice $A_{S,\alpha}, A_{\beta,S}, A_{\beta,\alpha}$.

1. $\varphi: \mathbb{R}^2 \to \mathbb{R}^3$, $A_S = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$,
   $\alpha = ((1,2);(-2,1))$, $\beta = ((1,1,1);(1,1,0);(1,2,0))$.

$$\mathbb{R}^3 \xleftarrow{P_{\beta,S}} \mathbb{R}^3 \leftarrow \varphi \leftarrow \mathbb{R}^2 \xleftarrow{P_{S,\alpha}} \mathbb{R}^2$$

$$\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_\beta \left[\varphi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]_S = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_S}_{A_{\beta,S}} \quad \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_S \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_\alpha}_{A_{S,\alpha}}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{A_{\beta,\alpha}}$$

$$P_{S,\alpha}: \left(\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right) \rightarrow P_{S,\alpha} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$P_{\beta,S}: \begin{array}{c} \\ -r_1 \\ -r_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right) \sim \begin{array}{c} -r_3 \\ +r_3 \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\sim \begin{array}{c} +r_2 \\ \\ \\ \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \begin{array}{c} \\ \cdot(-1) \\ \\ \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$P_{\beta,S} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$A_{S,\alpha} = A_S \cdot P_{S,\alpha} \cdot (u_\alpha) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$A_{\beta,S} = P_{\beta,S} \cdot A_S \cdot (\vec{v}_S) = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 5 & 0 \\ -2 & 0 \end{pmatrix}$$

$$A_{\beta,\alpha} = \underbrace{P_{\beta,S} \cdot A_S}_{A_{\beta,S}} \cdot P_{S,\alpha} \quad \left(A_S \cdot P_{S,\alpha} = A_{S,\alpha}\right) = \underbrace{\begin{pmatrix} -1 & 1 \\ 5 & 0 \\ -2 & 0 \end{pmatrix}}_{A_{\beta,S}} \cdot \underbrace{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}_{P_{S,\alpha}} = \begin{pmatrix} 1 & 3 \\ 5 & -10 \\ -2 & 4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}}_{P_{\beta,S}} \cdot \underbrace{\begin{pmatrix} 4 & -3 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}}_{A_{S,\alpha}} = \begin{pmatrix} 1 & 3 \\ 5 & -10 \\ -2 & 4 \end{pmatrix}$$

Příklad 3

Lineární transformace $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ je zadána maticí A_S ve standardní bázi prostoru $\mathbb{R}^3$. Pro bázi

$$\alpha = ((1,1,1);(1,1,0);(1,2,0))$$

prostoru $\mathbb{R}^3$ určete matice $A_{S,\alpha}, A_{\alpha,S}, A_{\alpha,\alpha}$.

1. $A_S = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix}$.

$$A_{\alpha,\alpha} = P_{\alpha,S} \cdot A_S \cdot P_{S,\alpha}$$

$$\varphi\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_\alpha = \begin{pmatrix} ? \end{pmatrix}_\alpha$$

$$P_{S,\alpha} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_{\alpha,S}: \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\text{Příklad 2.1}}{\overset{\text{viz. } P_{B,S}}{\sim \cdots \sim}} \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$A_{S,\alpha} = A_S \cdot P_{S,\alpha} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 1 & 0 \\ 9 & 5 & 6 \end{pmatrix}$$

$$A_{\alpha,S} = P_{\alpha,S} \cdot A_S = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 4 \\ -4 & 2 & -2 \\ 1 & -2 & 0 \end{pmatrix}$$

$$A_{\alpha,\alpha} = P_{\alpha,S} \cdot A_S \cdot P_{S,\alpha} = \begin{pmatrix} 4 & 1 & 4 \\ -4 & 2 & -2 \\ 1 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 5 & 6 \\ -4 & -2 & 0 \\ -1 & -1 & -3 \end{pmatrix}$$

$$\varphi\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_\alpha = A_{\alpha\alpha} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 & 5 & 6 \\ -4 & -2 & 0 \\ -1 & -1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}_\alpha$$

Příklad 4

Lineární transformace φ vektorového prostoru $\mathbb{R}^2$ je dána maticí A ve standardní bázi. Nalezněte vlastní čísla a jim odpovídající vlastní vektory lineární transformace φ.

a) $A = \begin{pmatrix} 2 & 6 \\ 6 & -3 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 6 \\ 6 & -3-\lambda \end{vmatrix} = (2-\lambda)\cdot(-3-\lambda) - 6 \cdot 6 = -6 - 2\lambda + 3\lambda + \lambda^2 - 36$$

$$= \lambda^2 + \lambda - 42 = 0$$

$$(\lambda + 7)(\lambda - 6) = 0 \Rightarrow \left.\begin{matrix} \lambda_1 = -7 \\ \lambda_2 = 6 \end{matrix}\right\} \text{vlastní čísla}$$

$\lambda_1 = -7$: $\vec{v}_1 = ?$

$$\left(\begin{array}{cc|c} 2-(-7) & 6 & 0 \\ 6 & -3-(-7) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 9 & 6 & 0 \\ 6 & 4 & 0 \end{array}\right)\begin{matrix} :3 \\ :2 \end{matrix} \sim \begin{array}{c} \begin{matrix} v_{11} & v_{12} & \end{matrix} \\ \left(\begin{array}{cc|c} 3 & 2 & 0 \\ \cancel{3} & \cancel{2} & \cancel{0} \end{array}\right) \end{array}$$

$$\left\{\begin{matrix} v_{12} = t \\ 3 \cdot v_{11} + 2t = 0 \\ v_{11} = -\frac{2}{3}t \end{matrix}\right.$$

$K = \{t \cdot (-\frac{2}{3}, 1), t \in \mathbb{R}\}$

$\vec{v}_1 = (-\frac{2}{3}, 1)$

$\lambda_2 = 6$: $\vec{v}_2 = ?$

$$\left(\begin{array}{cc|c} 2-6 & 6 & 0 \\ 6 & -3-6 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -4 & 6 & 0 \\ 6 & -9 & 0 \end{array}\right)\begin{matrix} :(-2) \\ :(3) \end{matrix} \sim \begin{array}{c} \begin{matrix} v_{21} & v_{22} & \end{matrix} \\ \left(\begin{array}{cc|c} 2 & -3 & 0 \\ \cancel{2} & \cancel{-3} & \cancel{0} \end{array}\right) \end{array}$$

$$\left\{\begin{matrix} v_{22} = t \\ 2 v_{21} - 3t = 0 \\ v_{21} = \frac{3}{2}t \end{matrix}\right.$$

$K = \{t \cdot (\frac{3}{2}, 1), t \in \mathbb{R}\}$

$\vec{v}_2 = (\frac{3}{2}, 1)$