Examples of tests

Brno 2007

1 Tests on parameters of normal distribution

Let $X_1, \dots, X_n$ be a sample from $\mathcal{N}(\xi, \sigma^2)$ with the density

$$(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{n\xi^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2}\sum x_i^2 + \frac{\xi}{\sigma^2}\sum x_i\right).$$

1.1 Tests on σ^2

Denote

$$\theta = -\frac{1}{2\sigma^2}, \quad \vartheta = \frac{n\xi}{\sigma^2}, \quad U(x) = \sum x_i^2,\ T(x) = \bar{x}.$$

Then the statistic

$$V = \sum (X_i - \bar{X})^2 = U - nT^2$$

is ancillary for θ and is increasing in U. Then $\mathbf{H}_1 : \sigma \le \sigma_0$ is equivalent to $\theta \le \theta_0$, and the UMP unbiased test rejects $\mathbf{H}_1$ if $\sum (X_i - \bar{X})^2 \ge C_\alpha$, where

$$\int_{C/\sigma_0^2}^{\infty} \chi^2_{n-1}(y)dy = \alpha.$$

Consider the hypothesis $\mathbf{H}_2 : \sigma = \sigma_0$. Because V is linear in U, the UMP unbiased test rejects $\mathbf{H}_2$ provided

$$\frac{\sum (x_i - \bar{x})^2}{\sigma_0^2} \le C_1 \quad \text{or} \quad \ge C_2$$

where the constants are determined so that

$$\int_{C_1}^{C_2} \chi^2_{n-1}(y)dy = \frac{1}{n-1}\int_{C_1}^{C_2} y\chi^2_{n-1}(y)dy = 1 - \alpha.$$

On the other hand,

$$\frac{1}{n-1} y\chi^2_{n-1}(y) = \frac{1}{(n-1)2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} y^{\frac{n-1}{2}} \mathrm{e}^{-\frac{y}{2}} = \frac{1}{2^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)} y^{\frac{n+1}{2}-1} \mathrm{e}^{-\frac{y}{2}} = \chi^2_{n+1}(y),$$

hence the constants $C_1,\ C_2$ are determined by the system of equations

$$\int_{C_1}^{C_2} \chi^2_{n-1}(y)dy = \int_{C_1}^{C_2} \chi^2_{n+1}(y)dy = 1 - \alpha.$$

1.2 Tests on ξ

Consider the reparametrization

$$\theta = \frac{n\xi}{\sigma^2}, \quad \vartheta = -\frac{1}{2\sigma^2}, \quad U(x) = \bar{x},\ T(x) = \sum x_i^2.$$

Consider the hypothesis $\mathbf{H}_3 : \xi = \xi_0$ and without loss of generality assume that $\xi_0 = 0$ (otherwise we put $X_i' = X_i - \xi_0$). Then $\theta \leq 0$ is equivalent to $\xi \leq 0$ and

$$V = \frac{\bar{X}}{\sqrt{\sum (X_i - \bar{X})^2}} = \frac{U}{\sqrt{T - nU^2}}$$

is under $\xi = 0$ independent of T, because under $\xi = 0$ its multiple has t_{n-1} distribution, hence independent of the nuisance parameter σ. Thus the UMP unbiased test for $\mathbf{H}_3$ rejects provided $t(x) \geq C_0$, where

$$t(x) = \frac{\sqrt{n}\bar{x}}{\sqrt{\frac{1}{n-1}\sum (x_i - \bar{x})^2}}.$$

Consider the hypothesis $\mathbf{H}_4 : \xi = \xi_0 = 0$ and the statistic

$$W = \frac{\bar{X}}{\sqrt{\sum X_i^2}} = \frac{U}{\sqrt{T}}.$$

Then W is also independent of T under $\xi = 0$ and it is linear in $U = \bar{X}$. Under $\xi = 0$, the distribution of W is symmetric around 0. Thus, the UMP unbiased test of $\mathbf{H}_4$ would reject it for $|W| \geq C$, where $\mathbb{P}_{\xi=0}(W \geq C') = \alpha$. Because

$$t(x) = \frac{\sqrt{(n-1)n}W(x)}{\sqrt{1 - nW^2(x)}},$$

then $|t(x)|$ is an increasing function of $|W(x)|$, and the rejection region of the UMP unbiased test is equivalent to

$$|t(x)| \geq C$$

and

$$\int_C^\infty t_{n-1}(y)dy = \frac{\alpha}{2}, \qquad \int_{C_0}^\infty t_{n-1}(y)dy = \alpha.$$

The tests of more general hypotheses with $\xi_0 \neq 0$ would have the criterion

$$t(x) = \frac{\sqrt{n}(\bar{x} - \xi_0)}{\sqrt{\frac{1}{n-1}\sum (x_i - \bar{x})^2}}.$$

2 Comparing the variances of two normal distributions

Let $\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be samples from the normal distributions $\mathcal{N}(\xi, \sigma^2)$ and $\mathcal{N}(\eta, \tau^2)$. Then the joint density of $(\mathbf{X}, \mathbf{Y})$ is

$$C(\xi, \eta, \sigma, \tau)\exp\left(-\frac{1}{2\sigma^2}\sum x_i^2 - \frac{1}{2\tau^2}\sum y_j^2 + \frac{m\xi}{\sigma^2}\bar{x} + \frac{n\eta}{\tau^2}\bar{y}\right).$$

Consider the reparametrization

$$\theta = -\frac{1}{2\tau^2}, \quad \vartheta_1 = -\frac{1}{2\sigma^2}, \quad \vartheta_2 = \frac{n\eta}{\tau^2}, \quad \vartheta_3 = \frac{m\xi}{\sigma^2}$$

and the sufficient statistics

$$U = \sum Y_j^2, \quad T_1 = \sum X_i^2, \quad T_2 = \bar{Y}, \quad T_3 = \bar{X}.$$

We want to test the hypothesis $\mathbf{H}_5 : \tau^2 \leq \Delta_0 \sigma^2$. It is convenient still to reparametrize

$$\theta^* = -\frac{1}{\tau^2} + \frac{1}{2\Delta_0 \sigma^2}, \quad \vartheta_i^* = \vartheta_i, \quad i = 1, 2, 3$$

and to consider the statistics

$$U^* = \sum Y_j^2, \quad T_1^* = \sum X_i^2 + \frac{1}{\Delta_0} \sum Y_j^2, \quad T_2^* = \bar{Y}, \quad T_3 = \bar{X}.$$

The test is based on the statistic

$$V = \frac{\sum (Y_j - \bar{Y})^2 / \Delta_0}{\sum (X_i - \bar{X})^2} = \frac{\sum (Y_j - \bar{Y})^2 / \tau^2}{\sum (X_i - \bar{X})^2 / \tau^2}.$$

Under $\tau^2 = \Delta_0 \sigma^2$, the distribution of V does not depend on ξ, η, τ, σ, hence V is stochastically independent of T_1^*, T_2^*, T_3^*. The rejection region of the UMP unbiased test of $\mathbf{H}_5$ can be written as

$$\mathcal{F} = \frac{\sum (Y_j - \bar{Y})^2 / \Delta_0 (n-1)}{\sum (X_i - \bar{X})^2 / (m-1)} \geq C_0.$$

When $\tau^2 = \Delta_0 \sigma^2$, then $\mathcal{F}$ has the F-distribution with $n-1$ and $m-1$ degrees of freedom, thus

$$\int_{C_0}^{\infty} F_{n-1,m-1}(y) dy = \alpha.$$

If we want to test the hypothesis $\mathbf{H}_6 : \tau^2 = \Delta_0 \sigma^2$, we should consider the statistic

$$W = \frac{\sum (Y_j - \bar{Y})^2 / \Delta_0}{\sum (X_i - \bar{X})^2 + \frac{1}{\Delta_0} \sum (Y_j - \bar{Y})^2}.$$

Under $\tau^2 = \Delta_0 \sigma^2$, it is also independent of T_1^*, T_2^*, T_3^* and it is linear in U^*. Under $\tau^2 = \Delta_0 \sigma^2$, the distribution of W is the *beta-distribution* with the density

$$B_{\frac{n-1}{2}, \frac{m-1}{2}}(w) = \frac{\Gamma[\frac{m+n-2}{2}]}{\Gamma[\frac{m-1}{2}]\Gamma[\frac{n-1}{2}]} w^{(n-3)/2} (1-w)^{(m-3)/2}, \quad 0 < w < 1$$

and $\mathbb{E}W = \frac{n-1}{m+n-2}$. We reject $\mathbf{H}_6$ provided $W \leq C_1$ or $W \geq C_2$ and C_1, C_2 are determined through the relations

$$\int_{C_1}^{C_2} B_{\frac{n-1}{2}, \frac{m-1}{2}}(w) dw = \int_{C_1}^{C_2} B_{\frac{n+1}{2}, \frac{m-1}{2}}(w) dw = 1 - \alpha.$$

The relation between the beta and F-distribution is such that $W = \frac{Y}{1+Y}$, where $\frac{(m-1)Y}{n-1}$ has the distribution $F_{n-1,m-1}$.

3 Comparing the means of two normal distributions

If we want to compare ξ and η and the variances σ^2 are τ^2 are unknown and unequal, then it is the *Behrens-Fisher problem*, that cannot be treated in similar way. Thus suppose that $\sigma = \tau$. Then the joint density of $(\mathbf{X}, \mathbf{Y})$ is

$$C(\xi, \eta, \sigma) \exp \left[-\frac{1}{2\sigma^2} (\sum x_i^2 + \sum y_j^2) + \frac{\xi}{\sigma^2} \sum x_i + \frac{\eta}{\sigma^2} \sum y_j \right].$$

Reparametrization:
$$\theta = \frac{\eta}{\sigma^2}, \quad \vartheta_1 = \frac{\xi}{\sigma^2}, \quad \vartheta_2 = -\frac{1}{2\sigma^2},$$
the sufficient statistics:
$$U = \sum Y_j, \quad T_1 = \sum X_i, \quad T_2 = \sum X_i^2 + \sum Y_j^2.$$
Consider the hypothesis $\mathbf{H}_7 : \eta - \xi \leq 0$. The it is better to reparametrize
$$\theta^* = \frac{\eta - \xi}{\left(\frac{1}{m} + \frac{1}{n}\right)\sigma^2}, \quad \vartheta_1^* = \frac{m\xi + n\eta}{(m+n)\sigma^2}, \quad \vartheta_2^* = \vartheta_2$$
with the sufficient statistics
$$U^* = \bar{Y} - \bar{X}, \quad T_1^* = m\bar{X} + n\bar{Y}, \quad T_2^* = \sum X_i^2 + \sum Y_j^2.$$
When $\eta = \xi$, then we have the statistic
$$V = \frac{\bar{Y} - \bar{X}}{\sqrt{\sum(X_i - \bar{X})^2 + \sum(Y_j - \bar{Y})^2}} = \frac{U^*}{\sqrt{T_2^* - \frac{1}{m+n}T_1^{*2} - \frac{mn}{m+n}U^{*2}}}$$
that does not depend on $\xi = \eta$ and on σ, hence it is independent of T_1^*, T_2^*. Thus, we reject $\mathbf{H}_7$ provided $t(X,Y) \geq C_0$, where
$$t(X,Y) = \frac{\bar{Y} - \bar{X} \Big/ \sqrt{\frac{1}{m} + \frac{1}{n}}}{\sqrt{[\sum(X_i - \bar{X})^2 + \sum(Y_j - \bar{Y})^2]/(m+n-2)}}$$
and $\int_{C_0}^{\infty} t_{m+n-2}(y)dy = \alpha$.

For the hypothesis $\mathbf{H}_8 : \eta - \xi = 0$, we should consider the statistics (linear in U^* with coefficients dependent only on T^*, with distribution symmetric around 0)
$$W = \frac{\bar{Y} - \bar{X}}{\sqrt{\sum X_i^2 + \sum Y_j^2 - \frac{1}{m+n}(\sum X_i + \sum Y_j)^2}}$$
that is related to V through
$$V = \frac{W}{\sqrt{1 - \frac{mn}{m+n}W^2}}.$$
Hence, we reject $\mathbf{H}_8$ provided $|t(X,Y)| \geq C$, and $\int_C^{\infty} t_{m+n-2}(y)dy = \frac{\alpha}{2}$.

3.1 Test of independence in bivariate normal distribution

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample from the bivariate normal distribution with density
$$\frac{1}{\left(2\pi\sigma\tau\sqrt{1-\rho^2}\right)^n}\exp\Big[-\frac{1}{2(1-\rho^2)}\Big(\frac{1}{\sigma^2}\sum(x_i - \xi)^2 - \frac{2\rho}{\sigma\tau}\sum(x_i - \xi)(y_i - \eta) + \frac{1}{\tau^2}\sum(y_i - \eta)^2\Big)\Big].$$
Consider the hypotheses $\mathbf{H}_9 : \rho \leq 0$ and $\mathbf{H}_{10} : \rho = 0$.
Reparametrization:
$$\theta = \frac{\rho}{\sigma\tau(1-\rho^2)}, \quad \vartheta_1 = \frac{-1}{2\sigma^2(1-\rho^2)}, \quad \vartheta_2 = \frac{-1}{2\tau^2(1-\rho^2)}$$
$$\vartheta_3 = \frac{1}{1-\rho^2}\left(\frac{\xi}{\sigma^2} - \frac{\eta\rho}{\sigma\tau}\right), \quad \vartheta_4 = \frac{1}{1-\rho^2}\left(\frac{\eta}{tau^2} - \frac{\xi\rho}{\sigma\tau}\right),$$

and the sufficient statistics

$$U = \sum X_i Y_i, \quad T_1 = \sum X_i^2, \quad T_2 = \sum Y_i^2, \quad T_3 \sum X_i, \quad \sum Y_i.$$

$\mathbf{H}_9$ is equivalent to $\theta \le 0$. The ancillary statististic is the sample corellation coefficient

$$R = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum (X_i - \bar{X})^2 \sum (Y_i - \bar{Y})^2}},$$

because it is invariant to transformations $X_i \mapsto \frac{X_i - \xi}{\sigma}$ and $Y_i \mapsto \frac{Y_i - \eta}{\tau}$ and hence under $\rho = \theta = 0$ its distribution is independent of $\vartheta_1, \ldots, \vartheta_4$. It is nondecreasing in U. Thus the UMP unbiased test of $\mathbf{H}_9$ rejects when $R \ge C_0$ or when

$$\frac{R}{\sqrt{1 - R^2}} \sqrt{(n-2)} \ge K_0,$$

where K_0 is determined so that $\int_{K_0}^{\infty} t_{n-2}(y)dy = \alpha$. The statistic R is linear in U, hence the UMP test of $\mathbf{H}_{10}$ rejects when

$$\frac{|R|}{\sqrt{1 - R^2}} \sqrt{(n-2)} \ge K_1, \quad \text{where} \quad \int_{K_1}^{\infty} t_{n-2}(y)dy = \frac{\alpha}{2}.$$

3.2 Regression

We have observations $(Y_1, x_1), \ldots, (Y_n, x_n)$ and consider the simple regression $\mathbb{E}[Y|x] = \alpha + \beta x$. Start with the transformations

$$v_i = \frac{x_i - \bar{x}}{\sqrt{\sum (x_j - \bar{x})^2}}$$

and denote $\gamma + \delta v_i = \alpha + \beta x_i$, thus $\sum v_i = 0, \quad \sum v_i^2 = 1$. Then

$$\alpha = \gamma - \delta \frac{\bar{x}}{\sqrt{\sum (x_j - \bar{x})^2}}, \quad \beta = \frac{\delta}{\sqrt{\sum (x_j - \bar{x})^2}}.$$

The joint density of $Y_1, \ldots, Y_n$ is

$$\frac{1}{(2\pi)^{n/2}\sigma^n} \exp\Big[-\frac{1}{2\sigma^2} \sum (y_i - \gamma - \delta v_i)^2) \Big].$$

Reparametrization:

$$\theta = \frac{\delta}{\sigma^2}, \quad \vartheta_1 = -\frac{1}{2\sigma^2}, \quad \vartheta_2 = \frac{\gamma}{\sigma^2},$$

sufficient statistics

$$U = \sum v_i Y_i, \quad T_1 = \sum Y_i^2, \quad T_2 = \sum Y_i.$$

The UMP test of hypothesis $\mathbf{H}_{11} : \beta = \delta = 0$ is based on the criterion

$$A = \frac{|\sum v_i Y_i|}{\sqrt{\frac{1}{n-2}[\sum (Y_i - \bar{Y})^2 - (\sum v_i Y_i)^2]}}$$

and rejects when $A \ge C$, where $\int_C^{\infty} t_{n-2}(y)dy = \frac{\alpha}{2}$.

4 Comparison of two Poisson distributions

Let X and Y be two independent variables with Poisson distributions $\mathcal{P}(\lambda)$ and $\mathcal{P}(\mu)$. Joint distribution of X, Y is

$$I\!\!P\left\{X = x, Y = y\right\} = \frac{\mathrm{e}^{-(x+y)}}{x!y!}\exp\Big[y \log\frac{\mu}{\lambda} + (x+y)\log\ \lambda\Big].$$

Reparametrization: $\theta = \log\frac{\mu}{\lambda}$, $\vartheta = \log\ \lambda$, sufficient statistics $U = Y$, $T = X + Y$. Consider the hypotheses $\mathbf{H}_{12} : \mu \leq \lambda \sim \theta \leq 0$ and $\mathbf{H}_{13} : \mu = \lambda \sim \theta = 0$. The test is conditional for given $T = t$. The conditional distribution of Y given $X + Y = t$ is

$$I\!\!P\Big(Y = y | X + Y = t\Big) = \binom{t}{y}\left(\frac{\mu}{\lambda+\mu}\right)^{y}\left(\frac{\lambda}{\lambda+\mu}\right)^{t-y},$$

the binomial $B\Big(t, p = \frac{\mu}{\lambda+\mu}\Big)$. The hypothesis $\mathbf{H}_{12}$ is equivalent to $p \leq \frac{1}{2}$ and is rejected when $Y > C(t)$ and rejected with probability $\gamma(t)$ when $Y = C(t)$, where

$$\sum_{i=C(t)+1}^{t}\binom{t}{y} + \gamma(t)\binom{t}{C(t)} = 2^{t}\alpha.$$

Similarly, the hypothesis $\mathbf{H}_{13}$ is equivalent to $p = \frac{1}{2}$ and is rejected when $Y < C_1(t)$ and $Y > C_2(t)$ and rejected with probability $\gamma_i(t)$ when $Y = C_i(t)$; due to the symmetry it holds $C_1(t) - \frac{t}{2} = \frac{t}{2} - C_2(t)$ and $\gamma_1(t) = \gamma_2(t)$.

5 Comparison of two binomial distributions

Let X, Y be two independent variables with binomial distributions $B(m, p_1)$ and $B(n, p_2)$, respectively. Their joint distribution is

$$\begin{aligned} I\!\!P\Big(X = x, Y = y\Big) &= \binom{m}{x} p_1^{x} q_1^{m-x}\binom{n}{y} p_2^{x} q_2^{n-y} \\ &= \binom{m}{x}\binom{n}{y} p_2^{x} q_2^{n-y} q_1^{m} q_2^{n} \exp\left[y\left(\log\frac{p_2}{q_2} - \log\frac{p_1}{q_1}\right) + (x+y)\log\frac{p_1}{q_1}\right]. \end{aligned}$$

Reparametrization:

$$\theta = \log\left(\frac{p_2}{q_2}\Big/\frac{p_1}{q_1}\right), \qquad \vartheta = \log\frac{p_1}{q_1},$$

sufficient statistics $U = Y$, $T = X + Y$. Under $\theta = 0$, the conditional distribution of Y given $X + Y = t$ is the hypergeometric distribution

$$I\!\!P(Y = y \Big| X + Y = t) = \frac{\binom{m}{t-y}\binom{n}{y}}{\binom{m+n}{t}}.$$

Hence, we reject the hypothesis $\mathbf{H}_{14} : p_2 \leq p_1 \ \sim\ \theta \leq 0$ when $Y > C(t)$ and reject with probability $\gamma(t)$ when $Y = C(t)$ where

$$I\!\!P(Y > C(t) \Big| X + Y = t) + \gamma(t) I\!\!P(Y = C(t) \Big| X + Y = t) = \alpha.$$

6 Test for independence in a 2×2 table

Consider the population of n individuals; for each individual we check whether it has properties A and B. The results we write in the table

	A	$\bar{A}$	sums
B	X	X'	M
$\bar{B}$	Y	Y'	N
sums	Q	Q'	s

where X is the number of individuals which have simultaneously A and B, etc. Denote p_{AB} the probability that an individual has both A and B, and similarly we denote the probabilities $p_{\bar{A}B}$, $p_{A\bar{B}}$, $p_{\bar{A}\bar{B}}$. Moreover, let p_A and p_B be the probabilities that an individual has property A, B, respectively. Then

$$p_A = p_{AB} + p_{A\bar{B}}, \quad p_B = p_{AB} + p_{\bar{A}B}.$$

The joint distribution of variables X, Y, X', Y' is multinomial and is given by

$$I\!P(X = x, X' = x', Y = y, Y' = y') = \frac{n!}{x!x'!y!y'!} p_{AB}^{x} p_{\bar{A}B}^{x'} p_{A\bar{B}}^{y} p_{\bar{A}\bar{B}}^{y'} \tag{6.1}$$

$$\frac{n!}{x!x'!y!y'!} (p_{\bar{A}\bar{B}})^n \exp\left(x \log \frac{p_{AB}}{p_{\bar{A}\bar{B}}} + x' \log \frac{p_{\bar{A}B}}{p_{\bar{A}\bar{B}}} + y \log \frac{p_{A\bar{B}}}{p_{\bar{A}\bar{B}}} \right).$$

We wish to test the hypothesis, that properties A, B appear independently in the population. More precisely, we wish to test the hypothesis of independence

$$\mathbf{H}_1 : \ p_{AB} = p_A \cdot p_B \ \text{ against } \ \mathbf{K}_1 : \ p_{AB} \neq p_A \cdot p_B,$$

or the hypothesis of positive dependence

$$\mathbf{H}_2 : \ p_{AB} \geq p_A \cdot p_B \ \text{ against } \ \mathbf{K}_2 : \ p_{AB} < p_A \cdot p_B.$$

Rewrite the distribution (6.1) as follows:

$$\frac{n!}{x!x'!y!y'!} (p_{\bar{A}\bar{B}})^n \exp\{\theta_1 T_1 + \theta_2 T_2 + |theta_3 T_3\}$$

where

$$\begin{aligned}
\theta_1 &= -\log \frac{p_{\bar{A}B}}{p_{\bar{A}\bar{B}}} - \log \frac{p_{A\bar{B}}}{p_{\bar{A}\bar{B}}} + \log \frac{p_{AB}}{p_{\bar{A}\bar{B}}} \\
\theta_2 &= \log \frac{p_{\bar{A}B}}{p_{\bar{A}\bar{B}}}, \quad \theta_3 = \log \frac{p_{A\bar{B}}}{p_{\bar{A}\bar{B}}}, \\
T_1 &= X, \qquad T_2 = X + X', \qquad T_3 = X + Y.
\end{aligned}$$

The hypotheses $\mathbf{H}_1$ and $\mathbf{H}_2$ can be then rewritten as

$$\begin{aligned}
\mathbf{H}_1 &: \ \theta_1 = 0, \qquad \mathbf{K}_1 : \ \theta \neq 0, \\
\mathbf{H}_2 &: \ \theta_1 \leq 0, \qquad \mathbf{K}_2 : \ \theta > 0.
\end{aligned}$$

The tests are conditional, based on the conditional distribution of T_1 given $T_2,\ T_3$ under $\theta_1 = 0$. First we get

$$\mathbb{P}_{\theta_1=0}\left(X = x, Y = y \mid X + X' = m\right) = \binom{m}{x}\binom{n-m}{y} p_A^{x+y}(1-p_A)^{n-x-y}$$

and this under condition $X + Y = q$ gives

$$\mathbb{P}_{\theta_1=0}\left(X = x \mid X + X' = m,\ X + Y = q\right) = \frac{\binom{m}{x}\binom{n-m}{q-x}}{\binom{n}{q}}.$$

The conditional test of $\mathbf{H}_2$ is called the *Fisher-Irwin test*; it has the form

$$\Phi(x, m, q) = \begin{cases} 1 & \text{if} \quad x < C_1(m, q) \\ \gamma(m, q) & \text{if} \quad x = C(m, q), \\ 0 & \text{if} \quad C_1(m, q) < x \end{cases}$$

where $C_1, \gamma_1, 2 \quad (C_1$, integer) are determined so that

$$\sum_{i \le C_1 - 1} \binom{m}{i}\binom{n-m}{q-i} + \gamma_1 \binom{m}{C_1}\binom{n-m}{q-C_1} = \alpha \binom{n}{q}.$$