6. Birth and Death Processes

6.1 Pure Birth Process (Yule-Furry Process)

6.2 Generalizations

6.3 Birth and Death Processes

6.4 Relationship to Markov Chains

6.5 Linear Birth and Death Processes

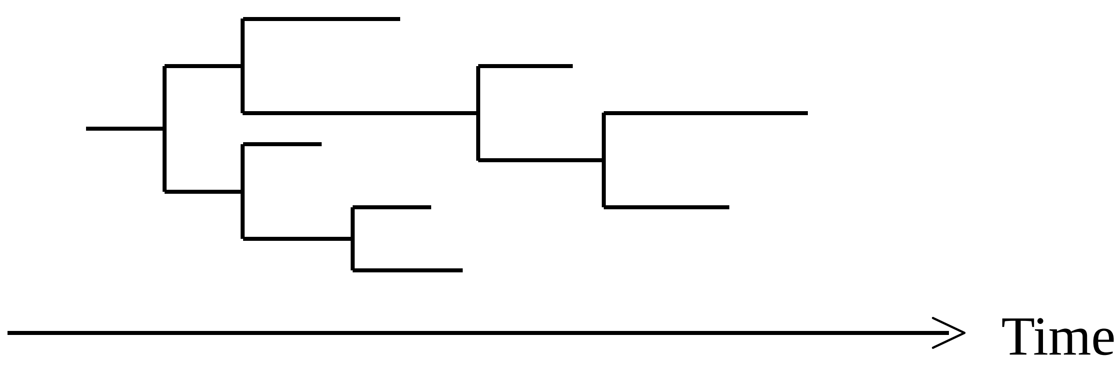
6.1 Pure Birth Process (Yule-Furry Process)

Example. Consider cells which reproduce according to the following rules:

i. A cell present at time t has probability $\lambda h + o(h)$ of splitting in two in the interval $(t, t+h)$

ii. This probability is independent of age.

iii. Events between different cells are independent



Non-Probablistic Analysis

$$n(t) = \text{no. of cells at time } t$$

$\Rightarrow$ $\lambda n(t)\Delta(t)$ births occur in $(t, t+\Delta t)$

where $\lambda =$ birth rate per single cell.

$$n(t+\Delta t) = n(t) + n(t)\lambda\Delta t$$

$$\frac{n(t+\Delta t) - n(t)}{\Delta t} \to n'(t) = n(t)\lambda$$

or

$$\frac{n'(t)}{n(t)} = \frac{d}{dt}\log n(t) = \lambda$$

$$\log n(t) = \lambda t + c$$

$$n(t) = Ke^{\lambda t}, \;\; n(0) = n_0$$

$$\boxed{n(t) = n_0 e^{\lambda t}}$$

Probabilistic Analysis

$N(t) \ = \ $ no. of cells at time t

$P\{N(t) = n\} = P_n(t)$

Prob. of birth in $(t, t+h)$ if $\{N(t) = n\} \ = \ n\lambda h + o(h)$

$$\begin{aligned} P_n(t+h) &= P_n(t)(1 - n\lambda h + o(h)) \\ &\qquad + P_{n-1}(t)((n-1)\lambda h + o(h)) \\ P_n(t+h) - P_n(t) &= -n\lambda h P_n(t) + P_{n-1}(t)(n-1)\lambda h + o(h) \\ \frac{P_n(t+h) - P_n(t)}{h} &= -n\lambda P_n(t) + P_{n-1}(t)(n-1)\lambda + o(h) \ \text{ as } h \to 0 \end{aligned}$$

$$\boxed{P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)}$$

Initial condition $\quad P_{n_0}(0) = P\{N(0) = n_0\} = 1$

$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t); \quad P_{n_0}(0) = 1$$

Solution:

(1) $$P_n(t) = \binom{n-1}{n-n_0} e^{-\lambda n_0 t}(1-e^{-\lambda t})^{n-n_0} \qquad n = n_0, n_0+1, \ldots$$

Solution is negative binomial distribution; i.e. Probability of obtaining exactly n_0 successes in n trials.

Suppose $p =$ prob. of success and $q = 1-p =$ prob. of failure. Then in first $(n-1)$ trials results in (n_0-1) successes and $(n-n_0)$ failures followed by success on n^{th} trial; i.e.

(2)

$$\binom{n-1}{n_0-1} p^{n_0-1} q^{n-n_0} \cdot p = \binom{n-1}{n-n_0} p^{n_0} q^{n-n_0} \quad n = n_0, n_0+1, ...$$

If $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$

$\Rightarrow$ (2) is same as (1).

Yule studied this process in connection with theory of evolution; i.e. population consists of the species within a genus and creation of new element is due to mutations. Neglects probability of species dying out and size of species.

Furry used same model for radioactive transmutations.

Notes on Negative Binomial Distribution

The geometric distribution is defined as the number of trials to achieve one success for a series of Bernoulli trials; i.e.

Geometric Distribution: $P\{N = n\} = pq^{n-1}, \;\; n = 1, 2, \ldots$

N is number of trials for 1 success

$$\phi_N(s) = E(e^{-sN}) = p\sum_{n=1}^{\infty} e^{-sn} q^{n-1} = \frac{p}{q}\sum_{n=1}^{\infty}(e^{-s}q)^n.$$

But

$$\sum_{n=1}^{\infty}(e^{-s}q)^n = \quad e^{-s}q/(1-e^{-s}q)$$

$$\phi_N(s) = \frac{p}{q}\cdot\frac{e^{-s}q}{1-e^{-s}q} \quad = \frac{pe^{-s}}{1-e^{-s}q} = \frac{pz}{1-qz} \quad \text{if } z = e^{-s}$$

$$\phi_N(z) \quad = pz(1-qz)^{-1}$$

$$\phi'_N(z) \quad = p\{(1-qz)^{-1} + z(1-qz)^{-2}q\}$$

$$\phi'_N(1) \quad = p\{(1-q)^{-1} + q(1-q)^{-2}\} = 1 + \frac{q}{p} = \frac{p+q}{p} = \frac{1}{p}$$

Similarly $\phi''_N(1) = \frac{2}{p^2} - \frac{2}{p} \;\Rightarrow\; V(n) = \frac{1}{p^2} - \frac{1}{p}$

Theorem: If N_i $(i = 1, 2, \dots, n_0)$ are iid geometric random variables with parameter p, then $N = N_1 + N_2 + \dots + N_{n_0}$ is a negative binomial distribution having generating function

$$\phi_N(z) = \left(\frac{pz}{1 - qz}\right)^{n_0}, \quad z = e^{-s}$$

$$\therefore \quad E(N) = n_0/p, \quad V(N) = n_0\left[\frac{1}{p^2} - \frac{1}{p}\right]$$

If $p = e^{-\lambda t}$ and $N(t)$ is a pure birth process

$$E[N(t)] = n_0 e^{\lambda t}$$

$$V[N(t)] = n_0[e^{2\lambda t} - e^{\lambda t}]$$

6.2 Generalization

In Poisson Process, the prob. of a change during $(t, t+h)$ is independent of number of changes in $(0, t)$. Assume instead that if n changes occur in $(0, t)$, the probability of new change to $n+1$ in $(t, t+h)$ is $\lambda_n h + o(h)$. The probability of more than one change is $o(h)$. Then

$$
\begin{aligned}
& P_n(t+h) = P_n(t)(1-\lambda_n h) + P_{n-1}(t)\lambda_{n-1}h + o(h),\ n \neq 0 \\
& P_0(t+h) = P_0(t)(1-\lambda_0 h) + o(h) \\
\Rightarrow \quad & P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1}P_{n-1}(t) \\
& P_0'(t) = -\lambda_0 P_0(t).
\end{aligned}
$$

Equations can be solved recursively with $P_0(t) = P_0(0)e^{-\lambda_0 t}$.

If the initial condition is $P_{n_0}(0) = 1$, then the resulting equations are:

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), \;\; n > n_0$$

$$P_{n_0}'(t) = -\lambda_{n_0} P_{n_0}(t)$$

Pure birth process assumed $\lambda_n = n\lambda$.

Change of Language

If n transitions take place during $(0, t)$, we may refer to the process as being in state E_n. Changes occur $E_n \to E_{n+1} \to E_{n+2} \to \ldots$ for the pure birth process.

6.3 Birth and Death Processes

Consider transitions $E_n \to E_{n-1}$ as well as $E_n \to E_{n+1}$ if $n \geq 1$.
If $n = 0$, we only allow $E_0 \to E_1$.

Assume that if the process at time t is in E_n, then during $(t, t+h)$ the transitions $E_n \to E_{n+1}$ have prob. $\lambda_n h + o(h)$, $E_n \to E_{n-1}$ have prob. $\mu_n h + o(h)$ and Prob. more than 1 change occurs = $o(h)$

$$P_n(t+h) = P_n(t)\{1 - \lambda_n h - \mu_n h\}$$

$$+ P_{n-1}(t)\{\lambda_{n-1} h\} + P_{n+1}(t)\{\mu_{n+1} h\} + o(h)$$

$$\Rightarrow \quad \boxed{P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)}$$

For $n = 0$

$$P_0(t+h) = P_0(t)\{1 - \lambda_0 h\} + P_1(t)\mu_1 h + o(h)$$

$$\Rightarrow \quad \boxed{P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)}$$

If the initial conditions $P_{n_0}(0) = 1$ in which case 0 in above is replaced by n_0.

If $\lambda_0 = 0$, then $E_0 \to E_1$ is impossible and E_0 is an absorbing state. If $\lambda_0 = 0$, then $P_0'(t) = \mu_1 P_1(t) \geq 0$ so that $P_0(t)$ increases monotonically.

Note: $\lim_{t \to \infty} P_0(t) = P_0(\infty) =$ Probability of being absorbed.

$$
\begin{aligned}
P_0'(t) &= -\lambda_0 P_0(t) + \mu_1 P_1(t) \\
P_n'(t) &= -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t)
\end{aligned}
$$

As $t \to \infty, P_n(t) \to P_n$ $(limit)$ hence $P_0'(t) = 0$ for large t and $P_n'(t) = 0$ for large t. Therefore

$$
\begin{aligned}
0 &= -\lambda_0 P_0 + \mu_1 P_1 \\
\Rightarrow \quad P_1 &= \frac{\lambda_0}{\mu_1} P_0 \\
0 &= -(\lambda_1 + \mu_1) P_1 + \lambda_0 P_0 + \mu_2 P_2 \\
\Rightarrow \quad P_2 &= \frac{\lambda_0 \lambda_1}{\mu_2 \mu_1} P_0 \\
\Rightarrow \quad P_3 &= \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 \qquad \text{etc.}
\end{aligned}
$$

Note that dependence on initial conditions has disappeared.

6.4 Relation to Markov Chains

Define for $t \to \infty$

$$\begin{aligned} P(E_{n+1}|E_n) &= \text{Prob. of transition } E_n \to E_{n+1} \\ &= \text{Prob. of going to } E_{n+1} \text{ conditional on being in } E_n. \end{aligned}$$

Similarly define $P(E_{n-1}|E_n)$.

$$P(E_{n+1}|E_n) \propto \lambda_n, \quad P(E_{n-1}|E_n) \propto \mu_n$$

$$\Rightarrow \quad P(E_{n+1}|E_n) = \frac{\lambda_n}{\lambda_n + \mu_n}, \; P(E_{n-1}|E_n) = \frac{\mu_n}{\lambda_n + \mu_n}$$

Same conditional probabilities hold if it is given that a transition will take place during $(t, t+h)$ conditional on being in E_n.

6.5 Linear birth and death processes

$$\lambda_n = n\lambda \ , \ \ \mu_n = n\mu$$

$$\Rightarrow \quad P_0'(t) = \mu P_1(t)$$

$$P_n'(t) = -(\lambda+\mu)nP_n(t) + \lambda(n-1)P_{n-1}(t) + \mu(n+1)P_{n+1}(t)$$

Steady state behavior is characterized by

$$\lim_{t\to\infty} P_0'(t) = 0 \Rightarrow P_1(\infty) = 0$$

Similarly as $t \to \infty$ $P_n'(\infty) = 0$

If $P_0(\infty) = 1 \Rightarrow$ Probability of ultimate extinction is 1.

If $P_0(\infty) = P_0 < 1$, the relations $P_1 = P_2 = P_3 \ldots = 0$ imply with prob. $1 - P_0$ the population can increase without bounds. The population must either die out or increase indefinitely.

$$P_n'(t) = -(\lambda+\mu)nP_n(t) + \lambda(n-1)P_{(n-1)}(t) + \mu(n+1)P_{n+1}(t)$$

Define Mean by $M(t) = \sum_{n=1}^{\infty} nP_n(t)$

and consider $M'(t) = \sum_{1}^{\infty} nP_n'(t)$.

$$M'(t) = -(\lambda+\mu)\sum_{1}^{\infty} n^2 P_n(t) + \lambda\sum_{1}^{\infty} (n-1)n\, P_{(n-1)}(t)$$

$$+ \mu\sum_{1}^{\infty} (n+1)nP_{n+1}(t)$$

Write $(n-1)n = (n-1)^2 + (n-1), \quad (n+1)n = (n+1)^2 - (n+1)$

$$
\begin{aligned}
M'(t) = &-(\lambda+\mu)\sum_1^\infty n^2 P_n(t) \\
&+\lambda\sum_1^\infty (n-1)^2 P_{n-1}(t) + \mu\left[\sum_1^\infty (n+1)^2 P_{n+1}(t) + 1\cdot P_1(t)\right] \\
&+\lambda\sum_1^\infty (n-1) P_{n-1}(t) - \mu\left[\sum_1^\infty (n+1) P_{n+1}(t) + P_1(t)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow M'(t) &= \lambda\sum_1^\infty n P_n(t) - \mu\sum_1^\infty n P_n(t) \\
&= (\lambda-\mu)M(t)
\end{aligned}
$$

$\boxed{M(t) = n_0 e^{(\lambda-\mu)t}}$ if $P_{n_0}(0) = 1$

$$\boxed{M(t) = n_0 e^{(\lambda-\mu)t}}$$

$M(t) \to 0$ or ∞ depending on $\lambda < \mu$ or $\lambda > \mu$.

Similarly if $M_2(t) = \sum_1^\infty n^2 P_n(t)$ one can show

$$M_2'(t) = 2(\lambda - \mu)M_2(t) + (\lambda + \mu)M(t)$$

and when $\lambda > \mu$, the variance is

$$\boxed{n_0 e^{2(\lambda-\mu)t} \left\{1 - e^{(\mu-\lambda)t}\right\} \frac{\lambda + \mu}{\lambda - \mu}}$$