Text 10: T. H. Kjeldsen (2011). "Does history have a significant role to play for the learning of mathematics? Multiple perspective approach to history, and the learning of meta level rules of mathematical discourse". In: *History and Epistemology in Mathematics Education. Proceedings of the Sixth European Summer University ESU 6.* Ed. by E. Barbin, M. Kronfellner, and C. Tzanakis. Vienna: Verlag Holzhausen GmbH, pp. 51–62.

DOES HISTORY HAVE A SIGNIFICANT ROLE TO PLAY FOR THE LEARNING OF MATHEMATICS?

Multiple perspective approach to history, and the learning of meta level rules of mathematical discourse

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ABSTRACT

In the present paper it will be argued that and proposed how the history of mathematics can play a significant role in mathematics education for the learning of meta rules of mathematical discourse. The theoretical argument is based on Sfard's theory of thinking as communicating. A multiple perspective approach to history of mathematics from the practice of mathematics will be introduced along with the notions of epistemic objects and techniques. It will be argued that by having students read and analyse mathematical texts from the past within this methodology, the texts can function as "interlocutors". In such learning situations the sources can assist in revealing meta rules of (past) mathematical discourses, making them explicit objects for students' reflections. The proposed methodology and the potential of history for the learning of meta-discursive rules of mathematical discourse is exemplified by analyses of four sources from the 17th century by Fermat and Newton belonging to the calculus, and it is demonstrated how meta level rules can be made objects of students' reflections. The paper ends with a proposal for a matrix-organised design for how the introduced approach to history of mathematics for elucidating meta-discursive rules might be implemented in upper secondary mathematics education.

1 Introduction

One can think of several purposes for using history in mathematics education: (1) For pedagogical reasons; it is often argued that history motivates students to learn mathematics by bringing in a human aspect. (2) As a didactical method for the learning and teaching of the subject matter of mathematics. (3) For the development of students' historical awareness and knowledge about the development of mathematics and its driving forces. (4) For general educational goals, with respect to which the so called cultural argument makes the strongest case for history, but history can also serve general educational goals in mathematics education of developing interdisciplinary competences as a counterpart to specialisation (Beckmann 2009). These purposes are not necessarily mutually independent. In carefully designed teaching sessions all four of the above mentioned purposes can be realized in varying degrees.[1]

Regarding the question whether history promotes students' learning of mathematics I have argued in (Kjeldsen 2011), that by adopting a multiple perspective approach to history from the practice of mathematics, history has potentials in developing students' mathematical competence while providing them with genuine historical insights. In the present paper, I will go a step further and suggest that history might have a much more

[1] See (Kjeldsen 2010) where it is shown how all these four purposes can be accomplished in problem oriented and student directed project work. In (Jankvist and Kjeldsen 2011) two avenues for integrating history in mathematics education are discussed with respect to the development of students' mathematical competence and historical awareness anchored in the subject matter of mathematics, respectively, both within a scholarly approach to history. In (Kjeldsen forthcoming) a didactical transposition of history from the academic research subject to history in mathematics education is proposed for developing a framework for integrating history of mathematics in mathematics education.

profound role to play for the learning of mathematics. This suggestion is based on Sfard's (2008) theory of commognition.

In the following it will be argued that, and proposed how, the history of mathematics can play a significant role in the teaching and learning of mathematics. The theoretical argument is outlined in section 2. In section 3, the multiple perspective approach to history of mathematics from its practice is presented along with some tools of historians'. The adaptation for mathematics education is discussed in section 4. The potential of history for the learning of meta-discursive rules of mathematical discourse is exemplified in section 5 through analyses of four sources from the 17th century by Fermat and Newton belonging to the calculus. In section 6 a proposal is outlined for a so called matrix-organised design for how such an approach to history of mathematics for elucidating meta-discursive rules might be implemented in upper secondary school. The paper ends with a concluding section 7.

2 The theoretical argument for the significance of history

In Sfard's (2008, 129) theory of *Thinking as Communicating* mathematics is seen as a discourse that is regulated by discursive rules, and where the objects of mathematics are discursive constructs. There are two kinds of discursive rules both of which are important for the learning of mathematics: object-level rules and meta-discursive rules.

The object-level rules have the content of the discourse as object. In mathematics they regard the properties of mathematical objects. The meta-discursive rules have the discourse itself as object. They govern proper communicative actions shaping the discourse. The meta-discursive rules are often tacit. They are implicitly present in discursive actions when we e.g. judge if a solution or proof of a mathematical problem or statement can count as a proper solution or proof (Sfard 2000, 167). The meta-discursive rules are not necessary; they are given historically.

The meta-discursive rules are connected to the object-level of the discourse and have an impact on how participants in the discourse interpret its content. As a consequence, developing proper meta-discursive rules are indispensable for the learning of mathematics (Sfard 2008, 202). This means that designing learning situations where meta-discursive rules are elucidated is an important aspect of mathematics education. History of mathematics is an obvious method for illuminating meta-discursive rules. Because of the contingency of these rules, they can be treated at the *object level* of history discourse, and thereby be made into explicit objects of reflection. Hence, history might have a significant role to play for the learning of mathematics, precisely because meta-discursive rules can be treated as objects of historical investigations. By reading historical sources students can be acquainted with episodes of past mathematics where other meta-discursive rules governed the discourse. If students study original sources in their historical context, and try to understand the work of past mathematicians, their views on mathematics, the way they formulated and argued for mathematical statements etc. the historical texts can play the role as "interlocutors", as discussants acting according to meta rules that are different than the ones that govern the discourse of our days mathematics and (maybe) of the students. By identifying meta rules that governed past mathematics and comparing them with the rules that govern e.g. their textbook, students can be engaged in learning processes where they can become aware of their own meta rules. In case a student is acting according to non-proper meta rules he or she might experience what Sfard calls a commognitive conflict, which is "a situation in which different discursants are acting according to different metarules" (Sfard 2008, 256). Such situations can initiate a metalevel change in the learner's discourse.

This, of course, presupposes a genuine approach to history. In section 3 and 4 it will be argued that within a multiple perspective approach to the history of the practice of mathematics, and by using historian of mathematics' tools such as the idea of epistemic objects and techniques, original sources can be used in mathematics education to have students investigate and reflect upon meta-discursive rules. For further discussion of this see (Kjeldsen and Blomhøj 2011), where also some student directed problem oriented project work performed by students at degree level mathematics are analysed with respect to students' reflections about meta-discursive rules to provide empirical evidence for the theoretical claim. These projects will not be presented here. Instead I will present a proposal (see section 6) for a so called matrix-organised design for how such an approach to history of mathematics for investigating meta-discursive rules might be implemented in upper secondary school.

3 A multiple perspective approach to history

The so called whig interpretation of history has been debated at length in the historiography of mathematics.[2] In mathematics education Schubring (2008) has pointed out how translations of sources, due to an underlying whig interpretation of history, have changed the mathematics of the source. In the whig interpretation history is written from the point of view of the present, as explained by the British historian Herbert Butterfield, who coined the term in the 1930s:

> It is part and parcel of the whig interpretation of history that it studies the past with reference to the present ... The whig historian stand on the summit of the twentieth century and organises his scheme of history from the point of view of his own day. (Butterfield 1931, 13)

If we want to use history to throw light on changes in meta rules from episodes of past mathematics to our days mathematics whig interpretations of history poses a problem, because, as it has been pointed out by Wilson and Ashplant (1988, 11) history then becomes "constrained by the perceptual and conceptual categories of the present, bound within the framework of the present, deploying a perceptual 'set' derived from the present". In this quote, Wilson and Ashplant emphasis exactly why one cannot design learning and teaching situations that focus on bringing out differences in meta rules of past episodes in the history of mathematics and modern ones within a whig interpretation of history. Historical sources cannot function as "interlocutors" that can be used to clarify differences in meta rules if the sources is interpreted within the framework of how mathematics is conceptualized and perceived of today.

The trap of whiggism can be avoided by investigating past mathematics as a historical product from its practice. This implies to study the sources in their proper historical context with respect to the intellectual workshop[3] of their authors, the particular mathematicians, to ask questions such as: how was mathematics viewed at the time? How did the mathematician, who wrote the source, view mathematics? What was his/hers

[2] Discussions of whig interpretations in the historiography of mathematics can be followed e.g. in the following papers (Unguru 1975), (van der Waerden 1976), (Freudenthal 1977), (Unguru and Rowe, 1981/82), (Grattan-Guiness 2004).

[3] See (Epple 2004).

intention? Why and how did mathematicians introduce certain concepts? How did they use them and for what purposes? Why and how did they work on the problems they did? Which kinds of tools were available for the mathematician (group of mathematicians)? Why and how did they employ certain strategies of proofs? Such questions can reveal underlying meta rules of the discourse at the time and place of the sources. By posing and answering such questions to the sources, possibilities for identifying meta rules that governed the mathematics of the source can emerge, and hereby also opportunities for turning meta rules into explicit objects of reflection in a teaching and learning situation.

As explained by Kjeldsen (2009b, 2011) one way of answering such questions and to provide explanations for historical processes of change is to adopt a multiple perspective approach to the history of the practice of mathematics. I have taken the term "a multiple perspective" approach from the Danish historian Jensen (2003). It signifies that episodes of the past can be studied from several perspectives, several points of observation, depending on which kind of insights into, or from, the past, we are searching for. Episodes in the history of mathematics can e.g. be studied from the perspective of sub-disciplines within mathematics to understand if, and if so, how other fields in mathematics have influenced the emergence and/or the development of the episode under consideration. They can be studied from an applied point of view to understand e.g. dynamics between pure and applied mathematics, or the role of mathematical modelling in the production of mathematical and/or scientific knowledge. They can be studied from a sociological perspective to understand the institutionalization of mathematics, its funding etc. They can be studied from a gender perspective, from a philosophical perspective and so on.

4 Adaptation for mathematics education

In mathematics education the above approach can be implemented on a small scale, by focusing on a limited amount of perspectives that address the intended learning. In the present context the purpose is to use past mathematics and history of mathematics as a means for elucidating meta discursive rules and make them into explicit objects of students' reflections. Hence, students should study the sources to answer clearly formulated historical questions that concern the underlying meta rules of the mathematics in the source.

Theoretical constructs that have been developed by historians of mathematics and/or science to investigate the history of scientific practices can be used to "open" the sources. With respect to the purpose of the present paper of uses of history to reveal meta rules of a (past) mathematical discourse by studying the history of mathematics from its practice, the notions of epistemic objects and techniques are promising tools. The term epistemic object refers to mathematical objects that are treated in a source, i.e. the object about which mathematicians were searching for new knowledge or were trying to grasp. The term epistemic technique refers to the methods employed in the source by the mathematicians to investigate the epistemic objects.[4] These theoretical constructs can give insights into the dynamics of concrete productions of pieces of mathematical knowledge, since they are constructed to distinguish between elements of the source that provide answers and elements that generate mathematical questions.[5]

[4] These notions have been adapted into the historiography of mathematics by Epple (2004) from Rheinberger's (1997) study of experimental science.

[5] For examples of uses of this methodological tool see (Epple 2004) and (Kjeldsen 2009a).

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The question is whether history dealt with in this way, where students study episodes from the history of mathematics from perspectives that pertain to meta rules of (past) discourses, ask historians' questions to the sources concerning the practice of mathematics, and answer them using theoretical constructs such as epistemic objects and techniques, can facilitate meta level learning in mathematics education. In the following section four texts from the 1600s will be analyzed to provide some answers to this question.

5 Analysis of four sources within the proposed methodology

Four texts from the 1600s will be used in the following; two by Pierre de Fermat (Fermat I and Fermat II) and two by Isaac Newton (Newton I and Newton II). Fermat I is Fermat's text on maxima and minima taken from Struik's (1969) *A Source Book in Mathematics, 1200-1800*, whereas Fermat II is called "A second method for finding maxima and minima", which is published in Fauvel's and Gray's (1988) reader in the history of mathematics. Newton I is Newton's demonstration of how he found a relation between the fluxions of some fluent quantities from a given relation between these. This text is the one prepared by Baron and Bos (1974), whereas Newton II is Newton's method of tangent taken from Whiteside's (1967) *The Mathematical Works of Isaac Newton*. The quality of these translations of sources can be criticised, and investigated for degrees of whiggism (Schubring 2008), but this will not be done in the present paper. In a teaching situation the students should work with the four texts, but in order to give the reader an impression of the texts, summaries of the four texts are inserted here:

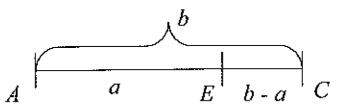
In Fermat I, Fermat stated a rule for the evaluation of maxima and minima and gave an example. The text is summarised below in Box 1.

Fermat I: On a method for the evaluation of max. and min.

Rule: let a be any unknown of the problem

- Indicate the max or min in terms of a
- Replace the unknown a by $a+e$ – express max./min. in terms of a and e
- "adequate" the two expressions for max./min. and remove common terms
- Both sides will contain terms with e – divide all terms by (powers of) e
- Suppress all terms in which e will still appear – and equate the others
- The solution of this equation will yield the value of a leading to max./min.

Example: To divide the segment AC at E so that AE x EC may be a maximum



Max: $a(b-a) = ab-aa$

$(a+e)b-(a+e)(a+e) = ab+eb-aa-2ae-ee$

$ab+eb-aa-2ae-ee \sim ab-aa$ *"adequate"*

$eb \sim 2ae + ee$ remove common terms

$b \sim 2a + e$; $b=2a$; $a=½b$; divide, suppress, solve

Box 1

If the above procedure is translated into modern mathematics using functions and the derivative it can be explained why Fermat reached the correct solution. But this does not explain how Fermat was thinking, since he knew neither our concept of a function nor our concept of derivatives. In Fermat II we can get a glimpse of how Fermat was thinking.

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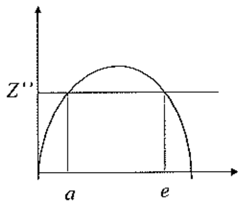
The text is summarised below in Box 2.

Fermat II: A second method for finding maxima and minima

- Here he explained why his "rule" leads to max./min.: correlative equations – Viete
- Resolving all the difficulties concerning limiting conditions

Example: To divide the line b such that the product of the segments shall be a max.

> If one proposes to divide the line b in such a way that the product of the segments [a and $(b-a)$] shall equal z'' ... there will be two points answering the question, and they will be found situated on one side and the other of the point corresponding to the max.



$ba-aa = z''$ and $be-ee = z''$

$ba-aa = be-ee$; $ba-be = aa-ee$

Divide by $a-e$

$b = a + e$

At the point of maximum we will have $a = e$, then

$b = a + a = 2a$, hence as before $a=½b$.

If we call the roots a and $a+e$ (instead of a and e) the procedure follows the rule from text I.

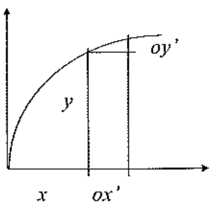
Box 2

In Newton I, Newton explained through an example, how, given a relation between fluent quantities, a relation between the fluxions of these quantities can be found. In Box 3 his procedure is summarised and illustrated with an example of a second degree equation instead of the third degree equation that Newton used in the text.

Newton I: Find relation between fluxions from fluents

Newton's fluxions and fluents

- Curves are trajectories (paths) for motions
- Variables are entities that change with time – fluents x , y
- The speed with which fluents change – fluxions x' , y' (*Newton: dots!*)
- Newton: All problems relating to curves can be reduced to two problems:
  1. Find the relation between the fluxions given the relation between the fluents.
  2. The opposite.



Example: $axx+bx+c-y=0$ substitute x, y with $x+x'o$, $y+y'o$

$a(x+x'o)(x+x'o)+b(x+x'o)+c-y-y'o=0$

$\underline{axx}+a2xx'o+ax'x'oo+\underline{bx}+bx'o+\underline{c-y}-y'o=0$

$a2xx'o+ax'x'oo+bx'o-y'o=0$

$a2xx'+\cancel{ax'x'o}+bx'-y'=0$ divided by o; cast out terms with o

$a2xx'+bx'-y'=0$ hence $y'/x'=2ax+b$

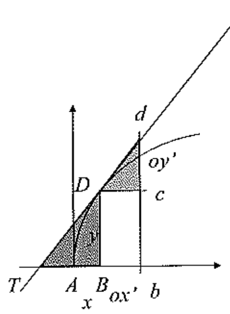
Box 3

In Newtons's terminology o denotes an infinitely small period of time, so ox' [Newton used a dot over x instead of x' to designate the fluxions] is the infinitely small addition by which x increases during the infinitely small interval of time.

Finally, in Newton II, Newton showed how to draw tangents to curves and illustrated it with the same example as he used in the first text. In Box 4 below the example is illustrated with reference to the example used in Box 3.

Newton II: To draw Tangents to Curves

Example:



Similar triangles: *dcD* and *DBT*

TB:BD = *Dc:cd* "infinitesimal triangle"

BT/y = *x'o/y'o* = *x'/y'*

x'/y' can be found by the method from Newton I

Box 4

The suggestion made in this paper is that these four sources can be used to exhibit changes in meta rules of mathematical discourse, if students read the sources from the perspective of rigor, and focus on entities and arguments. The following worksheet (Box 5) can be used to guide the students work. It consists of two sets of questions. The first set concerns questions that help the students to identify the epistemic objects and techniques of the two texts. The students are asked to compare and contrast the answers they get from studying Fermat, Newton, and their textbook, respectively.

Perspective

Rigor – entities, arguments

Worksheet: History from the practice of math. Compare/contrast Fermat and Newton

Questions:

- What mathematical objects are Fermat/Newton dealing with? Compare/contrast
- How do they perceive them? – compare with your textbook
- What are the problems they are trying to solve?
- What techniques are they using? – what do we do today?
- How do they argue for their claims? – how do we argue today?

Epistemic objects and techniques

- Can you find any changes in understandings of the involved mathematical concepts from Fermat over Newton to today? Explain
- Can you find any changes in the way of argumentation from Fermat over Newton to today? Explain
- What kind of objections do you think your math teacher would have to Fermat's and Newton's texts?

Meta-rules – explicit object of reflection
Opportunities provided by history

Box 5

The second set of questions refers directly to meta rules of the involved mathematical discourses.

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Regarding the first set of questions, an analysis of the four texts and the comparison between the objects that Fermat and Newton investigated, how they perceived them, the problems they tried to solve, the techniques they used and the arguments they employed might be summarised in the following scheme (Box 6):

Fermat:	**Newton:**
Objects: curves - algebraic expressions ex.: multiplication of line segments	Objects: any curve variables that change in time
Perceive: Area; geometrical problems treated by algebraic methods	Perceive: trajectories for moving particles
Problem: evaluate max/min	Problem: relations between fluxions (velocities) given relations between the fluents
Techniques: equations, roots, algebraic mani.	Techniques: algebraic mani; physics, geometry
Argue: Text 1: shows the method works on an example Text 2: heuristic arguments with roots in equations given by an example	Argue: Physical arguments about distance and velocity, algebraic arguments, infinitesimal triangle, *o*-infinitely small

Box 6

Regarding the second set of questions, which refers to meta rules of the discourse, the following changes can be discussed (se Box 7):

Changes in understanding:

Fermat: curves; algebraic expressions

Newton: curves, traced by a moving point, variables change in time

Today: functions, correspondence between variables in domains

Changes in the way of argumentation:

Fermat: ad hoc; "it works – its true"; heuristic argument, no infinitely small quantities

Newton: more general procedure, physical arguments, infinitesimal triangle, infinitely small quantities (*o*)

Today: limit, the real numbers, epsilon-delta proofs

Box 7

In Kjeldsen and Blomhøj (2011) we have analysed some student directed problem oriented project work conducted by students in a degree level university mathematics programme. Here we were able to demonstrate that history, used within the framework of a multiple perspective approach to the history of mathematics from its practice, can be used in mathematics education to give students insights into how meta rules of a mathematical discourse are established and why/how they change. These projects were made in a rather unique educational setting and the question is whether this methodology can be implemented in more traditional educational settings. The analyses of the sources guided by the worksheet (Box 5) and presented in Box 6 and Box 7 suggest that this approach can elucidate meta rules and turn them into explicit objects for students reflections. In the following section I present an outline for a so called matrix-organised design for how such a multiple perspective approach to history of mathematics from its practice might be implemented in upper secondary mathematics education.

6 Implementation in upper secondary school: A proposal

In the Danish upper secondary school system history of mathematics is part of the mathematics curriculum. The curriculum is comprised of a core curriculum which is mandatory and is tested in the national final, and a supplementary part, which should take up 1/3 of the teaching. History is mentioned explicitly in the supplementary part, which means that all upper secondary students should be taught some aspects of history of mathematics. The supplementary part of the curriculum is tested in an oral examination together with the core curriculum. In Box 8 below an outline is presented for a matrix organised design for how history could be (but has not yet been) implemented in a Danish upper secondary school for elucidating meta rules within the theoretical framework of section 2, 3 and 4, using the sources and the worksheet presented in section 5.

Implementation in a Danish high school: a proposal

Step 1: Six groups – basic groups (worksheets would have to be prepared for each group with respect to the intended learning)

1. The mathematical community in the 17^{th} century
2. The standard history of analysis
3. Who were Fermat and Newton?
4. The two texts of Fermat - the questions of the worksheet of Box 5
5. The two texts of Newton - the questions of the worksheet of Box 5
6. Berkeley's critique of Newton

Step 2: Six groups – expert groups (each group consists of at least one member from each of the basic groups)
The experts teach the other group members of what they learned in their basic group. Each expert group write a common report/prepare an oral presentation of the collected work from all six basic groups as it was discussed in their expert groups

Step 3: A plenary discussion lead by the teacher focuses on methods of argumentation, the development/changes in the perception of objects and techniques, compared with the standards of today.

Box 8

This design follows a three step implementation. First six groups (so called basic groups) are formed who look into some aspects of the historical episode in question. In Box 8 it is suggested e.g. that group 1 investigates what the mathematical community of the 17^{th} century looked like. Guided by a worksheet with questions relevant for the intended learning, the work in this group will provide the students with a sociological perspective on mathematics and its development. In step 2 new groups (so called expert groups) are formed. They consist of at least one member from each of the six basic groups. In this way each new group consists of individual experts. Each expert now teaches the other members of the new group what he/she learned in his/hers basic group, and based on their shared knowledge provided by the various experts they answer the second set of questions of the worksheet in Box 5. The design is referred to as being matrix organised because it can be illustrated with a matrix, where the members of basic group 1 is listed in column 1, the members of basic group in column 2, etc. In step 2 the expert groups are formed by taking the students in the rows, i.e. expert group 1 consists of the students listed in row 1; expert group 2 of the students listed in row 2, etc. In this way all expert groups consists of at least one member from each basic group. In such a set up it is possible to create complex teaching and learning situations where students work independently and autonomously in an inquire-like environment, developing general educational skills as well.[6]

7 Discussion and conclusion

The main question in the present paper is whether working with sources in the spirit of the worksheet of Box 5 within the methodology outlined in section 3 may give rise to situations where meta rules of (past) mathematical discourses are made into explicit objects of students' reflections, and whether this can assist the development of students' proper meta rules of mathematical discourse. As pointed out above, the analyses of the sources guided by the questions of the worksheet in Box 5, and the suggestions for answers outlined in Box 6 and 7, suggest that history and historical sources can be used within the methodological framework of section 2, 3 and 4 to elucidate meta rules and make them explicit objects for students reflections.

Regarding the second part of the question, whether such an approach to the use of history and historical sources in mathematics education also can assist the development of students' proper meta rules of our days mathematics is a complex question which is much more difficult to document. The framework and methodology outlined in this paper provide a theoretical argument for the claim that history has the potential for playing such a profound role for the learning of mathematics, but in order to realize this in practice more research needs to be done, and methodological tools for detecting students' meta rules and for monitoring any changes towards developing proper meta rules need to be developed.

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[6] Such a matrix organised design for using history in mathematics education to elucidate meta rules of past and present mathematics, using sources from the history of the development of the concept of a function, to have students reflect upon those, to develop students' mathematical competence, and general educational skills of independence and autonomy is being tried out in a pilot study in a Danish upper secondary class at the moment. Preliminary results from this study indicate that some of the students act according to meta discursive rules that coincide with Euler's; and that reading some of Dirichlet's text created obstacles for the students, that can be referenced to the differences in meta discursive rules. Results from the study will be published in forthcoming papers.

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