Other approaches to the formulation of the inverse scattering problem, with particular reference to the KdV equation, may be found in Lamb (1980), Ablowitz & Segur (1981), Calogero & Degasperis (1982), Dodd, Eilbeck, Gibbon & Morris (1982).

Exercises

Q3.1 Let y_1, y_2 be two solutions of the differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

Define the Wronskian W, of y_1 and y_2, and show that

$$W' + p(x)W = 0.$$

Hence deduce that either $W = 0$ for all x or W never vanishes (provided x and $p(x)$ remain finite).

Q3.2 Show that a continuous eigenfunction, $\hat{\psi}$, of equation (3.1), and any discrete eigenfunction, ψ_n, are orthogonal.

Q3.3 Find, if they exist, the eigenvalues and eigenfunctions of

$$\psi'' + (\lambda - U_0)\psi = 0,$$

where U_0 is any real constant.

Q3.4 *Two classical scattering problems.* Find the eigenvalues and eigenfunctions of

$$\psi'' + \{\lambda - u(x)\}\psi = 0$$

in these two cases:

(i) $$u(x) = \begin{cases} U_0, & 0 < x < 1, \\ 0, & x < 0,\ x > 1, \end{cases}$$

where U_0 is any real constant.

*(ii) $u(x) = -U_0\delta(x) - U_1\delta(x-1)$,
where U_0 and U_1 are positive constants, and show that there is only one discrete eigenfunction if $(U_0 + U_1)/(U_0U_1) > 1$.

Q3.5 *Another scattering problem.* Find the eigenvalues and eigenfunctions of

$$\psi'' + \{\lambda - u(x)\}\psi = 0,$$

if $u(x)$ is the step potential

$$u(x) = \begin{cases} 0, & x < 0 \\ U_0, & x > 0, \end{cases}$$

where U_0 is a positive constant. Show, in particular, that there is a continuous eigenfunction, no discrete eigenfunction, and an eigenfunction which decays as $x \to +\infty$ but is oscillatory in $x < 0$.

Q3.6 Relate the scattering problem with the potential $u(x) = -U_0 \operatorname{sech}^2 \beta x$, for some positive constant β, to the problem discussed in example (ii), §3.2.

Q3.7 *A reflectionless potential.* Find the discrete eigenfunctions for $N = 3$, where $u(x) = -N(N+1)\operatorname{sech}^2 x$ (see example (ii), §3.2).