where F is a solution of the pair of equations

$$F_{xx} - F_{zz} = (x - z)F,$$
$$3tF_t - F + F_{xxx} + F_{zzz} = xF_x + zF_z,$$

then

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0,$$

where

$$u(x,t) = -\frac{2}{(12t)^{2/3}}\frac{\partial}{\partial X}K(X,X;t) \qquad \text{with } X = \frac{x}{(12t)^{1/3}}.$$

(ii) Hence show that a solution for F is

$$F(x,z;t) = \int_{-\infty}^{\infty} f(yt^{1/3})\mathrm{Ai}(x+y)\mathrm{Ai}(y+z)\,\mathrm{d}y,$$

where f is an arbitrary function and Ai is the Airy function.

Q4.3 *Two-soliton solution of the KdV equation.* Obtain a solution for $F(x,z;t)$ (see Q4.1) which depends upon $x+z$ (but not $x-z$ since this would become trivial on $z=x$), and which is exponential in both $x+z$ and t (cf. examples (*i*) and (*ii*), §4.5).

Hence write down a solution for F which is the sum of two exponential terms, and construct the two-soliton solution of the KdV equation.

Q4.4 *Some initial-value problems.* Use the inverse scattering transform to find the solution of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

which satisfies $u(x,0) = f(x)$, $-\infty < x < \infty$, where

(*i*) $f(x) = -\frac{9}{2}\mathrm{sech}^2(\frac{3}{2}x)$;

(*ii*) $f(x) = -12\,\mathrm{sech}^2 x$;

*(*iii*) $f(x) = \begin{cases} -V \text{ for } -1 < x < 1, \\ 0 \text{ otherwise,} \end{cases}$

where $V > 0$ is a constant.

[Case (*iii*) is too difficult to solve explicitly, so just give a qualitative description of the solution for various V.]

Q4.5 *The character of two-soliton solutions.* Show that a special case of the two-soliton solution obtained in Q4.3 gives a sech^2 pulse at $t=0$ (see example (ii), §4.5). Show also that, for suitably defined x, the pulse at $t=0$ may have either one or two local maxima.

[You will find it convenient to define x so that a symmetric profile occurs at $t=0$.]

(Lax, 1968)

Q4.6 *Three-soliton solution.* Find the asymptotic form of the three-soliton solution (see Q4.4, (*ii*)) as $t \to \pm\infty$, and hence determine the phase shifts.

Q4.7 *Connection with Fourier transforms.* Consider the initial-value problem