Q5.10 *N-soliton solution.* Assume that $u(x, t)$ evolves, according to the KdV equation, into an N-soliton solution from a given initial profile, $u(x, 0)$. Consider the profile at $t = 0$, and the solution as $t \to \infty$, and hence show how the conserved quantities can be used to determine the amplitudes of the resulting solitons. Use the first two conservation laws, and then the first three, to verify your method for the two-soliton and three-soliton solutions, respectively.

(Berezin & Karpman, 1967)

Q5.11 *Unitarily equivalent operators.* Let L be a linear operator which acts on a Hilbert space H, and which depends on a parameter t. Define the adjoint, $\hat{\mathrm{U}}$, of a linear operator U on H, so that the inner product $(u, \mathrm{U}v) = (\hat{\mathrm{U}}u, v)$ for all $u, v \in \mathrm{H}$. Show that, if L(t) is *unitarily equivalent* to L(0), i.e. if

$$\hat{\mathrm{U}}(t)\mathrm{L}(t)\mathrm{U}(t) = \mathrm{L}(0) \qquad \text{and} \qquad \mathrm{U}(t)\hat{\mathrm{U}}(t) = \mathrm{I},$$

where I is the identity operator, then L(t) has the same eigenvalues as L(0).

Assuming that such an operator U does exist, show that there exists an operator M on H such that

$$\mathrm{U}_t = \mathrm{MU} \qquad \text{where } \hat{\mathrm{M}} = -\mathrm{M}.$$

Hence deduce that, if L is self-adjoint, then

$$\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0.$$

(Lax, 1968)

Q5.12 *Matrix Lax equation.* Show that, if L(t) and M(t) are $n \times n$ complex-valued matrices such that

$$\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0,$$

where L is Hermitian and $\lambda(t)$ is a real eigenvalue of L, then λ is constant.

Q5.13 *An ordinary differential system.* Cast the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = gy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -gx,$$

where $g(x, y, t)$ is a given continuous function, into the equivalent form

$$\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0,$$

finding L as some symmetric, and M as some antisymmetric, real matrix whose elements depend upon x, y and g (see Q5.12). Show that the eigenvalues of L are constant and hence deduce the (otherwise obvious) result that $x^2 + y^2$ is constant for each solution of the system.

Q5.14 *Two-dimensional KdV equation.* Show that the equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

can be obtained by choosing

$$\mathrm{L} = -\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} + u; \qquad \mathrm{M} = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x + 3\int^x u_y \,\mathrm{d}x.$$