

where F is a solution of the pair of equations

$$\begin{aligned} F_{xx} - F_{zz} &= (x - z)F, \\ 3tF_t - F + F_{xxx} + F_{zzz} &= xF_x + zF_z, \end{aligned}$$

then

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0,$$

where

$$u(x, t) = -\frac{2}{(12t)^{2/3}} \frac{\partial}{\partial X} K(X, X; t) \quad \text{with } X = \frac{x}{(12t)^{1/3}}.$$

(ii) Hence show that a solution for F is

$$F(x, z; t) = \int_{-\infty}^{\infty} f(yt^{1/3}) \text{Ai}(x + y) \text{Ai}(y + z) dy,$$

where f is an arbitrary function and Ai is the Airy function.

Q4.3 *Two-soliton solution of the KdV equation.* Obtain a solution for $F(x, z; t)$ (see Q4.1) which depends upon $x + z$ (but not $x - z$ since this would become trivial on $z = x$), and which is exponential in both $x + z$ and t (cf. examples (i) and (ii), §4.5).

Hence write down a solution for F which is the sum of two exponential terms, and construct the two-soliton solution of the KdV equation.

Q4.4 *Some initial-value problems.* Use the inverse scattering transform to find the solution of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

which satisfies $u(x, 0) = f(x)$, $-\infty < x < \infty$, where

$$(i) f(x) = -\frac{9}{2} \text{sech}^2\left(\frac{3}{2}x\right);$$

$$(ii) f(x) = -12 \text{sech}^2 x;$$

$$*(iii) f(x) = \begin{cases} -V & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $V > 0$ is a constant.

[Case (iii) is too difficult to solve explicitly, so just give a qualitative description of the solution for various V .]

Q4.5 *The character of two-soliton solutions.* Show that a special case of the two-soliton solution obtained in Q4.3 gives a sech^2 pulse at $t = 0$ (see example (ii), §4.5). Show also that, for suitably defined x , the pulse at $t = 0$ may have either one or two local maxima.

[You will find it convenient to define x so that a symmetric profile occurs at $t = 0$.]

(Lax, 1968)

Q4.6 *Three-soliton solution.* Find the asymptotic form of the three-soliton solution (see Q4.4, (ii)) as $t \rightarrow \pm \infty$, and hence determine the phase shifts.

Q4.7 *Connection with Fourier transforms.* Consider the initial-value problem