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M7160 Obyčejné diferenciální rovnice II

M7160 Ordinary Differential Equations II

Michal Veselý
Text vychází z přednášek prof. Lomtatidze

Masarykova univerzita, Přírodovědecká fakulta
Ústav matematiky a statistiky

Se sazbou v systému LaTeX pomáhali Jakub Juránek a Jiřina Šišoláková

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1. Auxiliary results

1.1 Notations

$\mathbb{R}_+$	non-negative real numbers
$x = (x_i)_{i=1}^m$	column vectors with coordinates $x_1, \dots, x_m$
$\mathbb{R}^m$	the space of real vectors $x = (x_i)_{i=1}^m$ with the norm $$\|x\| = \sum_{i=1}^m |x_i|$$
$B[x_0, r]$	the closed ball with the centre $x_0 \in \mathbb{R}^m$ and the radius $r \geq 0$, i.e., $$B[x_0, r] = \{x \in \mathbb{R}^m; \|x - x_0\| \leq r\}$$
$x \cdot y$	the Euclidean inner product of vectors $x, y \in \mathbb{R}^m$
$x \leq y$	the inequality between vectors $x = (x_i)_{i=1}^m, y = (y_i)_{i=1}^m \in \mathbb{R}^m$ such that $x_i \leq y_i$, $i \in \{1, \dots, m\}$
$\operatorname{sgn} x$	the vector $(\operatorname{sgn} x_i)_{i=1}^m$
$[a, b]$	the closed interval
(a, b)	the open interval
$m(A)$	the Lebesgue measure of a set $A \subseteq \mathbb{R}^m$
$\operatorname{int}(A)$	the interior of a set $A \subseteq \mathbb{R}^m$
∂A	the boundary of a set $A \subseteq \mathbb{R}^m$
I	a real interval, which is not degenerated to a point
$C(I, \mathbb{R}^m)$	the space of continuous and bounded vector functions $u\colon I \to \mathbb{R}^m$ with the norm $$\|u\|_C = \sup\{\|u(t)\|; t \in I\}$$
$C_{loc}(I, \mathbb{R}^m)$	the set of continuous vector functions $u\colon I \to \mathbb{R}^m$
$\tilde{C}([a, b], \mathbb{R}^m)$	the set of absolutely continuous vector functions $u\colon [a, b] \to \mathbb{R}^m$
$\tilde{C}_{loc}(I, \mathbb{R}^m)$	the set of vector functions $u\colon I \to \mathbb{R}^m$, which are absolutely continuous on any compact subinterval $[a, b] \subseteq I$
$\tilde{C}^n([a, b], \mathbb{R})$	the set of all functions $u\colon [a, b] \to \mathbb{R}$, whose n-th derivatives are absolutely continuous
$\tilde{C}^n_{loc}(I, \mathbb{R})$	the set of all functions $u\colon I \to \mathbb{R}$ such that $u \in \tilde{C}^n([a, b], \mathbb{R})$ for all interval $[a, b] \subseteq I$
$L(I, \mathbb{R}^m)$	the space of all vector functions $u\colon I \to \mathbb{R}^m$ which are strongly integrable in the Lebesgue sense with the norm $$\|u\|_L = \int_I \|u(s)\| \, \mathrm{d}s$$
$L_{loc}(I, \mathbb{R}^m)$	the set of all vector functions $u\colon I \to \mathbb{R}^m$ such that $u \in L([a, b], \mathbb{R}^m)$ for all interval $[a, b] \subseteq I$

Other notations are given by the range of considered values. For example, $L(I,D)$ is the set

$$\{u\colon I \to D; u \in L(I,\mathbb{R}^m)\},$$

where $D \subseteq \mathbb{R}^m$.

1.2 Carathéodory class

Definition 1.1. Let $A \subseteq \mathbb{R}^m$ and $D \subseteq \mathbb{R}^n$ be given. We say that a vector function $g\colon I \times A \to D$ belongs to the Carathéodory class and we write $g \in K(I \times A, D)$ if the following conditions:

1. the function $g(t,-)\colon A \to D$ is continuous for almost all $t \in I$;
2. the function $g(-,x)\colon I \to D$ is measurable for all $x \in A$;
3. for all $r > 0$, there exists $h_r \in L(I,\mathbb{R}_+)$ such that

$$\|g(t,x)\| \le h_r(t), \qquad t \in I,\, x \in A \cap B[0,r],$$

are satisfied.

Definition 1.2. We say that a vector function $g\colon I \times \mathbb{R}^m \to \mathbb{R}^n$ is from the set

$$K_{loc}(I \times \mathbb{R}^m, \mathbb{R}^n)$$

if

$$g \in K([a,b] \times \mathbb{R}^m, \mathbb{R}^n)$$

for all $[a,b] \subseteq I$.

Lemma 1.1. *If $g \in K(I \times \mathbb{R}^m, \mathbb{R}^n)$ and if $x \in C_{loc}(I,\mathbb{R}^m)$, then the vector function*

$$t \mapsto g(t,x(t)), \qquad t \in I,$$

is measurable.

Lemma 1.2. *Let $g \in K(I \times \mathbb{R}^m, \mathbb{R}^n)$ be given and let $A \subset \mathbb{R}^m$ be bounded. Let*

$$g^\star(t) = \sup\{\|g(t,x)\|; x \in A\}, \qquad t \in I.$$

Then, $g^\star \in L(I,\mathbb{R}_+)$.

Lemma 1.3. *Let $g \in K(I \times \mathbb{R}^m, \mathbb{R}^n)$. Then, there exists a function $h \in L(I,\mathbb{R}_+)$ and a non-decreasing function $\varphi \in C_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ such that*

$$\|g(t,x)\| \le h(t)\,\varphi(\|x\|), \qquad t \in I,\, x \in \mathbb{R}^m.$$

Lemma 1.4. *Let $g \in K(I \times \mathbb{R}^m, \mathbb{R}^n)$. For all $r > 0$, there exists a function $h_r \in L(I,\mathbb{R}_+)$ and a non-decreasing function $\varphi_r \in C_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ such that $\varphi_r(0) = 0$ and*

$$\|g(t,x) - g(t,y)\| \le h_r(t)\,\varphi_r(\|x-y\|), \qquad t \in I,\, x,y \in B[0,r].$$

Lemma 1.5. *Let $g \in K(I \times \mathbb{R}^m, \mathbb{R}^n)$. The operator*

$$F(x)(-) = g(-, x(-)), \qquad x \in C(I, \mathbb{R}^m),$$

maps the space $C(I, \mathbb{R}^m)$ into the space $L(I, \mathbb{R}^n)$ continuously.

Proof. For all $x \in C(I, \mathbb{R}^m)$, the function $F(x): I \to \mathbb{R}^n$ is measurable (see Lemma 1.1). According to Definition 1.1, we have that $F(x) \in L(I, \mathbb{R}^n)$. Thus, the operator maps the space $C(I, \mathbb{R}^m)$ into the space $L(I, \mathbb{R}^n)$.

Let $\{x_k\}_{k=1}^{\infty} \subseteq C(I, \mathbb{R}^m)$ and $x \in C(I, \mathbb{R}^m)$ satisfy

$$\lim_{k \to \infty} \|x_k - x\|_C = 0.$$

Let $r > 0$ be such that

$$\|x_k(t)\| \le r, \qquad \|x(t)\| \le r, \qquad t \in I, \, k \in \mathbb{N}.$$

Therefore, there exists a function $h_r \in L(I, \mathbb{R}_+)$ such that (see Lemma 1.4)

$$\|g(t, x_k(t)) - g(t, x(t))\| \le h_r(t), \qquad t \in I, \, k \in \mathbb{N},$$

i.e.,

$$\|F(x_k)(t) - F(x)(t)\| \le h_r(t), \qquad t \in I, \, k \in \mathbb{N}.$$

Since the function $g(t, -): \mathbb{R}^m \to \mathbb{R}^n$ is continuous for almost all $t \in I$, it holds

$$\lim_{k \to \infty} [F(x_k)(t) - F(x)(t)] = 0$$

for almost all $t \in I$. By the Lebesgue theorem, we have

$$\|F(x_k) - F(x)\|_L = \int_I \|F(x_k)(s) - F(x)(s)\| \, \mathrm{d}s \to 0 \text{ as } k \to \infty.$$

Thus, the operator F is continuous. $\square$

Lemma 1.6. *Let $g \in K(I \times \mathbb{R}^m, \mathbb{R}^n)$. For arbitrary $r > 0$, let a function $h_r \in L(I, \mathbb{R}_+)$ satisfy*

$$\|g(t, x)\| \le h_r(t), \qquad t \in I, \, x \in B[0, r].$$

Then, there exist functions $g_k: I \times \mathbb{R}^m \to \mathbb{R}^n$ for $k \in \mathbb{N}$ such that:

1. *all functions g_k have all partial derivatives with respect to the last m variables and all functions g_k and their partial derivatives belong to the class*

   $$K(I \times \mathbb{R}^m, \mathbb{R}^n);$$

2. *for all $r > 0$, the inequality*

   $$\|g_k(t, x)\| \le h_{r+1}(t), \qquad t \in I, \, x \in B[0, r], \, k \in \mathbb{N},$$

   holds;

3. *for almost all $t \in I$ and all $r > 0$, it holds*

   $$\lim_{k \to \infty} g_k(t, x) = g(t, x)$$

   uniformly in $B[0, r]$.

Remark 1.1. If
$$\|g(t,x)\| \le h(t), \qquad t \in I,\, x \in \mathbb{R}^m,$$
then one can assume that $h_r \equiv h$ for all $r > 0$.

Remark 1.2. The functions g_k from the statement of Lemma 1.6 have continuous partial derivatives. Therefore, they are locally Lipschitz, i.e., for all $r > 0$ and $k \in \mathbb{N}$, there exists a function $l_{r,k} \in L(I, \mathbb{R}_+)$ such that
$$\|g_k(t,x) - g_k(t,y)\| \le l_{r,k}(t)\, \|x - y\|, \qquad t \in I,\, x, y \in B[0,r].$$

Remark 1.3. The functions g_k from Lemma 1.6 can be chosen in such a way that they have the following property. For all $r > 0$, there exists a function $\omega_r \in K(I \times \mathbb{R}_+, \mathbb{R}_+)$ which is non-decreasing with respect to the second variable, $\omega_r(-,0) \equiv 0$, and
$$\|g_k(t,x) - g_k(t,y)\| \le \omega_r(t, \|x-y\|) + \omega_r\left(t, \frac{1}{k}\right), \qquad t \in I,\, x, y \in B[0,r],\, k \in \mathbb{N}.$$

Proof of Lemma 1.6. Let $\varphi_k \colon \mathbb{R}^m \to \mathbb{R}_+$ for $k \in \mathbb{N}$ be functions satisfying the following conditions:

a) the functions φ_k have continuous all partial derivatives on $\mathbb{R}^m$;

b) the functions φ_k satisfy
$$\varphi_k(x) = 0, \qquad x \in \mathbb{R}^m,\, \|x\| \ge \frac{1}{k},\, k \in \mathbb{N};$$

c) it holds
$$\int\limits_{\mathbb{R}^m} \varphi_k(x)\, \mathrm{d}x = 1, \qquad k \in \mathbb{N}.$$

Such functions can be constructed as follows. Let
$$\varphi(s) = \begin{cases} \mathrm{e}^{-\frac{1}{1-s}}, & 0 \le s < 1; \\ 0, & s \ge 1. \end{cases}$$

Let $\rho_k > 0$, $k \in \mathbb{N}$, be such that
$$\rho_k \int\limits_{\mathbb{R}^m} \varphi\left(m^2 k^2 x \cdot x\right) \mathrm{d}x = 1.$$

We put
$$\varphi_k(x) = \rho_k\, \varphi\left(m^2 k^2 x \cdot x\right), \qquad x \in \mathbb{R}^m,\, k \in \mathbb{N}.$$

Obviously, a) and c) are valid. It is known that
$$\|x\|^2 \le m^2 (x \cdot x), \qquad x \in \mathbb{R}^m.$$

If
$$\|x\| \ge \frac{1}{k},$$

then
$$m^2 k^2 (x \cdot x) \ge 1$$

and, consequently, $\varphi_k(x) = 0$. Therefore, b) is valid as well.

We define

$$g_k(t,x) = \int\limits_{\mathbb{R}^m} \varphi_k(y-x) g(t,y)\,\mathrm{d}y, \qquad t \in I,\, x \in \mathbb{R}^m,\, k \in \mathbb{N}.$$

The functions g_k satisfy the condition 1. Let $r > 0$ and $x \in B[0,r]$ be arbitrarily given. We have

$$g_k(t,x) = \int\limits_{B[0,r+1]} \varphi_k(y-x) g(t,y)\,\mathrm{d}y, \qquad t \in I,\, k \in \mathbb{N}.$$

We obtain

$$\begin{aligned}
\|g_k(t,x)\| &\le \int\limits_{B[0,r+1]} \varphi_k(y-x)\,\|g(t,y)\|\,\mathrm{d}y \\
&\le h_{r+1}(t) \int\limits_{B[0,r+1]} \varphi_k(y-x)\,\mathrm{d}y \\
&\le h_{r+1}(t) \int\limits_{\mathbb{R}^m} \varphi_k(y)\,\mathrm{d}y = h_{r+1}(t), \qquad t \in I,\, k \in \mathbb{N},
\end{aligned}$$

which gives the condition 2.

It remains to prove the condition 3. Let $r > 0$ and $x \in B[0,r]$ be arbitrarily given. Since

$$\int\limits_{\mathbb{R}^m} \varphi_k(y-x)\,\mathrm{d}y = 1, \qquad k \in \mathbb{N},$$

we have

$$g(t,x) = g(t,x) \int\limits_{\mathbb{R}^m} \varphi_k(y-x)\,\mathrm{d}y = \int\limits_{\mathbb{R}^m} \varphi_k(y-x) g(t,x)\,\mathrm{d}y, \qquad t \in I,\, k \in \mathbb{N}.$$

Therefore,

$$\begin{aligned}
g_k(t,x) - g(t,x) &= \int\limits_{\mathbb{R}^m} \varphi_k(y-x)\,[g(t,y) - g(t,x)]\,\mathrm{d}y \\
&= \int\limits_{B\left[x,\frac{1}{k}\right]} \varphi_k(y-x)\,[g(t,y) - g(t,x)]\,\mathrm{d}y, \qquad t \in I,\, k \in \mathbb{N}.
\end{aligned} \tag{1.1}$$

Due to Lemma 1.4, there exists a function $\omega_{r+1} \in K(I \times \mathbb{R}_+, \mathbb{R}_+)$ which is non-decreasing in the second variable, $\omega_{r+1}(-,0) \equiv 0$, and

$$\|g(t,y_1) - g(t,y_2)\| \le \omega_{r+1}(t, \|y_1 - y_2\|), \qquad t \in I,\, y_1, y_2 \in B[0,r+1].$$

Therefore,

$$\|g(t,x) - g(t,y)\| \le \omega_{r+1}\left(t, \frac{1}{k}\right), \qquad t \in I,\, y \in B\left[x, \frac{1}{k}\right].$$

From (1.1), we have

$$
\begin{aligned}
\|g_k(t,x)-g(t,x)\| &\leq \int\limits_{B\left[x,\frac{1}{k}\right]} \varphi_k(y-x)\,\|g(t,y)-g(t,x)\|\,\mathrm{d}y \\
&\leq \omega_{r+1}\left(t,\frac{1}{k}\right) \int\limits_{B\left[x,\frac{1}{k}\right]} \varphi_k(y-x)\,\mathrm{d}y \\
&\leq \omega_{r+1}\left(t,\frac{1}{k}\right) \int\limits_{\mathbb{R}^m} \varphi_k(y-x)\,\mathrm{d}y = \omega_{r+1}\left(t,\frac{1}{k}\right), \qquad t\in I,\, k\in\mathbb{N}.
\end{aligned}
$$

Thus, 3. is valid. □

1.3 Absolute continuity

Definition 1.3. We say that a function $x\colon [a,b]\to\mathbb{R}^m$ is absolutely continuous on $[a,b]$ if, for every $\varepsilon>0$, there exists $\delta>0$ such that, for any finite system of pairwise disjoint subintervals

$$(a_k,b_k)\subseteq[a,b], \qquad k\in\{1,\dots,n\},$$

satisfying

$$\sum_{k=1}^{n}(b_k-a_k)<\delta,$$

it holds

$$\sum_{k=1}^{n}\|x(b_k)-x(a_k)\|<\varepsilon.$$

We recall that the set of all absolutely continuous functions $x\colon [a,b]\to\mathbb{R}^m$ is denoted by

$$\tilde{C}([a,b],\mathbb{R}_m).$$

Theorem 1.1. *A function $x\colon [a,b]\to\mathbb{R}^m$ is called absolutely continuous if and only if the following conditions:*

1. *the function x is differentiable almost everywhere in $[a,b]$;*
2. *the derivative*
$$x'\in L([a,b],\mathbb{R}^m);$$
3. *it holds*
$$\int\limits_{\alpha}^{\beta} x'(s)\,\mathrm{d}s = x(\beta)-x(\alpha), \qquad \alpha,\beta\in[a,b],$$

are satisfied.

Remark 1.4. Let $t_0 \in [a,b]$ be arbitrarily given. If $h \in L([a,b],\mathbb{R}^m)$, then the function $x\colon [a,b] \to \mathbb{R}^m$ given by

$$x(t) = \int_{t_0}^{t} h(s)\,\mathrm{d}s, \qquad t \in [a,b],$$

is absolutely continuous and $x'(t) = h(t)$ for almost all $t \in [a,b]$.

2. Existence of solutions

Let us consider the equation

$$x' = f(t,x), \tag{2.1}$$

where $f \in K_{loc}(I \times \mathbb{R}^n, \mathbb{R}^n)$.

Definition 2.1. Let $I_0 \subseteq I$ be an interval. We say that a function $x\colon I_0 \to \mathbb{R}^n$ is a solution of Eq. (2.1) if:

1. $x \in \tilde{C}_{loc}(I_0, \mathbb{R}^n)$;
2. it holds
$$x'(t) = f(t, x(t))$$
for almost all $t \in I_0$.

Let $t_0 \in I_0$, $c_0 \in \mathbb{R}^n$. A solution $x\colon I_0 \to \mathbb{R}^n$ of Eq. (2.1) satisfying the condition

$$x(t_0) = c_0 \tag{2.2}$$

is called the solution of the Cauchy problem (2.1), (2.2).

Definition 2.2. Let A, B be sets of functions $x\colon I \to \mathbb{R}^n$ and let $t_0 \in I$. An operator $T\colon A \to B$ is called t_0-Volterra if, for all $x, y \in A$ and all $t \in I$ such that

$$x(s) = y(s), \qquad \min\{t_0, t\} \le s \le \max\{t_0, t\},$$

it holds

$$T(x)(t) = T(y)(t).$$

Lemma 2.1. *Let $t_0 \in [a,b]$, $c_0 \in \mathbb{R}^n$, $r > 0$, and let*

$$T\colon C([a,b], B[c_0, r]) \to C([a,b], B[c_0, r])$$

be a continuous t_0-Volterra operator such that $T(c_0)(t_0) = c_0$. Let there exist a function $\omega \in C([0, b-a], \mathbb{R}_+)$ such that $\omega(0) = 0$ and that

$$\|T(x)(t) - T(x)(s)\| \le \omega(|t-s|), \qquad t, s \in [a,b],\, x \in C([a,b], B[c_0, r]).$$

Then, the operator T has at least one fixed point, i.e., there exists a function

$$x \in C([a,b], B[c_0, r])$$

such that

$$T(x)(t) = x(t), \qquad t \in [a,b].$$

Proof. Without loss of generality, we can assume that the function ω is non-decreasing. We denote

$$I_{k,j} = \left[t_0 - \frac{j(t_0 - a)}{k}, t_0 + \frac{j(b - t_0)}{k}\right], \qquad j \in \{1, \dots, k-1\},\, k \in \mathbb{N}.$$

For $k \in \mathbb{N}$ and $t \in [a,b]$, we define

$$s_k(t) = \begin{cases} t + \dfrac{t_0 - a}{k}, & t < t_0 - \dfrac{t_0 - a}{k}; \\ t_0, & t \in I_{k,1}; \\ t - \dfrac{b - t_0}{k}, & t > t_0 + \dfrac{b - t_0}{k}. \end{cases}$$

Obviously, the functions $s_k \colon [a,b] \to [a,b]$, $k \in \mathbb{N}$, are continuous and

$$|s_k(t) - s_k(\tau)| \le |t - \tau|, \qquad t, \tau \in [a,b],\, k \in \mathbb{N}.$$

For $t \in [a,b]$, we denote

$$y_{k,0}(t) = c_0$$

and

$$y_{k,j}(t) = T(y_{k,j-1})(s_k(t)), \qquad j \in \{1, \ldots, k-1\},\, k \in \mathbb{N}.$$

We show that

$$y_{k,j}(t) = y_{k,j-1}(t), \qquad t \in I_{k,j},\, j \in \{1, \ldots, k-1\},\, k \ge 2,\, k \in \mathbb{N}. \tag{2.3}$$

For an arbitrarily given integer $k \ge 2$, using the induction, we prove that the identity

$$y_{k,j}(t) = y_{k,j-1}(t), \qquad t \in I_{k,j}, \tag{2.4}$$

is valid for all $j \in \{1, \ldots, k-1\}$. Obviously,

$$y_{k,1}(t) = T(y_{k,0})(s_k(t)) = T(c_0)(s_k(t)) = T(c_0)(t_0) = c_0 = y_{k,0}(t), \qquad t \in I_{k,1}.$$

We assume that (2.4) is valid for some $j \in \{1, \ldots, k-2\}$. We show that (2.4) is also valid for $j+1$. If $t \in I_{k,j+1}$, then $s_k(t) \in I_{k,j}$. Therefore,

$$y_{k,j+1}(t) = T(y_{k,j})(s_k(t)) = T(y_{k,j-1})(s_k(t)) = y_{k,j}(t), \qquad t \in I_{k,j+1}.$$

Hence, (2.4) is valid for all $j \in \{1, \ldots, k-1\}$. By the induction, we have proved (2.3).

We denote

$$x_k(t) = y_{k,k-1}(t), \qquad t \in [a,b],\, k \in \mathbb{N}.$$

One can see that

$$x_k \in C([a,b], B[c_0, r]), \qquad k \in \mathbb{N},$$

and that (see (2.3))

$$x_k(t) = y_{k,k-2}(t), \qquad t \in I_{k,k-1},\, k \ge 2,\, k \in \mathbb{N}.$$

Since

$$s_k(t) \in I_{k,k-1}, \qquad t \in [a,b],\, k \ge 2,\, k \in \mathbb{N},$$

we obtain

$$x_k(t) = T(y_{k,k-2})(s_k(t)) = T(x_k)(s_k(t)), \qquad t \in [a,b],\, k \ge 2,\, k \in \mathbb{N}. \tag{2.5}$$

Next, we get

$$\begin{aligned} \|x_k(t) - x_k(\tau)\| &= \|T(x_k)(s_k(t)) - T(x_k)(s_k(\tau))\| \\ &\le \omega(|s_k(t) - s_k(\tau)|) \\ &\le \omega(|t - \tau|), \qquad t, \tau \in [a,b],\, k \ge 2,\, k \in \mathbb{N}. \end{aligned}$$

Therefore, the functions x_k, $k \geq 2$, $k \in \mathbb{N}$, are uniformly bounded and equicontinuous. We can use the Arzelà–Ascoli theorem.

Without loss of generality, we can assume that

$$\lim_{k\to\infty} x_k(t) = x(t)$$

uniformly for $t \in [a,b]$, where

$$x \in C([a,b], B[c_0,r]).$$

We have

$$\begin{aligned}\|T(x_k)(s_k(t)) - T(x)(t)\| &\leq \|T(x_k)(s_k(t)) - T(x_k)(t)\| + \|T(x_k)(t) - T(x)(t)\| \\ &\leq \omega(|s_k(t) - t|) + \|T(x_k)(t) - T(x)(t)\|\end{aligned}$$

for all $t \in [a,b]$, $k \in \mathbb{N}$. Since the operator T is continuous, $\omega(0) = 0$, and since

$$\lim_{k\to\infty} s_k(t) = t, \qquad t \in [a,b],$$

we obtain (see (2.5))

$$x(t) = \lim_{k\to\infty} T(x_k)(s_k(t)) = T(x)(t), \qquad t \in [a,b],$$

i.e., x is a fixed point of the operator T. □

Theorem 2.1. *Let $r > 0$ and $[a,b] \subseteq I$ be such that $t_0 \in [a,b]$ and*

$$\left|\int_{t_0}^{t} f^{\star}_{c_0}(s,r)\,\mathrm{d}s\right| \leq r, \qquad t \in [a,b],$$

where

$$f^{\star}_{c_0}(t,r) = \sup\{\|f(t,x)\|; x \in B[c_0,r]\}, \qquad t \in [a,b].$$

Then, the problem (2.1)*,* (2.2) *has a solution on* $[a,b]$.

Proof. The problem (2.1), (2.2) is equivalent with the equation

$$x(t) = c_0 + \int_{t_0}^{t} f(\tau, x(\tau))\,\mathrm{d}\tau, \qquad t \in [a,b].$$

Especially, if $x \in C([a,b],\mathbb{R}^n)$ satisfies this integral equation, then $x \in \tilde{C}([a,b],\mathbb{R}^n)$. Therefore, it suffices to prove that there exists a function $x \in C([a,b],\mathbb{R}^n)$ satisfying this integral equation.

We define

$$T(x)(t) = c_0 + \int_{t_0}^{t} f(\tau, x(\tau))\,\mathrm{d}\tau, \qquad t \in [a,b],\, x \in C([a,b], B[c_0,r]).$$

It is obvious that the operator

$$T\colon C([a,b], B[c_0,r]) \to C([a,b],\mathbb{R}^n)$$

is t_0-Volterra. According to Lemma 1.5, the operator T is continuous. For

$$x \in C([a,b], B[c_0, r]),$$

we have

$$\|T(x)(t) - c_0\| \le \left| \int_{t_0}^{t} f_{c_0}^{\star}(\tau, r)\, \mathrm{d}\tau \right| \le r, \qquad t \in [a,b],$$

i.e.,

$$T(x)(t) \in B[c_0, r], \qquad t \in [a,b].$$

Thus,

$$T\colon C([a,b], B[c_0, r]) \to C([a,b], B[c_0, r]).$$

Further,

$$\|T(x)(t) - T(x)(s)\| \le \left| \int_{s}^{t} f_{c_0}^{\star}(\tau, r)\, \mathrm{d}\tau \right|, \qquad s,t \in [a,b],\, x \in C([a,b], B[c_0, r]).$$

We denote

$$\omega(\delta) = \max\left\{ \int_{t}^{t+\delta} f_{c_0}^{\star}(\tau, r)\, \mathrm{d}\tau; t \in [a, b-\delta] \right\}, \qquad \delta \in [0, b-a].$$

The function $\omega\colon [0, b-a] \to \mathbb{R}_+$ is continuous, $\omega(0) = 0$, and

$$\|T(x)(t) - T(x)(s)\| \le \omega(|t-s|), \qquad s,t \in [a,b],\, x \in C([a,b], B[c_0, r]).$$

Due to Lemma 2.1, there exists a function $x \in C([a,b], B[c_0, r])$ with the required property. □

Corollary 2.1. *For arbitrary $t_0 \in I$ and $c_0 \in \mathbb{R}^n$, there exists an interval $I_0 \subseteq I$ such that $t_0 \in I_0$ and the problem* (2.1), (2.2) *has at least one solution on the interval I_0. Moreover, if t_0 is an interior point of I, then the interval I_0 can be chosen so that t_0 is an interior point of I_0.*

Proof. The statement of the corollary follows from Theorem 2.1. □

Now, we consider the equation

$$u^{(n)} = f\left(t, u, u', \ldots, u^{(n-1)}\right), \tag{2.6}$$

where $f \in K_{loc}(I \times \mathbb{R}^n, \mathbb{R})$.

Definition 2.3. Let $I_0 \subseteq I$ be an interval. We say that a function $u\colon I_0 \to \mathbb{R}$ is a solution of Eq. (2.6) on I_0 if

1. $u \in \tilde{C}_{loc}^{n-1}(I_0, \mathbb{R})$;
2. it holds
$$u^{(n)}(t) = f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\right)$$
for almost all $t \in I_0$.

Let $t_0 \in I$, $c_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n-1\}$. A solution $u\colon I_0 \to \mathbb{R}$ of Eq. (2.6) satisfying the condition

$$u^{(i)}(t_0) = c_i, \qquad i \in \{0, 1, \ldots, n-1\}, \tag{2.7}$$

is called the solution of the Cauchy problem (2.6), (2.7).

Corollary 2.2. *For arbitrary $t_0 \in I$ and $c_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n-1\}$, there exists an interval $I_0 \subseteq I$ such that $t_0 \in I_0$ and that the problem* (2.6), (2.7) *has at least one solution on I_0. Moreover, if t_0 is an interior point of I, then the interval I_0 can be chosen so that t_0 is an interior point of I_0.*

Proof. The statement of the corollary follows from Corollary 2.1. □

3. Extendability of solutions

We consider the Cauchy problem

$$x' = f(t,x), \tag{3.1}$$

$$x(t_0) = c_0, \tag{3.2}$$

where $f \in K_{loc}(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

Definition 3.1. Let x be a solution of Eq. (3.1) on an interval $(a,b) \subseteq I$. We say that the solution x is right-extendable if there exist $b_1 > b$, $b_1 \in I$, and a solution y of Eq. (3.1) on the interval $(a,b_1) \subseteq I$ such that $y(t) = x(t)$, $t \in (a,b)$. The solution y is called a right-extension of the solution x. If any right-extension of the solution x does not exist, then we say that the solution x is not right-extendable.

Analogously, left-extendable solutions (and solutions which are not left-extendable) are defined. We say that a solution x is extendable if it is right-extendable or left-extendable. In the opposite case, we say that x is non-extendable.

Lemma 3.1. *Let $(\alpha,\beta) \subseteq I$, $r > 0$, $\delta \geq 0$, $c \in \mathbb{R}^n$, and let*

$$\delta + \int_{\alpha}^{\beta} f_c^\star(\tau,r)\,\mathrm{d}\tau < r,$$

where

$$f_c^\star(t,r) = \sup\{\|f(t,x)\|\,; x \in B[c,r]\}, \qquad t \in (\alpha,\beta).$$

Let x be a solution of Eq. (3.1) *on (α,β) satisfying*

$$\inf\{\|x(t)-c\|\,; t \in (\alpha,\beta)\} \leq \delta.$$

Then,

$$\|x(t)-c\| < r, \qquad t \in (\alpha,\beta),$$

and the limits

$$\lim_{t\to\alpha^+} x(t), \qquad \lim_{t\to\beta^-} x(t)$$

exist.

Proof. We prove the lemma by contradiction. There exists a point $t_0 \in (\alpha,\beta)$ such that

$$\|x(t_0)-c\| + \int_{\alpha}^{\beta} f_c^\star(\tau,r)\,\mathrm{d}\tau < r. \tag{3.3}$$

In addition, there exists $[\alpha_0,\beta_0] \subseteq (\alpha,\beta)$ such that $t_0 \in [\alpha_0,\beta_0]$ and that

$$\max\{\|x(t)-c\|\,; t \in [\alpha_0,\beta_0]\} = r. \tag{3.4}$$

From (3.1), we obtain

$$x(t) - c = x(t_0) - c + \int_{t_0}^{t} f(s,x(s))\,\mathrm{d}s, \qquad t \in (\alpha,\beta).$$

Therefore,

$$\|x(t)-c\| \leq \|x(t_0)-c\| + \int\limits_{\alpha_0}^{\beta_0} f_c^\star(s,r)\,\mathrm{d}s, \qquad t \in [\alpha_0, \beta_0]. \tag{3.5}$$

From (3.3) and (3.5), it follows that

$$\|x(t)-c\| < r, \qquad t \in [\alpha_0, \beta_0].$$

This is a contradiction with (3.4).

It remains to prove the existence of the limits. Since

$$f(-,x(-)) \in L((\alpha,\beta),\mathbb{R}^n),$$

the existence of the limits comes from

$$x(t) = x(t_0) + \int\limits_{t_0}^{t} f(s,x(s))\,\mathrm{d}s, \qquad t \in (\alpha,\beta).$$

□

Theorem 3.1. *Let x be a solution of Eq.* (3.1) *on an interval* $(a,b) \subseteq I$. *Then, x is right-extendable if and only if $b < \sup I$ and*

$$\liminf_{t\to b^-} \|x(t)\| < \infty.$$

Proof. Let $b < \sup I$ and let

$$\lim_{t\to b^-} \|x(t)\| \neq \infty.$$

There exists $c \in \mathbb{R}^n$ such that

$$\liminf_{t\to b^-} \|x(t)-c\| = 0.$$

We put $r = 1$, $\delta = 0$, and $\beta = b$. Let $\alpha \in (a,\beta)$ be such that the conditions of Lemma 3.1 are satisfied. Hence,

$$\lim_{t\to b^-} x(t) = c.$$

We consider the Cauchy problem

$$x' = f(t,x), \qquad x(b) = c.$$

From Corollary 2.1, it follows the existence of $b_1 > b$, $b_1 < \sup I$, and the existence of a solution $\bar{x}$ of the considered Cauchy problem on the interval $[b,b_1]$. We put

$$y(t) = \begin{cases} x(t), & t \in (a,b); \\ \bar{x}(t), & t \in [b,b_1). \end{cases}$$

Obviously,

$$y \in \tilde{C}_{loc}((a,b_1),\mathbb{R}^n)$$

and y is a solution of Eq. (3.1) on the interval (a,b_1), i.e., y is a right-extension of the solution x.

We consider the opposite implication. If x is a right-extendable solution and if $y\colon (a,b_1) \to \mathbb{R}^n$ is a right-extension of x, then

$$b < b_1 \le \sup I$$

and

$$\liminf_{t\to b^-} \|x(t)\| = \|y(b)\| < \infty.$$

□

Theorem 3.2. *Let x be a solution of Eq.* (3.1) *on an interval $(a,b) \subseteq I$. Then, x is left-extendable if and only if $a > \inf I$ and*

$$\liminf_{t\to a^+} \|x(t)\| < \infty.$$

Proof. Theorem 3.2 can be proved analogously as Theorem 3.1. □

Theorem 3.3. *The problem* (3.1)*,* (3.2) *has a non-extendable solution.*

Proof. We suppose that $t_0 < \sup I$. We show that the problem (3.1), (3.2) has a solution which is not right-extendable. Similarly, one can show the second case.

We consider an increasing sequence $\{b_k\}_{k=1}^{\infty} \subset (t_0, \sup I)$ with the property that

$$\lim_{k\to\infty} b_k = \sup I.$$

For $c \in \mathbb{R}^n$ and $r > 0$, we define

$$f_c^\star(t,r) = \sup\{\|f(t,x)\|\,; x \in B[c,r]\}, \qquad t \in I.$$

Obviously, there exists $t_1 \in (t_0, b_1]$ such that

$$\int_{t_0}^{t_1} f_{c_0}^\star(s,1)\,\mathrm{d}s \le 1.$$

According to Theorem 2.1, the problem (3.1), (3.2) has a solution x_0 on the interval $[t_0,t_1]$. We define

$$c_1 = x_0(t_1)$$

and

$$r_1 = \max\{\|x_0(t) - c_0\|\,; t \in [t_0,t_1]\}.$$

If

$$\int_{t_1}^{b_2} f_{c_0}^\star(s, r_1+1)\,\mathrm{d}s \le 1,$$

then we put $t_2 = b_2$. Otherwise, we choose $t_2 \in (t_1, b_2)$ such that

$$\int_{t_1}^{t_2} f_{c_0}^\star(s, r_1+1)\,\mathrm{d}s = 1.$$

We have

$$\int_{t_1}^{t_2} f^\star_{c_1}(s,1)\,\mathrm{d}s \le \int_{t_1}^{t_2} f^\star_{c_0}(s,r_1+1)\,\mathrm{d}s \le 1.$$

According to Theorem 2.1, the problem

$$x' = f(t,x), \qquad x(t_1) = c_1$$

has a solution x_1 on the interval $[t_1,t_2]$.

We continue in this process. We obtain the sequences $\{t_k\}_{k=1}^\infty$, $\{x_k\}_{k=0}^\infty$, $\{c_k\}_{k=1}^\infty$, $\{r_k\}_{k=1}^\infty$ for which:

1. $t_k \in (t_{k-1}, b_k]$, $k \in \mathbb{N}$;
2. $c_k = x_{k-1}(t_k)$, $k \in \mathbb{N}$;
3. x_k for $k \in \mathbb{N} \cup \{0\}$ is a solution of the problem

   $$x' = f(t,x), \qquad x(t_k) = c_k$$

   on the interval $[t_k, t_{k+1}]$;
4. it holds

   $$r_k = \max\left\{\|x_{k-1}(t) - c_0\|\,; t \in [t_{k-1}, t_k]\right\}, \qquad k \in \mathbb{N};$$
5. if

   $$\int_{t_k}^{t_{k+1}} f^\star_{c_0}(s, r_k+1)\,\mathrm{d}s < 1$$

   for some $k \in \mathbb{N}$, then $t_{k+1} = b_{k+1}$.

We put

$$b = \lim_{k\to\infty} t_k$$

and

$$x(t) = x_k(t), \qquad t \in [t_k, t_{k+1}),\, k \in \mathbb{N} \cup \{0\}.$$

Considering 3., we see that x is a solution of the problem (3.1), (3.2) on the interval $[t_0, b)$.

By contradiction, we show that the solution x is not right-extendable. We assume that the finite limit

$$\lim_{t\to b^-} x(t)$$

exists and that

$$b < \sup I.$$

The function x is bounded, i.e.,

$$r = \sup\left\{\|x(t) - c_0\|\,; t \in [t_0, b)\right\} < \infty.$$

According to 4.,

$$r_k \le r, \qquad k \in \mathbb{N}.$$

Thus,

$$\int_{t_k}^{t_{k+1}} f_{c_0}^{\star}(s, r_k + 1)\,\mathrm{d}s \le \int_{t_k}^{b} f_{c_0}^{\star}(s, r + 1)\,\mathrm{d}s \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore, there exists $k_0 \in \mathbb{N}$ such that

$$\int_{t_k}^{t_{k+1}} f_{c_0}^{\star}(s, r_k + 1)\,\mathrm{d}s < 1, \qquad k \ge k_0,\, k \in \mathbb{N}.$$

Now, from 5., it follows that

$$t_{k+1} = b_{k+1}, \qquad k \ge k_0,\, k \in \mathbb{N}.$$

Hence,

$$b = \sup I.$$

The obtained contradiction proves that the solution x is not right-extendable. $\square$

4. Set of solutions

We consider the Cauchy problem

$$x' = f(t,x), \tag{4.1}$$

$$x(t_0) = c_0, \tag{4.2}$$

where $f \in K_{loc}(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

Lemma 4.1. *Let $r_0 \geq 0$, $r > 0$, $t_0 \leq a_0 < b_0$, where $b_0 \in I$, be such that*

$$\int_{a_0}^{b_0} f_0^\star(s, r_0 + r)\,\mathrm{d}s < r,$$

where

$$f_0^\star(t,r) = \sup\{\|f(t,x)\|\,; x \in B[0,r]\}, \qquad t \in I.$$

Then, for all solution x of Eq. (4.1) *on the interval $[t_0, b_0)$ which satisfies*

$$\|x(t)\| \leq r_0, \qquad t \in [t_0, a_0],$$

it holds

$$\|x(t)\| < r_0 + r, \qquad t \in [t_0, b_0).$$

Moreover, the limit

$$\lim_{t \to b_0^-} \|x(t)\|$$

exists.

Proof. We prove the lemma by contradiction. We suppose that there exist a solution x of Eq. (4.1) on the interval $[t_0, b_0)$ and $t_1 \in (a_0, b_0)$ such that

$$\|x(t)\| \leq r_0, \qquad t \in [t_0, a_0],$$

$$\|x(t)\| < r_0 + r, \qquad t \in [a_0, t_1),$$

and that

$$\|x(t_1)\| = r_0 + r.$$

The contradiction comes from

$$\|x(t_1)\| \leq \|x(a_0)\| + \int_{a_0}^{t_1} \|f(s, x(s))\|\,\mathrm{d}s \leq r_0 + \int_{a_0}^{t_1} f_0^\star(s, r_0 + r)\,\mathrm{d}s < r_0 + r.$$

Note that the obtained statement guarantees

$$f(-, x(-)) \in L([a_0, b_0), \mathbb{R}^n).$$

Therefore, the existence of the limit

$$\lim_{t \to b_0^-} \|x(t)\|$$

follows from

$$x(t) = x(a_0) + \int\limits_{a_0}^{t} f(s, x(s))\,\mathrm{d}s, \qquad t \in [a_0, b_0).$$

$\square$

Theorem 4.1. *Let $[a,b] \subseteq I$, $t_0 \in [a,b]$, and let any non-extendable solution of the problem* (4.1), (4.2) *exist on $[a,b]$. Let $X_{[a,b]}$ be the set of the restrictions of all non-extendable solutions of the problem* (4.1), (4.2) *to the interval $[a,b]$. Then, the set $X_{[a,b]}$ is bounded in the space $C([a,b],\mathbb{R}^n)$.*

Proof. We assume that $t_0 < b$. We show that the set $X_{[t_0,b]}$ is bounded in the space $C([t_0,b],\mathbb{R}^n)$. For $t_0 > a$, one can similarly show that the set $X_{[a,t_0]}$ is bounded in the space $C([a,t_0],\mathbb{R}^n)$.

We put

$$\rho(t) = \sup\left\{\|x(s)\|\,; s \in [t_0,t], x \in X_{[t_0,b]}\right\}, \qquad t \in [t_0,b].$$

We choose $t_1 \in (t_0, b]$ such that

$$\int\limits_{t_0}^{t_1} f_0^\star(s, \|c_0\| + 1)\,\mathrm{d}s < 1,$$

where $f_0^\star$ is from Lemma 4.1. According to Lemma 4.1, we have

$$\rho(t) \le \rho(t_1) \le \|c_0\| + 1, \qquad t \in [t_0, t_1].$$

We show that $\rho(b) < \infty$. By contradiction, we assume that $\rho(b) = \infty$. Then, $t_1 < b$ and there exists $t^\star \in (t_1, b]$ such that

$$\rho(t) = \infty, \qquad t \in (t^\star, b],$$

and that

$$\rho(t) < \infty, \qquad t \in [t_0, t^\star).$$

We assume the existence of a sequence $\{\tau_k\}_{k=1}^{\infty} \subset [t_0, t^\star)$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of solutions of the problem (4.1), (4.2) on the interval $[t_0, t^\star]$ such that

$$\lim_{k\to\infty} \|x_k(\tau_k)\| = \infty.$$

In addition, for all $\beta \in [t_0, t^\star)$, we have

$$\|x_k(t)\| \le \rho(\beta), \qquad t \in [t_0, \beta], k \in \mathbb{N}.$$

It is obvious that the functions x_k, $k \in \mathbb{N}$, are uniformly bounded on any compact subinterval of the interval $[t_0, t^\star)$. At the same time, for any $\beta \in [t_0, t^\star)$, we have

$$\begin{aligned} \|x_k(t) - x_k(s)\| &\le \left|\int\limits_{s}^{t} \|f(\xi, x_k(\xi))\|\,\mathrm{d}\xi\right| \\ &\le \left|\int\limits_{s}^{t} f_0^\star(\xi, \rho(\beta))\,\mathrm{d}\xi\right|, \qquad s,t \in [t_0,\beta], k \in \mathbb{N}. \end{aligned}$$

Therefore, the functions x_k, $k \in \mathbb{N}$, are also equicontinuous on any compact subinterval of the interval $[t_0, t^\star)$. Due to the Arzelà–Ascoli theorem, without loss of generality, we can assume that the sequence $\{x_k\}_{k=1}^\infty$ is uniformly convergent on any compact subinterval of the interval $[t_0, t^\star)$. We put

$$\lim_{k\to\infty} x_k(t) = x(t), \qquad t \in [t_0, t^\star).$$

By the Lebesgue theorem, one can verify that x is a solution of the problem (4.1), (4.2) on the interval $[t_0, t^\star)$. Since $t^\star \leq b$ and since any non-extendable solution of the problem (4.1), (4.2) exists on the interval $[t_0, b]$, the finite limit

$$\lim_{t\to t^{\star -}} \|x(t)\|$$

exists. Hence,

$$r_0 = \sup\{\|x(t)\|\,; t \in [t_0, t^\star)\} < \infty. \tag{4.3}$$

We choose $t_\star \in [t_0, t^\star)$ so that

$$\int_{t_\star}^{t^\star} f_0^\star(s, r_0 + 2)\,\mathrm{d}s < 1.$$

From the construction of the solution x and from (4.3), it follows the existence of $k_0 \in \mathbb{N}$ with the property that

$$\|x_k(t)\| \leq r_0 + 1, \qquad t \in [t_0, t_\star],\, k \geq k_0,\, k \in \mathbb{N}.$$

Therefore, considering Lemma 4.1, we obtain

$$\|x_k(t)\| < r_0 + 2, \qquad t \in [t_0, t^\star),\, k \geq k_0,\, k \in \mathbb{N}.$$

Hence,

$$\rho(t^\star) < \infty.$$

Now, it is enough to consider again Lemma 4.1 (for $r_0 + 3$). □

Remark 4.1. From the Arzelà–Ascoli theorem, from the proof of Theorem 4.1, and from the Lebesgue theorem, it follows that the set $X_{[a,b]}$ from the statement of Theorem 4.1 is even compact in $C([a,b], \mathbb{R}^n)$.

5. Upper and lower solutions

We consider the Cauchy problem

$$x' = f(t,x), \tag{5.1}$$
$$x(t_0) = c_0, \tag{5.2}$$

where $f \in K_{loc}(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

Definition 5.1. Let $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$, where $f = (f_i)_{i=1}^n$. We say that f is quasi non-decreasing in the last n variables if, for all $i \in \{1,\dots,n\}$ and almost all $t \in I$, it holds

$$f_i(t,x_1,\dots,x_n) \le f_i(t,y_1,\dots,y_n), \qquad x_k \le y_k,\, k \in \{1,\dots,n\},\, k \ne i,\, x_i = y_i.$$

Lemma 5.1. *Let the map*

$$(t,x) \mapsto f(t,x)\,\mathrm{sgn}(t-t_0), \qquad (t,x) \in I \times \mathbb{R}^n,$$

be quasi non-decreasing in the last n variables. Let $[a,b] \subseteq I$, $t_0 \in [a,b]$, and let any non-extendable solution of the problem (5.1)*,* (5.2) *exist on the interval $[a,b]$. Then, for any function $y \in \tilde{C}([a,b],\mathbb{R}^n)$ satisfying*

$$y(t_0) \le c_0$$

and

$$\left[y'(t) - f(t,y(t))\right]\,\mathrm{sgn}(t-t_0) \le 0$$

for almost all $t \in [a,b]$, there exists a solution x of the problem (5.1)*,* (5.2) *on the interval $[a,b]$ such that*

$$y(t) \le x(t), \qquad t \in [a,b].$$

Proof. Let $y \in \tilde{C}([a,b],\mathbb{R}^n)$ be an arbitrary function from the statement of the lemma. We suppose that $t_0 < b$. We prove the existence of a solution x of the problem (5.1), (5.2) on the interval $[t_0,b]$ with the property that

$$y(t) \le x(t), \qquad t \in [t_0,b].$$

In the second case, we can proceed analogously.

For all $i \in \{1,\dots,n\}$, we put

$$\chi_i(t,z) = \begin{cases} y_i(t), & z \le y_i(t); \\ z, & z > y_i(t); \end{cases} \qquad t \in [t_0,b],\, z \in \mathbb{R}.$$

We define

$$\chi(t,x) = (\chi_i(t,x_i))_{i=1}^n, \qquad t \in [t_0,b],\, x = (x_i)_{i=1}^n \in \mathbb{R}^n,$$

and

$$\tilde{f}(t,x) = f(t,\chi(t,x)), \qquad t \in [t_0,b],\, x \in \mathbb{R}^n.$$

Obviously,

$$\tilde{f} \in K([t_0,b] \times \mathbb{R}^n, \mathbb{R}^n).$$

We consider the Cauchy problem

$$x' = \tilde{f}(t,x), \qquad x(t_0) = c_0.$$

We consider that $x = (x_i)_{i=1}^n$ is a solution of this problem, which is not right-extendable, and that $[t_0,b_0) \subseteq [t_0,b]$ is the maximal interval, where the solution x exists (see Theorem 3.3). Let $i \in \{1,\dots,n\}$ be arbitrarily given. We prove that

$$y_i(t) \le x_i(t), \qquad t \in [t_0,b_0).$$

Let us consider the opposite, i.e., let there exist $[\alpha,\beta] \subseteq [t_0,b_0)$ such that

$$y_i(\alpha) = x_i(\alpha)$$

and that

$$y_i(t) > x_i(t), \qquad t \in (\alpha,\beta].$$

We define

$$u(t) = y_i(t) - x_i(t), \qquad t \in [\alpha,\beta].$$

Obviously, $u(\alpha) = 0$ and

$$u(t) > 0, \qquad t \in (\alpha,\beta].$$

We have

$$\begin{aligned} u'(t) &= y_i'(t) - x_i'(t) \\ &\le f_i(t,y(t)) - \tilde{f}_i(t,x(t)) \\ &= f_i(t,y_1(t),\dots,y_n(t)) - f_i(t,\chi_1(t,x_1(t)),\dots,\chi_n(t,x_n(t))) \end{aligned}$$

for almost all $t \in (\alpha,\beta)$. From the definition of the function χ, it follows that

$$y_i(t) = \chi_i(t,x_i(t)), \qquad t \in [\alpha,\beta],$$

and that

$$y_k(t) \le \chi_k(t,x_k(t)), \qquad t \in [\alpha,\beta],\, k \ne i.$$

Since the map

$$(t,x) \mapsto f(t,x)\,\mathrm{sgn}(t-t_0), \qquad (t,x) \in I \times \mathbb{R}^n,$$

is quasi non-decreasing in the last n variables, $u'(t) \le 0$ for almost all $t \in (\alpha,\beta)$, which gives a contradiction. The contradiction (together with the arbitrariness of $i \in \{1,\dots,n\}$) proves that

$$y(t) \le x(t), \qquad t \in [t_0,b_0).$$

Thus,

$$\tilde{f}(t,x(t)) = f(t,x(t)), \qquad t \in [t_0,b_0),$$

i.e., x is a solution of the problem (5.1), (5.2) on $[t_0,b_0)$.

Now, we prove that $b_0 = b$. Let $b_0 < b$. Since x is a solution of the problem

$$x' = \tilde{f}(t,x), \qquad x(t_0) = c_0$$

on the interval $[t_0,b_0)$, which is not right-extendable, from Theorem 3.1, it follows

$$\lim_{t \to b_0^-} \|x(t)\| = \infty.$$

At the same time, from Theorem 3.1, it follows a contradiction with an assumption of the lemma. Thus, we have proved that $b_0 = b$, i.e.,

$$y(t) \leq x(t), \qquad t \in [t_0, b).$$

Due to the assumptions of the lemma, there exists the finite limit

$$\lim_{t \to b^-} x(t),$$

i.e., x is a solution of the problem (5.1), (5.2) on $[t_0, b]$ and

$$y(t) \leq x(t), \qquad t \in [t_0, b].$$

□

Lemma 5.2. *Let the map*

$$(t,x) \mapsto f(t,x) \operatorname{sgn}(t - t_0), \qquad (t,x) \in I \times \mathbb{R}^n,$$

be quasi non-decreasing in the last n variables. Let $[a,b] \subseteq I$, $t_0 \in [a,b]$, and let any non-extendable solution of the problem (5.1)*,* (5.2) *exist on the interval $[a,b]$. Then, for any function $y \in \tilde{C}([a,b], \mathbb{R}^n)$ satisfying*

$$y(t_0) \geq c_0$$

and

$$\left[y'(t) - f(t, y(t))\right] \operatorname{sgn}(t - t_0) \geq 0$$

for almost all $t \in [a,b]$, there exists a solution x of the problem (5.1)*,* (5.2) *on the interval $[a,b]$ with the property that*

$$y(t) \geq x(t), \qquad t \in [a,b].$$

Proof. The lemma is possible to prove analogously as Lemma 5.1. □

Definition 5.2. Let $x^\star$ be a solution of the problem (5.1), (5.2) on the interval $I_0 \subseteq I$, where $t_0 \in I_0$. We say that $x^\star$ is the upper (lower) solution of the problem (5.1), (5.2) on the interval $I_0 \subseteq I$ if, for all interval $I_1 \subseteq I_0$, where $t_0 \in I_1$, and any solution x of the problem (5.1), (5.2) on the interval I_1, it holds

$$x(t) \leq x^\star(t) \quad (x(t) \geq x^\star(t)), \qquad t \in I_1.$$

Theorem 5.1. *Let the map*

$$(t,x) \mapsto f(t,x) \operatorname{sgn}(t - t_0), \qquad (t,x) \in I \times \mathbb{R}^n,$$

be quasi non-decreasing in the last n variables. Let $[a,b] \subseteq I$, $t_0 \in [a,b]$, and let any non-extendable solution of the problem (5.1)*,* (5.2) *exist on the interval $[a,b]$. Then, the problem* (5.1)*,* (5.2) *has the upper solution and the lower solution on the interval $[a,b]$.*

Proof. We prove only the existence of the upper solution on the interval $[a,b]$. In the second case, it is possible to proceed analogously. As X, we denote the set of all solutions of the problem (5.1), (5.2) on the interval $[a,b]$. We define

$$x_i^\star(t) = \sup\left\{x_i(t); (x_k)_{k=1}^n \in X\right\}, \qquad t \in [a,b], i \in \{1, \ldots, n\}.$$

According to Theorem 4.1, the set X is bounded in the space $C([a,b],\mathbb{R}^n)$. Therefore,

$$x_i^\star(t) < \infty, \qquad t \in [a,b],\, i \in \{1,\dots,n\}\,.$$

We show that $x^\star = (x_i^\star)_{i=1}^n$ is the upper solution of the problem (5.1), (5.2) on the interval $[a,b]$. It is obvious that

$$x(t) \le x^\star(t), \qquad t \in [a,b],\, x \in X.$$

Firstly, we prove that the function $x^\star$ is absolutely continuous. Since the set X is bounded in $C([a,b],\mathbb{R}^n)$, there exists a function $h \in L([a,b],\mathbb{R}_+)$ with the property that

$$\|f(t,x)\| \le h(t), \qquad t \in [a,b],\, x \in X.$$

Let $s,t \in [a,b]$, $s < t$. Then,

$$\begin{aligned} x_i(t) &= x_i(s) + \int_s^t f_i(\tau, x(\tau))\,\mathrm{d}\tau \\ &\le x_i(s) + \int_s^t h(\tau)\,\mathrm{d}\tau \\ &\le x_i^\star(s) + \int_s^t h(\tau)\,\mathrm{d}\tau, \qquad x = (x_i)_{i=1}^n \in X,\, i \in \{1,\dots,n\}\,, \end{aligned}$$

i.e.,

$$x_i^\star(t) \le x_i^\star(s) + \int_s^t h(\tau)\,\mathrm{d}\tau, \qquad i \in \{1,\dots,n\}\,.$$

Analogously, one can obtain

$$x_i^\star(s) \le x_i^\star(t) + \int_s^t h(\tau)\,\mathrm{d}\tau, \qquad i \in \{1,\dots,n\}\,,$$

which gives

$$|x_i^\star(t) - x_i^\star(s)| \le \left| \int_s^t h(\tau)\,\mathrm{d}\tau \right|, \qquad i \in \{1,\dots,n\}\,, s,t \in [a,b].$$

Therefore, $x^\star \in \tilde{C}([a,b],\mathbb{R}^n)$.

Now, we show that $x^\star$ is a solution of the problem (5.1), (5.2). We know that one can find $r_0 > 0$ such that

$$\|x(t)\| \le r_0, \qquad t \in [a,b],\, x \in X.$$

According to Lemma 1.4, for r_0, there exist a function $h_0 \in L([a,b],\mathbb{R}_+)$ and a non-decreasing function $\omega_0 \in C_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ such that $\omega_0(0) = 0$ and

$$\|f(t,x) - f(t,y)\| \le h_0(t)\,\omega_0(\|x-y\|), \qquad t \in [a,b],\, x,y \in B[0,r_0].$$

As I_0, we denote the set of all $s \in [a,b]$ for which the following conditions:

1. there exists $(x^\star)'(s)$;
2. there exists $v'(s)$ and
$$v'(s) = f(s, x^\star(s)),$$
where
$$v(t) = \int_{t_0}^{t} f(\tau, x^\star(\tau))\,\mathrm{d}\tau, \qquad t \in [a,b];$$
3. there exists $v_0'(s)$ and
$$v_0'(s) = h_0(s),$$
where
$$v_0(t) = \int_{t_0}^{t} h_0(\tau)\,\mathrm{d}\tau, \qquad t \in [a,b],$$

are valid. Obviously, $m(I_0) = b - a$. Let $i \in \{1, \ldots, n\}$ and $s \in I_0$, where $s > t_0$, be arbitrary. Due to Remark 4.1, X is a compact set in the space $C([a,b], \mathbb{R}^n)$. Therefore, there exists $\tilde{x} = (\tilde{x}_k)_{k=1}^n \in X$ such that $\tilde{x}_i(s) = x_i^\star(s)$. We put
$$\varepsilon(t) = \max\{|\tilde{x}_i(\tau) - x_i^\star(\tau)|\,; t \le \tau \le s\}, \qquad t \in [t_0, s].$$

It is seen that the function $\varepsilon\colon [t_0, s] \to \mathbb{R}_+$ is continuous, non-increasing, and $\varepsilon(s) = 0$. We have
$$\begin{aligned}
f_i(t, \tilde{x}(t)) &\le f_i\left(t, x_1^\star(t), \ldots, x_{i-1}^\star(t), \tilde{x}_i(t), x_{i+1}^\star(t), \ldots, x_n^\star(t)\right) \\
&= f_i(t, x^\star(t)) + f_i\left(t, x_1^\star(t), \ldots, x_{i-1}^\star(t), \tilde{x}_i(t), x_{i+1}^\star(t), \ldots, x_n^\star(t)\right) - f_i(t, x^\star(t)) \\
&\le f_i(t, x^\star(t)) + h_0(t)\,\omega_0(|\tilde{x}_i(t) - x_i^\star(t)|) \\
&\le f_i(t, x^\star(t)) + h_0(t)\,\omega_0(\varepsilon(t)), \qquad t \in (t_0, s].
\end{aligned}$$

Hence (consider $\tilde{x}_i(s) = x_i^\star(s)$), we obtain
$$\begin{aligned}
x_i^\star(s) = \tilde{x}_i(s) &= \tilde{x}_i(t) + \int_t^s f_i(\tau, \tilde{x}(\tau))\,\mathrm{d}\tau \\
&\le x_i^\star(t) + \int_t^s f_i(\tau, x^\star(\tau))\,\mathrm{d}\tau + \int_t^s h_0(\tau)\omega_0(\varepsilon(\tau))\,\mathrm{d}\tau \\
&\le x_i^\star(t) + \int_t^s f_i(\tau, x^\star(\tau))\,\mathrm{d}\tau + \omega_0(\varepsilon(t)) \int_t^s h_0(\tau)\,\mathrm{d}\tau, \qquad t \in (t_0, s].
\end{aligned}$$

Therefore,
$$\frac{x_i^\star(s) - x_i^\star(t)}{s - t} \le \frac{1}{s-t} \int_t^s f_i(\tau, x^\star(\tau))\,\mathrm{d}\tau + \frac{1}{s-t}\,\omega_0(\varepsilon(t)) \int_t^s h_0(\tau)\,\mathrm{d}\tau, \qquad t \in (t_0, s).$$

Since $s \in I_0$ is arbitrary and $\omega_0(\varepsilon(s)) = 0$, we obtain
$$(x_i^\star)'(s) \le f_i(s, x^\star(s)), \qquad s \in I_0,\, s > t_0.$$

Analogously, one can show

$$(x_i^\star)'(s) \geq f_i(s, x^\star(s)), \qquad s \in I_0,\, s < t_0.$$

Thus,

$$\left[(x^\star)'(t) - f(t, x^\star(t))\right] \operatorname{sgn}(t - t_0) \leq 0$$

for almost all $t \in [a,b]$. According to Lemma 5.1, there exists $x_0 \in X$ such that

$$x^\star(t) \leq x_0(t), \qquad t \in [a,b].$$

At the same time,

$$x(t) \leq x^\star(t), \qquad t \in [a,b],\, x \in X, \tag{5.3}$$

which gives

$$x_0(t) \leq x^\star(t), \qquad t \in [a,b].$$

Therefore, $x^\star \equiv x_0$ and $x^\star$ is a solution of the problem (5.1), (5.2). Moreover, (5.3) means that $x^\star$ is the upper solution of this problem on the interval $[a,b]$. □

Corollary 5.1. *Let the map*

$$(t,x) \mapsto f(t,x) \operatorname{sgn}(t - t_0), \qquad (t,x) \in I \times \mathbb{R}^n,$$

be quasi non-decreasing in the last n variables. Let $[a,b] \subseteq I$, $t_0 \in [a,b]$, and let any non-extendable solution of the problem (5.1), (5.2) *exist on the interval $[a,b]$.*

1. *For any function $y \in \tilde{C}([a,b], \mathbb{R}^n)$ satisfying*

   $$y(t_0) \leq c_0$$

   and

   $$\left[y'(t) - f(t, y(t))\right] \operatorname{sgn}(t - t_0) \leq 0$$

   for almost all $t \in [a,b]$, it holds

   $$y(t) \leq x^\star(t), \qquad t \in [a,b],$$

   where $x^\star$ is the upper solution of the problem (5.1), (5.2) *on the interval $[a,b]$.*

2. *For any function $y \in \tilde{C}([a,b], \mathbb{R}^n)$ satisfying*

   $$y(t_0) \geq c_0$$

   and

   $$\left[y'(t) - f(t, y(t))\right] \operatorname{sgn}(t - t_0) \geq 0$$

   for almost all $t \in [a,b]$, it holds

   $$y(t) \geq x_\star(t), \qquad t \in [a,b],$$

   where $x_\star$ is the lower solution of the problem (5.1), (5.2) *on the interval $[a,b]$.*

Proof. The corollary follows directly from Lemmas 5.1 and 5.2 and from Theorem 5.1. □

Definition 5.3. Let $f\colon I\times\mathbb{R}^n\to\mathbb{R}^n$ be given. We say that f is non-decreasing in the last n variables if

$$f(t,x)\le f(t,y),\qquad t\in I,\, x\le y.$$

Corollary 5.2. *Let the map*

$$(t,x)\mapsto f(t,x)\,\mathrm{sgn}(t-t_0),\qquad (t,x)\in I\times\mathbb{R}^n,$$

be non-decreasing in the last n variables. Let $[a,b]\subseteq I$, $t_0\in[a,b]$, and let any non-extendable solution of the problem (5.1)*,* (5.2) *exist on the interval $[a,b]$.*

1. *For any function $y\in C([a,b],\mathbb{R}^n)$ satisfying*

$$y(t)\le c_0+\int_{t_0}^{t} f(s,y(s))\,\mathrm{d}s,\qquad t\in[a,b],$$

it holds

$$y(t)\le x^\star(t),\qquad t\in[a,b],$$

where $x^\star$ is the upper solution of the problem (5.1)*,* (5.2) *on the interval $[a,b]$.*

2. *For any function $y\in C([a,b],\mathbb{R}^n)$ satisfying*

$$y(t)\ge c_0+\int_{t_0}^{t} f(s,y(s))\,\mathrm{d}s,\qquad t\in[a,b],$$

it holds

$$y(t)\ge x_\star(t),\qquad t\in[a,b],$$

where $x_\star$ is the lower solution of the problem (5.1)*,* (5.2) *on the interval $[a,b]$.*

Proof. We prove only the first part. The second part can be proved analogously. We put

$$z(t)=c_0+\int_{t_0}^{t} f(s,y(s))\,\mathrm{d}s,\qquad t\in[a,b].$$

Obviously,

$$z\in\tilde{C}([a,b],\mathbb{R}^n)$$

and

$$y(t)\le z(t),\qquad t\in[a,b].$$

The identity

$$z'(t)=f(t,y(t))$$

holds for almost all $t\in[a,b]$. We have

$$\begin{aligned} z'(t)\,\mathrm{sgn}(t-t_0)&=f(t,y(t))\,\mathrm{sgn}(t-t_0)\\ &\le f(t,z(t))\,\mathrm{sgn}(t-t_0)\end{aligned}$$

for almost all $t\in[a,b]$, i.e.,

$$\big[z'(t)-f(t,z(t))\big]\,\mathrm{sgn}(t-t_0)\le 0$$

for almost all $t \in [a,b]$. According to Corollary 5.1, we obtain

$$z(t) \leq x^\star(t), \qquad t \in [a,b],$$

where $x^\star$ is the upper solution of the problem (5.1), (5.2) on the interval $[a,b]$. Finally, the inequality

$$y(t) \leq z(t), \qquad t \in [a,b],$$

gives

$$y(t) \leq x^\star(t), \qquad t \in [a,b].$$

□

6. Wintner theorem

We consider the Cauchy problem

$$x' = f(t,x), \tag{6.1}$$

$$x(t_0) = c_0, \tag{6.2}$$

where $f \in K_{loc}(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

Theorem 6.1 (Wintner)**.** *Let there exist a function $h \in K_{loc}(I \times \mathbb{R}_+, \mathbb{R}_+)$ such that*

$$f(t,x) \cdot \operatorname{sgn}((t-t_0)x) \le h(t, \|x\|), \qquad (t,x) \in I \times \mathbb{R}^n,$$

where the problem

$$\rho' = h(t,\rho) \operatorname{sgn}(t-t_0), \tag{6.3}$$

$$\rho(t_0) = \|c_0\| \tag{6.4}$$

has the upper solution on the interval I. Then, the problem (6.1), (6.2) *has a solution on I. Moreover, all non-extendable solutions of the problem* (6.1), (6.2) *exist on I.*

Proof. We suppose that $t_0 \in \operatorname{int}(I)$. If t_0 is an endpoint of the interval I, then one can proceed analogously.

From Theorem 3.3, it follows that there exists at least one non-extendable solution of the problem (6.1), (6.2) on a subinterval J of I. Let x be an arbitrary non-extendable solution of the problem (6.1), (6.2) on $\operatorname{int}(J) = (a,b) \subseteq I$. We show that $a = \inf I$, $b = \sup I$. We denote

$$u(t) = \|x(t)\|, \qquad t \in (a,b).$$

Obviously,

$$u \in \tilde{C}_{loc}((a,b), \mathbb{R}_+),$$

$$u(t_0) = \|c_0\|,$$

and

$$u'(t) = x'(t) \cdot \operatorname{sgn} x(t)$$

for almost all $t \in (a,b)$. Therefore,

$$u'(t) \operatorname{sgn}(t-t_0) \le h(t, u(t))$$

for almost all $t \in (a,b)$. Thus,

$$\left[u'(t) - h(t,u(t)) \operatorname{sgn}(t-t_0)\right] \operatorname{sgn}(t-t_0) \le 0 \tag{6.5}$$

for almost all $t \in (a,b)$. Since the problem (6.3), (6.4) has the upper solution on I, all non-extendable solutions of the problem (6.3), (6.4) exist on I. Therefore, considering $u(t_0) = \|c_0\|$ together with (6.5), from Corollary 5.1, it follows that

$$u(t) \le \rho^\star(t), \qquad t \in (a,b),$$

where $\rho^\star$ is the upper solution of the problem (6.3), (6.4) on the interval I. Using Theorems 3.1 and 3.2, we obtain

$$a = \inf I, \qquad b = \sup I.$$

If $a \notin I$, $b \notin I$, then the proof is done. Let us consider that

$$a \in I \qquad \text{or} \qquad b \in I.$$

We show that the finite limit

$$\lim_{t\to a^+} x(t) \qquad \text{or} \qquad \lim_{t\to b^-} x(t)$$

exists. We know that

$$f \in K([a,t_0]\times\mathbb{R}^n,\mathbb{R}^n) \qquad \text{or} \qquad f \in K([t_0,b]\times\mathbb{R}^n,\mathbb{R}^n).$$

Therefore, from

$$u(t) \le \rho^\star(t), \qquad t \in (a,b),$$

and from the fact that $\rho^\star$ is the upper solution of the problem (6.3), (6.4) on I, it follows

$$f(-,x(-)) \in L([a,t_0],\mathbb{R}^n) \qquad \text{or} \qquad f(-,x(-)) \in L([t_0,b],\mathbb{R}^n).$$

The existence of the limit

$$\lim_{t\to a^+} x(t) \qquad \text{or} \qquad \lim_{t\to b^-} x(t)$$

comes from

$$x(t) = c_0 + \int_{t_0}^{t} f(s,x(s))\,\mathrm{d}s, \qquad t \in (a,b).$$

□

Corollary 6.1. *Let $h_0 \in L_{loc}(I,\mathbb{R}_+)$ and let $\omega \in C_{loc}(\mathbb{R}_+,(0,\infty))$ satisfy*

$$\int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty.$$

If the inequality

$$f(t,x)\cdot \operatorname{sgn}((t-t_0)x) \le h_0(t)\,\omega(\|x\|)$$

holds on the set $I\times\mathbb{R}^n$, then the problem (6.1)*,* (6.2) *has a solution on I. Moreover, all non-extendable solutions of the problem* (6.1)*,* (6.2) *exist on I.*

Proof. Since the problem

$$\rho' = h_0(t)\,\omega(\rho)\,\operatorname{sgn}(t-t_0),$$
$$\rho(t_0) = \|c_0\|$$

has the upper solution on the interval I, the statement of the corollary comes directly from Theorem 6.1. □

7. Uniqueness of solutions

We consider the Cauchy problem

$$x' = f(t,x), \tag{7.1}$$

$$x(t_0) = c_0, \tag{7.2}$$

where $f \in K_{loc}(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

Definition 7.1. We say that the problem (7.1), (7.2) is uniquely solvable if, for arbitrary solutions x_1 and x_2 on intervals I_1 and I_2, respectively, it holds

$$x_1(t) = x_2(t), \qquad t \in I_1 \cap I_2.$$

Definition 7.2. We say that a function $g\colon I \setminus \{t_0\} \times \mathbb{R}_+ \to \mathbb{R}$ is an element of the set $K_{loc}(I \setminus \{t_0\} \times \mathbb{R}_+, \mathbb{R})$ if $g \in K(I_0 \times \mathbb{R}_+, \mathbb{R})$ for any compact interval $I_0 \subseteq I \setminus \{t_0\}$.

Definition 7.3. Let $g \in K_{loc}(I \setminus \{t_0\} \times \mathbb{R}_+, \mathbb{R})$ and let $I_0 \subseteq I$ be such that $t_0 \in I_0$. We say that a function $x\colon I_0 \setminus \{t_0\} \to \mathbb{R}_+$ is a solution of the equation $x' = g(t,x)$ if the following conditions:

1. $x \in \tilde{C}(J, \mathbb{R})$ for any compact interval $J \subseteq I_0 \setminus \{t_0\}$;
2. $x'(t) = g(t, x(t))$ for almost all $t \in I_0$

are fulfilled.

Lemma 7.1. *Let $\Delta > 0$, $\lambda \in [0,1)$, $h \in L([t_0, t_0+\Delta], (0,\infty))$, and let a function $\varphi \in C_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be such that*

$$\lim_{s \to 0^+} \frac{\varphi(s)}{s^\lambda} = 0.$$

If a function $u \in C([t_0, t_0+\Delta], \mathbb{R}_+)$ satisfies

$$u(t) \le \int_{t_0}^{t} h(\tau)\varphi(u(\tau))\,\mathrm{d}\tau, \qquad t \in [t_0, t_0+\Delta],$$

then

$$\lim_{t \to t_0^+} \frac{[u(t)]^{1-\lambda}}{\int_{t_0}^{t} h(s)\,\mathrm{d}s} = 0.$$

Proof. It is seen that $u(t_0) = 0$. Let $\varepsilon > 0$ be arbitrary. There exists $t_\varepsilon \in (t_0, t_0+\Delta]$ with the property that

$$\varphi(u(\tau)) \le \varepsilon\,[u(\tau)]^\lambda, \qquad \tau \in [t_0, t_\varepsilon].$$

Next, we obtain

$$u(t) \le \varepsilon \int_{t_0}^{t} h(\tau)\,[u(\tau)]^\lambda\,\mathrm{d}\tau, \qquad t \in [t_0, t_\varepsilon]. \tag{7.3}$$

We denote

$$v(t) = \left(\varepsilon(1-\lambda) \int_{t_0}^{t} h(\tau)\, \mathrm{d}\tau \right)^{\frac{1}{1-\lambda}}, \qquad t \in [t_0, t_\varepsilon].$$

One can directly verify that v is "the unique positive solution" of the Cauchy problem

$$v' = \varepsilon h(t) v^{\lambda}, \qquad v(t_0) = 0$$

for $t \in [t_0, t_\varepsilon]$. Therefore, Corollary 5.2 and (7.3) give

$$u(t) \leq v(t), \qquad t \in [t_0, t_\varepsilon].$$

Thus,

$$\limsup_{t \to t_0^+} \frac{[u(t)]^{1-\lambda}}{\int_{t_0}^{t} h(s)\, \mathrm{d}s} \leq \limsup_{t \to t_0^+} \frac{\varepsilon(1-\lambda) \int_{t_0}^{t} h(s)\, \mathrm{d}s}{\int_{t_0}^{t} h(s)\, \mathrm{d}s} \leq \varepsilon.$$

Now, it suffices to consider the arbitrariness of $\varepsilon > 0$. □

Lemma 7.2. *Let $\Delta > 0$, $\lambda \in [0,1)$, $h \in L([t_0 - \Delta, t_0], (0, \infty))$, and let a function $\varphi \in C_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be such that*

$$\lim_{s \to 0^+} \frac{\varphi(s)}{s^{\lambda}} = 0.$$

If a function $u \in C([t_0 - \Delta, t_0], \mathbb{R}_+)$ satisfies

$$u(t) \leq \int_{t}^{t_0} h(\tau) \varphi(u(\tau))\, \mathrm{d}\tau, \qquad t \in [t_0 - \Delta, t_0],$$

then

$$\lim_{t \to t_0^-} \frac{[u(t)]^{1-\lambda}}{\int_{t}^{t_0} h(s)\, \mathrm{d}s} = 0.$$

Proof. The lemma can be proved analogously as Lemma 7.1. □

Theorem 7.1. *Let $\delta > 0$ and $\varepsilon > 0$ be such that*

$$[f(t,x) - f(t,y)] \cdot \operatorname{sgn}[(t - t_0)(x - y)] \leq h(t) \varphi(\|x - y\|), \qquad t \in J,\, x, y \in B[c_0, \delta],$$

where $J = [t_0 - \varepsilon, t_0 + \varepsilon] \cap I$, $h \in L_{loc}(J, (0, \infty))$, and $\varphi \in C([0, 2\delta], \mathbb{R}_+)$ has the property that

$$\lim_{s \to 0^+} \frac{\varphi(s)}{s^{\lambda}} = 0$$

for some $\lambda \in [0,1)$. If, for any $r > 0$, there exists a function

$$\omega_r \in K_{loc}(I \setminus \{t_0\} \times \mathbb{R}_+, \mathbb{R})$$

such that

$$\omega_r(-, 0) \equiv 0$$

and that

$$[f(t,x)-f(t,y)]\cdot\operatorname{sgn}[(t-t_0)(x-y)] \le \omega_r(t,\|x-y\|), \qquad t\in I\setminus\{t_0\}, x,y\in B[c_0,r],$$

and if the problem

$$\rho' = \omega_r(t,\rho)\operatorname{sgn}(t-t_0), \qquad \lim_{t\to t_0}\frac{[\rho(t)]^{1-\lambda}}{\int\limits_{t_0}^{t} h(s)\,\mathrm{d}s} = 0$$

has only the zero solution, then the problem (7.1), (7.2) *is uniquely solvable.*

Proof. Let x_1 and x_2 be non-extendable solutions of the problem (7.1), (7.2) on intervals I_1 and I_2, respectively. Our aim is to prove

$$x_1(t) = x_2(t), \qquad t\in I_1\cap I_2.$$

We denote

$$u(t) = \|x_1(t)-x_2(t)\|, \qquad t\in I_1\cap I_2.$$

We consider that there exists $t_1\in I_1\cap I_2$ such that $u(t_1)\neq 0$. Without loss of generality, we can assume that $t_1>t_0$. It is obvious that

$$u\in\tilde{C}([t_0,t_1],\mathbb{R}_+)$$

and that

$$u'(t) = \left[x_1'(t)-x_2'(t)\right]\cdot\operatorname{sgn}(x_1(t)-x_2(t)) \tag{7.4}$$

for almost all $t\in[t_0,t_1]$. Thus, we obtain

$$u(t) = \int\limits_{t_0}^{t} u'(\tau)\,\mathrm{d}\tau \le \int\limits_{t_0}^{t} h(\tau)\varphi(u(\tau))\,\mathrm{d}\tau, \qquad t\in[t_0,t_0+\Delta],$$

where $\Delta>0$ is sufficiently small. According to Lemma 7.1, we have

$$\lim_{t\to t_0^+}\frac{[u(t)]^{1-\lambda}}{\int\limits_{t_0}^{t} h(s)\,\mathrm{d}s} = 0.$$

Next, we denote

$$r = \max\{\|x_1(t)-c_0\|+\|x_2(t)-c_0\|\,;t\in[t_0,t_1]\}.$$

For this number r, there exists a function $\omega_r\in K_{loc}(I\setminus\{t_0\}\times\mathbb{R}_+,\mathbb{R})$ from the statement of the theorem. From (7.4), it follows

$$u'(t)\le\omega_r(t,u(t))$$

for almost all $t\in(t_0,t_1]$. We put

$$\bar{\omega}_r(t,y) = \begin{cases}\omega_r(t,u(t)), & y>u(t);\\ \omega_r(t,y), & 0\le y\le u(t);\\ 0, & y<0;\end{cases} \qquad t\in(t_0,t_1],\, y\in\mathbb{R}.$$

Evidently,

$$\bar{\omega}_r \in K_{loc}((t_0,t_1]\times\mathbb{R},\mathbb{R}).$$

From Theorem 3.3, it follows that there exist $a \in [t_0,t_1)$ and a non-extendable solution ρ of the problem

$$\rho' = \bar{\omega}_r(t,\rho), \qquad \rho(t_1) = \frac{1}{2}u(t_1)$$

on the interval $(a,t_1]$.

We show that

$$\rho(t) \le u(t), \qquad t \in (a,t_1].$$

By contradiction, we assume that there exist $t_2,t_3 \in (a,t_1)$, where $t_2 < t_3$, such that

$$\rho(t) > u(t), \qquad t \in [t_2,t_3),$$

and that

$$\rho(t_3) = u(t_3).$$

From the definition of $\bar{\omega}_r$, it follows

$$\rho'(t) = \omega_r(t,u(t))$$

for almost all $t \in [t_2,t_3]$. From the inequality

$$u'(t) \le \omega_r(t,u(t))$$

which is valid for almost all $t \in (t_0,t_1]$, we obtain

$$\rho(t) \le u(t), \qquad t \in [t_2,t_3].$$

We have a contradiction.

Now, we show that

$$\rho(t) > 0, \qquad t \in (a,t_1].$$

By contradiction, we assume that there exists $t_4 \in (a,t_1)$ for which

$$\rho(t) > 0, \qquad t \in (t_4,t_1],$$

and for which

$$\rho(t_4) = 0.$$

We recall that

$$\rho(t) \le u(t), \qquad t \in (a,t_1].$$

Due to the definition of $\bar{\omega}_r$, the function

$$\bar{\rho}(t) = \begin{cases} 0, & t \in (t_0,t_4]; \\ \rho(t), & t \in (t_4,t_1] \end{cases}$$

is a non-zero solution of the problem

$$\rho'(t) = \omega_r(t,\rho)\,\mathrm{sgn}\,(t-t_0), \qquad \lim_{t\to t_0} \frac{[\rho(t)]^{1-\lambda}}{\int_{t_0}^{t} h(s)\,\mathrm{d}s} = 0 \tag{7.5}$$

on the interval $(t_0,t_1]$, which is a contradiction.

We have proved that

$$0<\rho(t)\le u(t), \qquad t\in(a,t_1].$$

From Theorem 3.2, it follows that $a=t_0$, i.e.,

$$0<\rho(t)\le u(t), \qquad t\in(t_0,t_1].$$

Due to the definition of $\bar{\omega}_r$, we have a non-zero solution of the problem (7.5) on the interval $(t_0,t_1]$, which is a contradiction with the assumption of the theorem. □

Corollary 7.1 (Osgood)**.** *If there exist functions $l_r\in L(I,\mathbb{R}_+)$ and $\eta_r\in C_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ for any $r>0$ such that*

$$\eta_r(0)=0, \qquad \eta_r(s)>0, \qquad s>0, \qquad \int_0^{2r}\frac{\mathrm{d}s}{\eta_r(s)}=\infty,$$

and that

$$[f(t,x)-f(t,y)]\cdot\operatorname{sgn}[(t-t_0)(x-y)]\le l_r(t)\eta_r(\|x-y\|), \qquad t\in I,\, x,y\in B[c_0,r],$$

then the problem (7.1)*,* (7.2) *is uniquely solvable.*

Proof. We put

$$\begin{gathered}\delta=1, \qquad J=I, \qquad \lambda=0,\\ h(t)=l_1(t)+1, \qquad t\in J,\\ \varphi(x)=\eta_1(x), \qquad x\in[0,2],\end{gathered}$$

and

$$\omega_r(t,x)=l_r(t)\eta_r(x), \qquad t\in I,\, x\in\mathbb{R}_+,\, r>0.$$

One can directly verify that all conditions of Theorem 7.1 are fulfilled. Note that, for any $r>0$, the problem

$$\rho'=l_r(t)\eta_r(\rho)\,\operatorname{sgn}(t-t_0), \qquad \rho(t_0)=0$$

has only the zero solution. Thus, for any $r>0$, the problem

$$\rho'=l_r(t)\eta_r(\rho)\,\operatorname{sgn}(t-t_0), \qquad \lim_{t\to t_0}\frac{\rho(t)}{\int_{t_0}^{t}h(s)\,\mathrm{d}s}=0$$

has only the zero solution as well. Therefore, the corollary follows from Theorem 7.1. □

Corollary 7.2 (Nagumo–Perron)**.** *Let $t_0\in\operatorname{int}(I)$. Let $h\in L_{loc}(I,(0,\infty))$ be such that the function*

$$f_0(t,x)=\frac{f(t,x)}{h(t)}, \qquad (t,x)\in I\times\mathbb{R}^n,$$

is continuous in a neighbourhood of $[t_0,c_0]$. If the inequality

$$[f(t,x)-f(t,y)]\cdot\operatorname{sgn}[(t-t_0)(x-y)]\le\frac{h(t)}{\left|\int_{t_0}^{t}h(\tau)\,\mathrm{d}\tau\right|}\|x-y\|, \qquad t\neq t_0,$$

is valid on the set $I\times\mathbb{R}^n$, then the problem (7.1)*,* (7.2) *is uniquely solvable.*

Proof. We put $\lambda = 0$ and

$$\omega_r(t,x) = \frac{h(t)\,x}{\left|\int\limits_{t_0}^{t} h(\tau)\,\mathrm{d}\tau\right|}, \qquad t \neq t_0,\, t \in I,\, x \in \mathbb{R}_+,\, r > 0.$$

Since the function f_0 is continuous in some neighbourhood of the point $[t_0, c_0]$, there exist

$$\varepsilon, \delta \in \left(0, \frac{1}{2}\right)$$

such that the function f_0 is uniformly continuous on $[t_0-\varepsilon, t_0+\varepsilon] \times B[c_0,\delta]$. Let $J = [t_0-\varepsilon, t_0+\varepsilon]$. For $s \in [0, 2\delta] \subseteq [0,1]$, we put

$$\varphi(s) = \max\left\{\|f_0(t,x) - f_0(t,y)\|\,; t \in J, x, y \in B[c_0,\delta], \|x-y\| \leq s\right\}.$$

Obviously, $\varphi \in C([0,2\delta], \mathbb{R}_+)$ and $\varphi(0) = 0$. One can easily verify that all conditions of Theorem 7.1 are fulfilled. For example, by a direct computation, one can verify that the problem

$$\rho' = \frac{h(t)}{\int\limits_{t_0}^{t} h(\tau)\,\mathrm{d}\tau}\,\rho, \qquad \lim_{t \to t_0} \frac{\rho(t)}{\int\limits_{t_0}^{t} h(\tau)\,\mathrm{d}\tau} = 0$$

has only the zero solution. Therefore, the corollary also follows from Theorem 7.1. $\square$

8. Krein theorem

Let I be a compact interval. Now, we consider the Cauchy problem

$$x' = f_0(t,x), \tag{8.1.0}$$

$$x(t_0) = c_0, \tag{8.2.0}$$

where $f_0 \in K(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

For $m \in \mathbb{N}$, together with the problem (8.1.0), (8.2.0), we consider the perturbed problem

$$x' = f_m(t,x), \tag{8.1.m}$$

$$x(t_m) = c_m, \tag{8.2.m}$$

where $f_m \in K(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_m \in I$, $c_m \in \mathbb{R}^n$.

For $m \in \mathbb{N} \cup \{0\}$, by the symbol $X(f_m, t_m, c_m)$, we denote the set of all non-extendable solutions of the problem (8.1.m), (8.2.m).

Definition 8.1. Let $Y \subseteq C(I, \mathbb{R}^n)$. Then, the ε-neighbourhood of the set Y is the set

$$Y_\varepsilon = \bigcup_{y \in Y} B(y, \varepsilon),$$

where

$$B(y, \varepsilon) = \{x \in C(I, \mathbb{R}^n); \|x - y\|_C < \varepsilon\}.$$

Definition 8.2. Let $Y \subseteq C(I, \mathbb{R}^n)$ and let $Y^m \subseteq C(I, \mathbb{R}^n)$ for all sufficiently large $m \in \mathbb{N}$. If, for any $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$Y^m \subseteq Y_\varepsilon, \qquad m \geq m_0,$$

then we write

$$\lim_{m \to \infty} Y^m \subseteq Y.$$

Lemma 8.1. *Let the following conditions:*

$\bar{1}$. *for all $x \in \mathbb{R}^n$, it holds*

$$\lim_{m \to \infty} \int_{t_0}^{t} f_m(\tau, x)\, \mathrm{d}\tau = \int_{t_0}^{t} f_0(\tau, x)\, \mathrm{d}\tau$$

uniformly on I;

$\bar{2}$. *for any $r > 0$, there exists $\omega_r \in K(I \times \mathbb{R}_+, \mathbb{R}_+)$ such that*

$$\omega_r(-, 0) \equiv 0$$

and that

$$\|f_m(t,x) - f_m(t,y)\| \leq \omega_r(t, \|x - y\|), \qquad t \in I,\, x, y \in B[0,r],\, m \in \mathbb{N},$$

be fulfilled. If $\{x_m\}_{m=0}^{\infty}$ *is a sequence of functions from the space* $C(I,\mathbb{R}^n)$ *such that*

$$\lim_{m\to\infty} x_m(t) = x_0(t)$$

uniformly on I, then

$$\lim_{m\to\infty} \int_{t_0}^{t} f_m(\tau, x_m(\tau))\,\mathrm{d}\tau = \int_{t_0}^{t} f_0(\tau, x_0(\tau))\,\mathrm{d}\tau$$

uniformly on I.

Proof. Since the sequence $\{x_m\}_{m=1}^{\infty}$ is uniformly convergent, $\{\|x_m\|_C\}_{m=1}^{\infty}$ is bounded. Let $r \in \mathbb{R}$ satisfy

$$1 + \|x_m\|_C \le r, \qquad m \in \mathbb{N} \cup \{0\}.$$

Let ω_r be from the condition $\bar{2}$. Without loss of generality, we can assume that the function ω_r is non-decreasing in the second variable. Using Lemma 1.4, we can also assume that

$$\|f_0(t,x) - f_0(t,y)\| \le \omega_r(t, \|x-y\|), \qquad t \in I,\, x, y \in B[0,r].$$

We denote

$$y_m(t) = \int_{t_0}^{t} f_m(\tau, x_m(\tau))\,\mathrm{d}\tau - \int_{t_0}^{t} f_0(\tau, x_0(\tau))\,\mathrm{d}\tau, \qquad t \in I,\, m \in \mathbb{N}.$$

Let $\varepsilon > 0$ be arbitrarily given. We choose $\eta \in (0,1]$ so that (see also Lemma 4.1)

$$\int_I \omega_r(\tau, \eta)\,\mathrm{d}\tau < \frac{\varepsilon}{2}.$$

Since the function x_0 is continuous on the compact interval, there exists a system $\{t_i\}_{i=1}^{k}$ ($k \ge 2$, $k \in \mathbb{N}$) of points of the interval I such that

$$\min I = t_1 < t_2 < \cdots < t_k = \max I$$

and that

$$\|x_0(t) - x_0(t_i)\| < \frac{\eta}{2}, \qquad t \in [t_i, t_{i+1}],\, i \in \{1, \ldots, k-1\}.$$

We put

$$\tilde{x}(t) = x_0(t_i), \qquad t \in [t_i, t_{i+1}),\, i \in \{1, \ldots, k-1\},$$

and

$$\tilde{x}(t_k) = x_0(t_{k-1}).$$

Obviously,

$$\|\tilde{x}(t) - x_0(t)\| < \frac{\eta}{2}, \qquad t \in I.$$

There exists $m_0 \in \mathbb{N}$ with the property that

$$\|x_m - x_0\|_C < \frac{\eta}{2}, \qquad m \ge m_0,\, m \in \mathbb{N}.$$

Therefore,

$$\|x_m(t) - \tilde{x}(t)\| < \eta, \qquad t \in I,\, m \geq m_0,\, m \in \mathbb{N}.$$

Next, we obtain

$$\begin{aligned}
\|y_m(t)\| \leq & \left| \int_{t_0}^{t} \|f_m(\tau, x_m(\tau)) - f_m(\tau, \tilde{x}(\tau))\| \,\mathrm{d}\tau \right| \\
& + \left\| \int_{t_0}^{t} f_m(\tau, \tilde{x}(\tau)) - f_0(\tau, \tilde{x}(\tau)) \,\mathrm{d}\tau \right\| \\
& + \left| \int_{t_0}^{t} \|f_0(\tau, \tilde{x}(\tau)) - f_0(\tau, x_0(\tau))\| \,\mathrm{d}\tau \right| \\
\leq & \left\| \int_{t_0}^{t} f_m(\tau, \tilde{x}(\tau)) - f_0(\tau, \tilde{x}(\tau)) \,\mathrm{d}\tau \right\| + 2 \int_I \omega_r(\tau, \eta) \,\mathrm{d}\tau
\end{aligned}$$

for all $t \in I$, $m \geq m_0$, $m \in \mathbb{N}$. We define

$$\gamma_{i,m} = \max \left\{ \left\| \int_{t_0}^{t} f_m(\tau, x_0(t_i)) - f_0(\tau, x_0(t_i)) \,\mathrm{d}\tau \right\| ; t \in I \right\}$$

for all $i \in \{1, \ldots, k-1\}$, $m \in \mathbb{N}$.

Using the condition $\bar{1}$, one can easily verify that

$$\lim_{m \to \infty} \gamma_{i,m} = 0, \qquad i \in \{1, \ldots, k-1\}.$$

Since

$$\int_I \omega_r(\tau, \eta) \,\mathrm{d}\tau < \frac{\varepsilon}{2},$$

the definitions of $\tilde{x}$ and $\gamma_{i,m}$ give

$$\|y_m(t)\| \leq \varepsilon + \sum_{i=1}^{k-1} \gamma_{i,m}, \qquad t \in I,\, m \geq m_0,\, m \in \mathbb{N}.$$

From this inequality, we obtain the statement of the lemma. It is enough to consider the arbitrariness of $\varepsilon > 0$ and

$$\lim_{m \to \infty} \gamma_{i,m} = 0, \qquad i \in \{1, \ldots, k-1\}.$$

□

Theorem 8.1 (Krein). *If the following conditions:*

1. *it holds*

$$\lim_{m \to \infty} t_m = t_0, \qquad \lim_{m \to \infty} c_m = c_0;$$

2. *for all $x \in \mathbb{R}^n$, it holds*

$$\lim_{m\to\infty} \int_{t_0}^{t} f_m(\tau,x)\,\mathrm{d}\tau = \int_{t_0}^{t} f_0(\tau,x)\,\mathrm{d}\tau$$

uniformly on I;

3. *for any $r > 0$, there exists a function $\omega_r \in K(I \times \mathbb{R}_+, \mathbb{R}_+)$ such that*

$$\omega_r(-,0) \equiv 0$$

and that

$$\|f_m(t,x) - f_m(t,y)\| \le \omega_r(t, \|x-y\|), \qquad t \in I,\, x,y \in B[0,r],\, m \in \mathbb{N};$$

4. *every non-extendable solution of the problem* (8.1.0), (8.2.0) *exists on I, i.e.,*

$$X(f_0,t_0,c_0) \subset C(I,\mathbb{R}^n),$$

are fulfilled, then there exists $m_0 \in \mathbb{N}$ such that, for all $m > m_0$, $m \in \mathbb{N}$, all non-extendable solution of the problem (8.1.m), (8.2.m) *exists on I and*

$$\lim_{m\to\infty} X(f_m,t_m,c_m) \subseteq X(f_0,t_0,c_0).$$

Proof. According to Theorem 4.1, the set $X(f_0,t_0,c_0)$ is bounded in the space $C(I,\mathbb{R}^n)$. Thus, there exists $r_0 > 0$ such that

$$\|x\|_C \le r_0, \qquad x \in X(f_0,t_0,c_0).$$

We define the function

$$\chi(x) = \begin{cases} x, & \|x\| \le r_0 + 1,\, x \in \mathbb{R}^n; \\ (r_0+1)\dfrac{x}{\|x\|}, & \|x\| > r_0 + 1,\, x \in \mathbb{R}^n, \end{cases}$$

and we put

$$\tilde{f}_m(t,x) = f_m(t,\chi(x)), \qquad (t,x) \in I \times \mathbb{R}^n,\, m \in \mathbb{N} \cup \{0\}.$$

It is obvious that

$$\tilde{f}_m \in K(I \times \mathbb{R}^n, \mathbb{R}^n), \qquad m \in \mathbb{N} \cup \{0\},$$

and that

$$\left\|\tilde{f}_m(t,x)\right\| \le f_m^\star(t), \qquad t \in I, x \in \mathbb{R}^n, m \in \mathbb{N} \cup \{0\},$$

where

$$f_m^\star(t) = \sup\{\|f_m(t,x)\|\,; x \in B[0,r_0+1]\}, \qquad t \in I.$$

Due to Lemma 1.2, $f_m^\star(t) \in L(I,\mathbb{R}_+)$ for all considered m. For all $m \in \mathbb{N} \cup \{0\}$, we consider the equation

$$x' = \tilde{f}_m(t,x). \tag{8.3.m}$$

As $X\left(\tilde{f}_m,t_m,c_m\right)$, we denote the set of all non-extendable solutions of the problem (8.3.m), (8.2.m).

Now, we show that

$$X\left(\tilde{f}_0, t_0, c_0\right) = X\left(f_0, t_0, c_0\right).$$

Obviously,

$$X(f_0, t_0, c_0) \subseteq X\left(\tilde{f}_0, t_0, c_0\right).$$

We assume the existence of $\tilde{x} \in X\left(\tilde{f}_0, t_0, c_0\right)$ such that $\tilde{x} \notin X(f_0, t_0, c_0)$. We have

$$\|\tilde{x}\|_C > r_0 + 1, \qquad \|\tilde{x}(t_0)\| = \|c_0\| < r_0 + 1.$$

Therefore, there exists an interval $I_0 \subset I$ such that $t_0 \in I_0$ and that

$$\sup\{\|\tilde{x}(t)\|; t \in I_0\} = r_0 + 1.$$

Now, it is seen that the function $\tilde{x}$ is a solution of (8.1.0), (8.2.0) on the interval I_0. Due to 4. from the statement of the theorem, we know that there exists an extension y of the solution $\tilde{x}$ to I. Then,

$$y \in X(f_0, t_0, c_0)$$

and

$$\|y\|_C \geq r_0 + 1.$$

This is a contradiction which proves

$$X\left(\tilde{f}_0, t_0, c_0\right) \subseteq X(f_0, t_0, c_0).$$

Since

$$\left\|\tilde{f}_m(t,x)\right\| \leq f_m^{\star}(t), \qquad t \in I, x \in \mathbb{R}^n, m \in \mathbb{N},$$

according to Corollary 6.1, for arbitrary $m \in \mathbb{N}$, any element of the set $X\left(\tilde{f}_m, t_m, c_m\right)$ exists on I, i.e.,

$$X\left(\tilde{f}_m, t_m, c_m\right) \subset C(I, \mathbb{R}^n), \qquad m \in \mathbb{N}.$$

Next, we prove that

$$\lim_{m \to \infty} X\left(\tilde{f}_m, t_m, c_m\right) \subseteq X(f_0, t_0, c_0).$$

We assume the opposite. Then, there exist $\varepsilon_0 > 0$, an increasing sequence $\{m_k\}_{k=1}^{\infty}$ of positive integers, and a sequence $\{x_k\}_{k=1}^{\infty}$ of functions from the space $C(I, \mathbb{R}^n)$ such that

$$x_k \in X\left(\tilde{f}_{m_k}, t_{m_k}, c_{m_k}\right), \qquad k \in \mathbb{N},$$

and that

$$x_k \notin X_{\varepsilon_0}(f_0, t_0, c_0), \qquad k \in \mathbb{N}.$$

Evidently,

$$x_k(t) = c_{m_k} + y_k(t) + z_k(t), \qquad t \in I, k \in \mathbb{N},$$

where

$$y_k(t) = \int_{t_{m_k}}^{t} \tilde{f}_{m_k}(\tau, x_k(\tau)) - f_{m_k}(\tau, 0)\,\mathrm{d}\tau, \qquad t \in I, k \in \mathbb{N},$$

and

$$z_k(t) = \int_{t_{m_k}}^{t} f_{m_k}(\tau, 0)\,\mathrm{d}\tau, \qquad t \in I, k \in \mathbb{N}.$$

From the condition 3., we obtain

$$\begin{aligned}\left\|\tilde{f}_m(t,x)-f_m(t,0)\right\| &= \|f_m(t,\chi(x))-f_m(t,0)\|\\ &\le \omega_{r_0+1}(t,\|\chi(x)\|)\\ &\le \omega_{r_0+1}(t,r_0+1), \qquad t\in I,\, x\in\mathbb{R}^n,\, m\in\mathbb{N},\end{aligned}$$

because, without loss of generality, we can assume that the function ω_{r_0+1} is non-decreasing in the second variable. Thus,

$$\left\|y_k'(t)\right\| \le \omega_{r_0+1}(t,r_0+1)$$

for almost all $t\in I$, $k\in\mathbb{N}$. Hence, the functions y_k, $k\in\mathbb{N}$, are equicontinuous. Moreover, we have

$$\begin{aligned}\|y_k(t)\| &\le \left|\int_{t_{m_k}}^{t}\left\|y_k'(\tau)\right\|\mathrm{d}\tau\right|\\ &\le \int_I \omega_{r_0+1}(\tau,r_0+1)\,\mathrm{d}\tau, \qquad t\in I,\, k\in\mathbb{N},\end{aligned}$$

i.e., the functions y_k, $k\in\mathbb{N}$, are uniformly bounded. According to the Arzelà–Ascoli theorem, without loss of generality, we can assume that the sequence $\{y_k\}_{k=1}^{\infty}$ is uniformly convergent.

At the same time, for $t\in I$, $k\in\mathbb{N}$, we have

$$\begin{aligned}\left\|z_k(t)-\int_{t_0}^{t} f_0(\tau,0)\,\mathrm{d}\tau\right\| &\le \left\|\int_{t_0}^{t_{m_k}} f_{m_k}(\tau,0)\,\mathrm{d}\tau\right\| + \left\|\int_{t_0}^{t} f_{m_k}(\tau,0)\,\mathrm{d}\tau - \int_{t_0}^{t} f_0(\tau,0)\,\mathrm{d}\tau\right\|\\ &\le \left\|\int_{t_0}^{t_{m_k}} f_{m_k}(\tau,0)\,\mathrm{d}\tau - \int_{t_0}^{t_{m_k}} f_0(\tau,0)\,\mathrm{d}\tau\right\| + \left\|\int_{t_0}^{t_{m_k}} f_0(\tau,0)\,\mathrm{d}\tau\right\|\\ &\qquad + \left\|\int_{t_0}^{t} f_{m_k}(\tau,0)\,\mathrm{d}\tau - \int_{t_0}^{t} f_0(\tau,0)\,\mathrm{d}\tau\right\|\\ &\le \int_{t_0}^{t_{m_k}} \|f_0(\tau,0)\|\,\mathrm{d}\tau\\ &\qquad + 2\max\left\{\left\|\int_{t_0}^{s} f_{m_k}(\tau,0)\,\mathrm{d}\tau - \int_{t_0}^{s} f_0(\tau,0)\,\mathrm{d}\tau\right\|; s\in I\right\}.\end{aligned}$$

Thus, from the conditions 1. and 2., we obtain

$$\lim_{k\to\infty} z_k(t) = \int_{t_0}^{t} f_0(\tau,0)\,\mathrm{d}\tau$$

uniformly on I. From

$$x_k(t) = c_{m_k} + y_k(t) + z_k(t), \qquad t\in I,\, k\in\mathbb{N},$$

from the condition 1., and from the uniform convergences of the sequences $\{y_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ obtained above, it follows that there exists $\tilde{x} \in C(I, \mathbb{R}^n)$ such that

$$\lim_{k\to\infty} x_k(t) = \tilde{x}(t)$$

uniformly on I.

Now, we show that

$$\tilde{x} \in X\left(\tilde{f}_0, t_0, c_0\right).$$

From

$$x_k \in X\left(\tilde{f}_{m_k}, t_{m_k}, c_{m_k}\right), \qquad k \in \mathbb{N},$$

it follows

$$x_k(t) = x_k(t_0) + \int_{t_0}^{t} f_{m_k}(\tau, \chi(x_k(\tau)))\,\mathrm{d}\tau, \qquad t \in I, k \in \mathbb{N}. \tag{8.1}$$

Since

$$\begin{aligned}\|x_k(t_0) - c_0\| &\le \|x_k(t_0) - \tilde{x}(t_0)\| + \left\|\tilde{x}(t_0) - \tilde{x}(t_{m_k})\right\| \\ &\quad + \left\|\tilde{x}(t_{m_k}) - x_k(t_{m_k})\right\| + \left\|x_k(t_{m_k}) - c_0\right\| \\ &\le 2\,\|x_k - \tilde{x}\|_C + \left\|\tilde{x}(t_0) - \tilde{x}(t_{m_k})\right\| + \left\|c_{m_k} - c_0\right\|\end{aligned}$$

for all $k \in \mathbb{N}$, from the condition 1., from the uniform limit

$$\lim_{k\to\infty} x_k(t) = \tilde{x}(t)$$

on I, and from the continuity of $\tilde{x}$, we obtain

$$\lim_{k\to\infty} x_k(t_0) = c_0.$$

Next,

$$\lim_{k\to\infty} \chi(x_k(t)) = \chi(\tilde{x}(t))$$

uniformly on I. Thus, (8.1) and Lemma 8.1 give

$$\tilde{x}(t) = c_0 + \int_{t_0}^{t} \tilde{f}_0(\tau, \tilde{x}(\tau))\,\mathrm{d}\tau, \qquad t \in I,$$

i.e.,

$$\tilde{x} \in X\left(\tilde{f}_0, t_0, c_0\right).$$

Therefore,

$$\tilde{x} \in X\left(f_0, t_0, c_0\right).$$

Since $\tilde{x}$ is the uniform limit of the sequence $\{x_k\}_{k=1}^{\infty}$, there exists $k_0 \in \mathbb{N}$ such that

$$\|\tilde{x} - x_k\|_C < \varepsilon_0, \qquad k \ge k_0, k \in \mathbb{N},$$

and, consequently,

$$x_k \in X_{\varepsilon_0}\left(f_0, t_0, c_0\right), \qquad k \ge k_0, k \in \mathbb{N},$$

which is a contradiction with

$$x_k \notin X_{\varepsilon_0}\left(f_0, t_0, c_0\right), \qquad k \in \mathbb{N}.$$

The contradiction proves

$$\lim_{m\to\infty} X\left(\tilde{f}_m, t_m, c_m\right) \subseteq X\left(f_0, t_0, c_0\right).$$

From Definition 8.2, it follows the existence of $m_0 \in \mathbb{N}$ for which

$$X\left(\tilde{f}_m, t_m, c_m\right) \subseteq X_1\left(f_0, t_0, c_0\right), \qquad m \geq m_0,\, m \in \mathbb{N}.$$

Let $m \geq m_0$ ($m \in \mathbb{N}$) and $y \in X\left(\tilde{f}_m, t_m, c_m\right)$ be arbitrary. Then, $y \in X_1\left(f_0, t_0, c_0\right)$, i.e., there exists $x \in X\left(f_0, t_0, c_0\right)$ such that $\|y - x\|_C < 1$. Thus,

$$\|y\|_C < \|x\|_C + 1 \leq r_0 + 1,$$

i.e.,

$$y \in X(f_m, t_m, c_m).$$

We have obtained that

$$X\left(\tilde{f}_m, t_m, c_m\right) \subseteq X\left(f_m, t_m, c_m\right), \qquad m \geq m_0,\, m \in \mathbb{N}.$$

The opposite inclusion is also valid, which gives

$$X\left(\tilde{f}_m, t_m, c_m\right) = X\left(f_m, t_m, c_m\right), \qquad m \geq m_0,\, m \in \mathbb{N}.$$

We have proved that

$$X\left(\tilde{f}_m, t_m, c_m\right) \subset C(I, \mathbb{R}^n), \qquad m \in \mathbb{N}.$$

Hence,

$$X\left(f_m, t_m, c_m\right) \subset C(I, \mathbb{R}^n), \qquad m \geq m_0,\, m \in \mathbb{N}.$$

Now, it is enough to consider

$$\lim_{m\to\infty} X\left(\tilde{f}_m, t_m, c_m\right) \subseteq X\left(f_0, t_0, c_0\right),$$

i.e.,

$$\lim_{m\to\infty} X\left(f_m, t_m, c_m\right) \subseteq X\left(f_0, t_0, c_0\right).$$

□

Corollary 8.1. *Let the assumptions 1.–4. from Theorem 8.1 be valid and let*

$$x_m \in X(f_m, t_m, c_m), \qquad m \in \mathbb{N}.$$

Then, the sequence $\{x_m\}_{m=1}^{\infty}$ *has a subsequence* $\left\{x_{m_k}\right\}_{k=1}^{\infty}$ *such that*

$$\lim_{k\to\infty} x_{m_k}(t) = x(t)$$

uniformly on I, where

$$x \in X(f_0, t_0, c_0).$$

Proof. From Theorem 8.1, it follows the existence of $m_0 \in \mathbb{N}$ such that

$$x_m \in C(I, \mathbb{R}^n), \qquad m \geq m_0, m \in \mathbb{N}.$$

In addition,

$$\lim_{m\to\infty} X(f_m, t_m, c_m) \subseteq X(f_0, t_0, c_0).$$

Thus, there exists $r_0 > 0$ for which

$$\|x_m(t)\| \leq r_0, \qquad t \in I, m \geq m_0, m \in \mathbb{N}.$$

We recall that the set $X(f_0, t_0, c_0)$ is bounded in the space $C(I, \mathbb{R}^n)$ (due to Theorem 4.1). We have obtained that the functions of the sequence $\{x_m\}_{m=m_0}^{\infty}$ are uniformly bounded.

Obviously,

$$x_m(t) = x_m(t_0) + y_m(t) + z_m(t), \qquad t \in I, m \geq m_0, m \in \mathbb{N},$$

where

$$y_m(t) = \int_{t_0}^{t} f_m(\tau, x_m(\tau)) - f_m(\tau, 0)\,\mathrm{d}\tau, \qquad t \in I, m \geq m_0, m \in \mathbb{N},$$

and

$$z_m(t) = \int_{t_0}^{t} f_m(\tau, 0)\,\mathrm{d}\tau, \qquad t \in I, m \geq m_0, m \in \mathbb{N}.$$

From the condition 3., we obtain

$$\begin{aligned} \|f_m(t,x) - f_m(t,0)\| &\leq \omega_{r_0}(t, \|x\|) \\ &\leq \omega_{r_0}(t, r_0), \qquad t \in I, x \in B[0, r_0], m \geq m_0, m \in \mathbb{N}, \end{aligned}$$

because we can assume (without loss of generality) that the function ω_{r_0} is non-decreasing in the second variable. At the same time, we have

$$\left\|y_m'(t)\right\| \leq \omega_{r_0}(t, r_0)$$

for almost all $t \in I$ and all $m \geq m_0$, $m \in \mathbb{N}$. Hence, the functions of the sequence $\{y_m\}_{m=m_0}^{\infty}$ are equicontinuous. From the condition 2., we get

$$\lim_{m\to\infty} z_m(t) = \int_{t_0}^{t} f_0(\tau, 0)\,\mathrm{d}\tau$$

uniformly on I. Thus, the functions of the sequence $\{z_m\}_{m=m_0}^{\infty}$ are equicontinuous. Therefore, the functions of the sequence $\{x_m\}_{m=m_0}^{\infty}$ are equicontinuous as well.

Using the Arzelà–Ascoli theorem, from the sequence $\{x_m\}_{m=m_0}^{\infty}$, one can extract a subsequence $\left\{x_{m_k}\right\}_{k=1}^{\infty}$ satisfying

$$\lim_{k\to\infty} x_{m_k}(t) = x(t)$$

uniformly on I, where $x \in C(I, \mathbb{R}^n)$.

It remains to prove that $x \in X(f_0,t_0,c_0)$. Let $\varepsilon > 0$ be arbitrary. Then, there exists $k_0 \in \mathbb{N}$ such that

$$x_{m_{k_0}} \in X_{\frac{\varepsilon}{2}}(f_0,t_0,c_0)$$

and that

$$\left\| x_{m_{k_0}} - x \right\|_C < \frac{\varepsilon}{2}.$$

Therefore, one can choose $y \in X(f_0,t_0,c_0)$ satisfying

$$x_{m_{k_0}} \in B\left(y, \frac{\varepsilon}{2}\right),$$

i.e.,

$$\left\| y - x_{m_{k_0}} \right\|_C < \frac{\varepsilon}{2}.$$

Thus, $x \in B(y,\varepsilon)$. From the arbitrariness of $\varepsilon > 0$, it follows that x is an element of the closure of the set $X(f_0,t_0,c_0)$. This set is closed (see Remark 4.1). Therefore,

$$x \in X(f_0,t_0,c_0).$$

$\square$

Remark 8.1. From the proof of Theorem 8.1 and from Corollary 8.1, it follows that the condition 3. can be replaced by the following one:

$\bar{3}$. for any $r > 0$, there exists a function $\omega_r \in K(I \times \mathbb{R}_+, \mathbb{R}_+)$ such that

$$\omega_r(-,0) \equiv 0$$

and that

$$\|f_m(t,x) - f_m(t,y)\| \leq \omega_r(t, \|x-y\|) + \omega_r\left(t, \frac{1}{m}\right)$$

for all $t \in I$, $x,y \in B[0,r]$, $m \in \mathbb{N}$.

9. Kneser theorem

Let I be a compact interval. We consider the Cauchy problem

$$x' = f(t,x), \tag{9.1}$$
$$x(t_0) = c_0, \tag{9.2}$$

where $f \in K(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$. As $X(f,t_0,c_0)$, we denote the set of all non-extendable solutions of the problem (9.1), (9.2).

Theorem 9.1 (Kneser)**.** *If any non-extendable solution of the problem* (9.1), (9.2) *exists on I, then the set $X(f,t_0,c_0)$ is compact and connected in the space $C(I,\mathbb{R}^n)$.*

Proof. According to Remark 4.1, the set $X(f,t_0,c_0)$ is compact in the space $C(I,\mathbb{R}^n)$. We show that it is connected. We suppose the opposite, i.e., let $X(f,t_0,c_0)$ be disconnected. Thus, there exist non-empty closed sets $X_1, X_2 \subset X(f,t_0,c_0)$ such that

$$X_1 \cap X_2 = \emptyset, \qquad X_1 \cup X_2 = X(f,t_0,c_0).$$

We denote

$$\delta = \inf\{\|y-x\|_C ; y \in X_1, x \in X_2\}.$$

We know that $\delta > 0$. The sets X_1, X_2 are compact. Hence, there exist $x_1 \in X_1$, $x_2 \in X_2$ such that

$$\|x_1 - x_2\|_C = \delta.$$

From the compactness of $X(f,t_0,c_0)$ in $C(I,\mathbb{R}^n)$, it follows the existence of $r > 0$ such that

$$\|x\|_C \leq r-1, \qquad x \in X(f,t_0,c_0).$$

Now, we use Lemma 1.6 and Remarks 1.2 and 1.3. There exists a sequence $\{f_m\}_{m=1}^{\infty}$ of functions satisfying the following conditions:

a. it holds
$$f_m \in K(I \times \mathbb{R}^n, \mathbb{R}^n), \qquad m \in \mathbb{N};$$

b. for any $\rho > 0$ and $m \in \mathbb{N}$, there exists a function $l_{\rho,m} \in L(I,\mathbb{R}_+)$ such that
$$\|f_m(t,x) - f_m(t,y)\| \leq l_{\rho,m}(t)\,\|x-y\|, \qquad t \in I,\, x,y \in B[0,\rho];$$

c. for almost all $t \in I$ and any $\rho > 0$, it holds
$$\lim_{m\to\infty} f_m(t,x) = f(t,x)$$
uniformly on $B[0,\rho]$;

d. for any $\rho > 0$, there exists a function $h_\rho \in L(I,\mathbb{R}_+)$ such that
$$\|f_m(t,x)\| \leq h_\rho(t), \qquad t \in I,\, x \in B[0,\rho],\, m \in \mathbb{N};$$

e. for any $\rho > 0$, there exists a function $\omega_\rho \in K(I \times \mathbb{R}_+, \mathbb{R}_+)$ such that

$$\omega_\rho(-,0) \equiv 0$$

and that

$$\|f_m(t,x) - f_m(t,y)\| \le \omega_\rho(t, \|x-y\|) + \omega_\rho\left(t, \frac{1}{m}\right)$$

for all $t \in I$, $x, y \in B[0,\rho]$, $m \in \mathbb{N}$.

For all $\lambda \in [0,1]$ and $m \in \mathbb{N}$, we put

$$\begin{aligned} f_{\lambda,m}(t,x) = f_m(t,x) &+ (1-\lambda)\left[f(t,x_1(t)) - f_m(t,x_1(t))\right] \\ &+ \lambda\left[f(t,x_2(t)) - f_m(t,x_2(t))\right], \qquad t \in I, x \in \mathbb{R}^n. \end{aligned}$$

One can easily verify that the functions $f_{\lambda,m}$ satisfy the following conditions:

1. it holds
$$f_{\lambda,m} \in K(I \times \mathbb{R}^n, \mathbb{R}^n), \qquad \lambda \in [0,1], m \in \mathbb{N};$$

2. for any $\rho > 0$ and $m \in \mathbb{N}$, there exists a function $l_{\rho,m} \in L(I, \mathbb{R}_+)$ such that
$$\left\|f_{\lambda,m}(t,x) - f_{\lambda,m}(t,y)\right\| \le l_{\rho,m}(t)\ \|x-y\|, \qquad t \in I, x, y \in B[0,\rho], \lambda \in [0,1];$$

3. for almost all $t \in I$ and any $\rho > 0$, it holds
$$\lim_{m\to\infty} f_{\lambda,m}(t,x) = f(t,x)$$
uniformly with respect to $x \in B[0,\rho]$, $\lambda \in [0,1]$;

4. for any $\rho > 0$, it holds
$$\left\|f_{\lambda,m}(t,x)\right\| \le \tilde{h}_\rho(t), \qquad t \in I, x \in B[0,\rho], \lambda \in [0,1], m \in \mathbb{N},$$
where
$$\tilde{h}_\rho(t) = h_\rho(t) + h_{r-1}(t) + \|f(t,x_1(t))\| + \|f(t,x_2(t))\|, \qquad t \in I;$$

5. for any $\rho > 0$, there exists a function $\omega_\rho \in K(I \times \mathbb{R}_+, \mathbb{R}_+)$ with the property that
$$\omega_\rho(-,0) \equiv 0$$
and that
$$\left\|f_{\lambda,m}(t,x) - f_{\lambda,m}(t,y)\right\| \le \omega_\rho(t, \|x-y\|) + \omega_\rho\left(t, \frac{1}{m}\right)$$
for all $t \in I$, $x, y \in B[0,\rho]$, $\lambda \in [0,1]$, $m \in \mathbb{N}$.

For all $\lambda \in [0,1]$ and $m \in \mathbb{N}$, we consider the problem

$$x' = f_{\lambda,m}(t,x), \tag{9.3}$$
$$x(t_0) = c_0. \tag{9.2}$$

Due to the condition 2., from Corollary 7.1, it follows that the problem (9.3), (9.2) is uniquely solvable for any $\lambda \in [0,1]$ and $m \in \mathbb{N}$. As $x_{\lambda,m}$, we denote the non-extendable solution of this problem. Since the convergence is uniform with respect to λ in the condition 3. and since the majorants are also independent on λ in the conditions 4. and 5., the assumptions of Theorem 8.1 (Remark 8.1) are satisfied (uniformly with respect to λ). Thus, there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, $m \in \mathbb{N}$, and $\lambda \in [0,1]$, the solution $x_{\lambda,m}$ exists on the interval I. Moreover, due to the inequality

$$\|x\|_C \leq r-1, \qquad x \in X(f,t_0,c_0),$$

the proof of Theorem 8.1 guarantees that, without loss of generality, we can assume the inequality

$$\left\|x_{\lambda,m}\right\|_C \leq r, \qquad m \geq m_0, m \in \mathbb{N}, \lambda \in [0,1].$$

We denote

$$\eta_m(\lambda) = \inf\left\{\left\|x_{\lambda,m} - y\right\|_C ; y \in X_1\right\}, \qquad \lambda \in [0,1], m \geq m_0, m \in \mathbb{N}.$$

From the definition of the function $f_{\lambda,m}$ and the uniqueness of the solution $x_{\lambda,m}$, it follows that

$$x_{0,m} \equiv x_1, \qquad x_{1,m} \equiv x_2, \qquad m \in \mathbb{N}.$$

Therefore, $\eta_m(0) = 0$, $\eta_m(1) = \delta$, $m \geq m_0$, $m \in \mathbb{N}$. Let $m \geq m_0$ ($m \in \mathbb{N}$) be arbitrarily given. Let $\lambda_0 \in [0,1]$ be also arbitrarily given and let $\{\lambda_k\}_{k=1}^{\infty} \subset [0,1]$ be a sequence for which

$$\lim_{k\to\infty} \lambda_k = \lambda_0.$$

Using Theorem 8.1, one can verify that

$$\lim_{k\to\infty} x_{\lambda_k,m}(t) = x_{\lambda_0,m}(t)$$

uniformly on I. Since the set X_1 is compact, from the definition of the function η_m, it follows that

$$\lim_{k\to\infty} \eta_m(\lambda_k) = \eta_m(\lambda_0),$$

i.e., the function η_m is continuous on $[0,1]$. Therefore,

$$\eta_m(0) = 0, \qquad \eta_m(1) = \delta, \qquad m \geq m_0, m \in \mathbb{N},$$

implies that, for any $m \geq m_0$, $m \in \mathbb{N}$, there exists $\lambda_m \in (0,1)$ with the property that

$$\eta_m(\lambda_m) = \frac{\delta}{2},$$

i.e.,

$$\inf\left\{\left\|x_{\lambda_m,m} - y\right\|_C ; y \in X_1\right\} = \frac{\delta}{2}, \qquad m \geq m_0, m \in \mathbb{N}.$$

Due to the condition 4. and

$$\left\|x_{\lambda,m}\right\|_C \leq r, \qquad m \geq m_0, m \in \mathbb{N}, \lambda_0 \in [0,1],$$

the functions of the sequence $\left\{x_{\lambda_m,m}\right\}_{m=m_0}^{\infty}$ have to be uniformly bounded and equicontinuous. According to the Arzelà–Ascoli theorem, without loss of generality, we can assume that this sequence is convergent, i.e., there exists $\tilde{x} \in C(I,\mathbb{R}^n)$ such that

$$\lim_{m\to\infty} x_{\lambda_m,m}(t) = \tilde{x}(t)$$

uniformly on I.

Next, Theorem 8.1 gives

$$\tilde{x} \in X(f, t_0, c_0).$$

Since the set X_1 is compact, we have

$$\inf\{\|\tilde{x} - y\|_C ; y \in X_1\} = \frac{\delta}{2}.$$

Thus, we know that $\tilde{x} \notin X_1$ and, consequently, $\tilde{x} \in X_2$, which is a contradiction. The contradiction proves that the set $X(f, t_0, c_0)$ is connected. $\square$

10. Fukuhara theorems

Let I be a compact interval. We consider the Cauchy problem

$$x' = f(t,x), \tag{10.1}$$
$$x(t_0) = c_0, \tag{10.2}$$

where $f \in K(I \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 \in I$, $c_0 \in \mathbb{R}^n$. As $X(f,t_0,c_0)$, we denote the set of all non-extendable solutions of the problem (10.1), (10.2). For $M \subseteq \mathbb{R}^m$, we denote

$$X(f,t_0,M) = \bigcup_{c \in M} X(f,t_0,c).$$

Theorem 10.1 (1. Fukuhara)**.** *Let M be a closed and connected subset of $\mathbb{R}^n$. If, for any $c_0 \in M$, all non-extendable solution of the problem* (10.1)*,* (10.2) *exists on I, then the set $X(f,t_0,M)$ is closed and connected in the space $C(I,\mathbb{R}^n)$. Moreover, if the set M is bounded, then the set $X(f,t_0,M)$ is compact.*

Proof. The set $X(f,t_0,M)$ is closed in the space $C(I,\mathbb{R}^n)$. Indeed, it suffices to consider Theorem 8.1 and the fact that the set M is closed.

We show that the set $X(f,t_0,M)$ is connected. We suppose the opposite. Then, there exist non-empty closed sets $X_1, X_2 \subset X(f,t_0,M)$ such that

$$X_1 \cap X_2 = \emptyset, \qquad X_1 \cup X_2 = X(f,t_0,M).$$

Let $c_0 \in M$ be arbitrary. We denote

$$Y_i = X(f,t_0,c_0) \cap X_i, \qquad i \in \{1,2\}.$$

If $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$, then Y_1 and Y_2 are non-empty closed subsets of $X(f,t_0,c_0)$ such that

$$Y_1 \cap Y_2 = \emptyset, \qquad Y_1 \cup Y_2 = X(f,t_0,c_0).$$

Next, from Theorem 9.1, it follows that the set $X(f,t_0,c_0)$ is a subset of X_i for some $i \in \{1,2\}$. We denote

$$M_i = \{c_0 \in M; X(f,t_0,c_0) \subseteq X_i\}, \qquad i \in \{1,2\}.$$

Obviously, $M_1 \neq \emptyset$, $M_2 \neq \emptyset$, $M_1 \cap M_2 = \emptyset$, and $M_1 \cup M_2 = M$.

We prove that the set M_1 is closed. Let $\{c_m\}_{m=1}^{\infty} \subseteq M_1$ be such that

$$\lim_{m\to\infty} c_m = c_0.$$

Then, $c_0 \in M$ (M is closed) and there exists a sequence $\{x_m\}_{m=1}^{\infty} \subseteq X_1$ such that

$$x_m \in X(f,t_0,c_m), \qquad m \in \mathbb{N}.$$

Without loss of generality (see Corollary 8.1), we can assume that

$$\lim_{m\to\infty} x_m(t) = \tilde{x}(t)$$

uniformly on I, where

$$\tilde{x} \in X(f, t_0, c_0).$$

Since the set X_1 is closed, we see that $\tilde{x} \in X_1$. Thus, $c_0 \in M_1$, which means that M_1 is closed.

Analogously, one can prove that M_2 is closed as well. Hence, we have proved that M_1, M_2 are non-empty closed subsets of the set M satisfying

$$M_1 \cap M_2 = \emptyset, \qquad M_1 \cup M_2 = M,$$

which is a contradiction with the assumption that M is connected. The contradiction means that the set $X(f, t_0, M)$ is connected.

Now, we consider that the set M is bounded. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence of elements of $X(f, t_0, M)$. Then, there exists a sequence $\{c_m\}_{m=1}^{\infty} \subseteq M$ such that

$$x_m \in X(f, t_0, c_m), \qquad m \in \mathbb{N}.$$

Due to the compactness of the set M, without loss of generality, we can assume that

$$\lim_{m \to \infty} c_m = c_0,$$

where $c_0 \in M$. From Corollary 8.1, it follows the existence of a subsequence $\left\{x_{m_k}\right\}_{k=1}^{\infty}$ of the sequence $\{x_m\}_{m=1}^{\infty}$ satisfying

$$\lim_{k \to \infty} x_{m_k}(t) = x(t)$$

uniformly on I, where

$$x \in X(f, t_0, c_0).$$

Since $c_0 \in M$, we see that

$$x \in X(f, t_0, M),$$

i.e., the set $X(f, t_0, M)$ is compact in the space $C(I, \mathbb{R}^n)$. □

Let

$$X(f, t_0, c_0) \subseteq C(I, \mathbb{R}^n).$$

We denote

$$W(f, t_0, c_0) = \{(t, x(t)); t \in I, x \in X(f, t_0, c_0)\}$$

and, for $t \in I$, we put

$$W_t(f, t_0, c_0) = \{x \in \mathbb{R}^n; (t, x) \in W(f, t_0, c_0)\}.$$

Theorem 10.2 (2. Fukuhara)**.** *If any non-extendable solution of the problem* (10.1), (10.2) *exists on I, then, for all $\bar{t} \in I$ and $\bar{c} \in \partial W_{\bar{t}}(f, t_0, c_0)$, there exists $\bar{x} \in X(f, t_0, c_0)$ such that $\bar{x}(\bar{t}) = \bar{c}$ and the graph of the function $\bar{x}$ between t_0 and $\bar{t}$ is on the boundary of the set $W(f, t_0, c_0)$, i.e.,*

$$(t, \bar{x}(t)) \in \partial W(f, t_0, c_0), \qquad \min\{t_0, \bar{t}\} \le t \le \max\{t_0, \bar{t}\}.$$

Proof. If we use the boundedness of the set $X(f,t_0,c_0)$ (see Theorem 4.1), then, without loss of generality, we can assume the existence of a function $h \in L(I,\mathbb{R}_+)$ for which

$$\|f(t,x)\| \le h(t), \qquad t \in I, x \in \mathbb{R}^n.$$

We prove the theorem in the case when $\bar{t} > t_0$ (in the second case, the proof is analogical). According to Theorem 9.1, $W_{\bar{t}}(f,t_0,c_0)$ is a connected and compact subset of $\mathbb{R}^n$. Therefore, there exists a sequence $\{c_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ such that

$$c_k \notin W_{\bar{t}}(f,t_0,c_0), \qquad k \in \mathbb{N},$$

and that

$$\lim_{k\to\infty} c_k = \bar{c}.$$

For $k \in \mathbb{N}$, we consider the problem

$$x' = f(t,x), \tag{10.1}$$

$$x(\bar{t}) = c_k. \tag{10.3}$$

We assume that

$$\|f(t,x)\| \le h(t), \qquad t \in I, x \in \mathbb{R}^n.$$

Therefore, for all $k \in \mathbb{N}$, there exists a solution x_k of the problem (10.1), (10.3) on the interval I, where (see also Corollary 6.1)

$$x_k \in X(f,\bar{t},c_k), \qquad k \in \mathbb{N}.$$

By contradiction, we show that

$$(t,x_k(t)) \notin W(f,t_0,c_0), \qquad t \in [t_0,\bar{t}], k \in \mathbb{N}.$$

If, for some $k \in \mathbb{N}$, there exists $t^\star \in [t_0,\bar{t})$ such that

$$(t^\star,x_k(t^\star)) \in W(f,t_0,c_0),$$

then one can find $\tilde{x} \in X(f,t_0,c_0)$ satisfying $\tilde{x}(t^\star) = x_k(t^\star)$. We denote

$$y(t) = \begin{cases} \tilde{x}(t), & t \in [\inf I, t^\star]; \\ x_k(t), & t \in (t^\star, \sup I]. \end{cases}$$

Obviously,

$$y \in X(f,t_0,c_0).$$

Therefore,

$$y(\bar{t}) \in W_{\bar{t}}(f,t_0,c_0).$$

At the same time,

$$y(\bar{t}) = c_k \notin W_{\bar{t}}(f,t_0,c_0),$$

which is a contradiction.

Corollary 6.1 gives that all non-extendable solution of the problem

$$x' = f(t,x), \tag{10.1}$$

$$x(\bar{t}) = \bar{c} \tag{10.4}$$

exists on the interval I. Therefore, with regard to Corollary 8.1, without loss of generality, we can assume that

$$\lim_{k\to\infty} x_k(t) = x_0(t)$$

uniformly on I, where

$$x_0 \in X(f,\bar{t},\bar{c}).$$

Let $s \in (t_0,\bar{t})$ be arbitrarily given. We consider the sets $W_s(f,t_0,c_0)$ and $W_s(f,\bar{t},\bar{c})$. We have proved that $x_0(s)$ is not an inner point of the set $W_s(f,t_0,c_0)$, i.e.,

$$x_0(s) \notin W_s(f,t_0,c_0) \qquad \text{or} \qquad x_0(s) \in \partial W_s(f,t_0,c_0).$$

But, it is valid that

$$W_s(f,t_0,c_0) \cap W_s(f,\bar{t},\bar{c}) \neq \emptyset.$$

Indeed, since $(\bar{t},\bar{c}) \in W(f,t_0,c_0)$, there exists $\tilde{x} \in X(f,t_0,c_0)$ such that $\tilde{x}(\bar{t}) = \bar{c}$. This fact means that

$$\tilde{x} \in X(f,\bar{t},\bar{c}).$$

Thus,

$$\tilde{x}(s) \in W_s(f,t_0,c_0) \cap W_s(f,\bar{t},\bar{c}).$$

The sets $W_s(f,t_0,c_0)$ and $W_s(f,\bar{t},\bar{c})$ are connected and closed and they are not disjoint. There exists a point belonging to the set $W_s(f,\bar{t},\bar{c})$ which is not in the interior of the set $W_s(f,t_0,c_0)$. Therefore, there exists a point $c_s \in \mathbb{R}^n$ such that

$$c_s \in \partial W_s(f,t_0,c_0) \cap W_s(f,\bar{t},\bar{c}).$$

Hence, there exists a solution $\tilde{x}_1$ of the problem (10.1), (10.2) passing through the point $[s,c_s]$ and, at the same time, there exists a solution $\tilde{x}_2$ of the problem (10.1), (10.4) passing through the point $[s,c_s]$. If we put

$$y(t) = \begin{cases} \tilde{x}_1(t), & t \in [\inf I, s]; \\ \tilde{x}_2(t), & t \in (s, \sup I], \end{cases}$$

then

$$y \in X(f,t_0,c_0), \qquad y(s) = c_s, \qquad y(\bar{t}) = \bar{c}.$$

We have proved that, for any $s \in (t_0,\bar{t})$, there exists $y \in X(f,t_0,c_0)$ such that

$$y(\bar{t}) = \bar{c}, \qquad (s,y(s)) \in \partial W(f,t_0,c_0).$$

Thus, for an arbitrary set

$$\{s_1,\ldots,s_m\} \subset (t_0,\bar{t}),$$

there exists $y_m \in X(f,t_0,c_0)$ with the property that

$$y_m(\bar{t}) = \bar{c}, \qquad (s_i,y_m(s_i)) \in \partial W(f,t_0,c_0), \qquad i \in \{1,\ldots,m\}. \tag{10.5}$$

Let $\{s_k\}_{k=1}^{\infty}$ be dense in $(t_0,\bar{t})$. It is obvious that, for any $m \in \mathbb{N}$, there exists $y_m \in X(f,t_0,c_0)$ satisfying (10.5). Since the set $X(f,t_0,c_0)$ is compact in the space $C(I,\mathbb{R}^n)$, without loss of generality, we can assume that the sequence $\{y_m\}_{m=1}^{\infty}$ is convergent, i.e.,

$$\lim_{m\to\infty} y_m(t) = \bar{x}(t)$$

uniformly on I, where

$$\bar{x} \in X(f, t_0, c_0).$$

Using (10.5), one can easily verify that

$$(t, \bar{x}(t)) \in \partial W(f, t_0, c_0), \qquad t \in [t_0, \bar{t}].$$

The proof is complete. $\square$

REFERENCES

[1] Y. Deng: Lectures, problems and solutions for ordinary differential equations. Hackensack: World Scientific, 2018.

[2] S. P. Hastings: Classical methods in ordinary differential equations: with applications to boundary value problems. Providence: American Mathematical Society, 2012.

[3] D. W. Jordan: Nonlinear ordinary differential equations—problems and solutions: a sourcebook for scientists and engineers. Oxford: Oxford University Press, 2007.

[4] A. K. Nandakumaran, P. S. Datti, R. K. George: Ordinary differential equations: principles and applications. New York: Cambridge University Press, 2017.

[5] C. E. Roberts: Ordinary differential equations: applications, models, and computing. Boca Raton: Chapman and Hall/CRC, 2010.

[6] D. G. Schaeffer: Ordinary differential equations: basics and beyond. New York: Springer, 2016.