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Introduction to Cosmology

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ABSTRACT: This paper contains notes devoted to the Introduction to cosmology

KEYWORDS: Cosmology.

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1. Basic Principles

1.1 Units

We mostly use the natural system of units where the Planck constant, speed of light and the Boltzman constant are equal to one

$$\hbar = c = k_B = 1 \ . \tag{1.1}$$

Then the mass M, energy E and temperature T have the same dimensions since

$$[E] = [Mc^2] = [M] \tag{1.2}$$

and also we have

$$[E] = [k_B T] = [T] = [M] \ . \tag{1.3}$$

Time t and length l have in natural system dimension $[M]^{-1}$ as follows from the fact that

$$[E] = [\hbar\omega] = [\omega] = [t^{-1}] \tag{1.4}$$

so that $[t] = [M]^{-1}$. In the same way we have

$$[l] = [ct] = [t] = [M]^{-1} \ . \tag{1.5}$$

It is useful to know coeficients that relate various units

Quantity	SI dimensions	Natuaral dimensions	Conversions
mass	kg	M	$1GeV = 1.8 \times 10^{-27} kg$
length	m	M^{-1}	$1GeV^{-1} = 0.197 \times 10^{-15} m$
time	s	M^{-1}	$1GeV^{-1} = 6.58 \times 10^{-25} s$
energy	$kg \cdot m^2 \cdot s^{-2}$	M	$1GeV = 5.39 \times 10^{-19} kg \cdot m \cdot s^{-1}$
momentum	$kg \cdot m \cdot s^{-1}$	M	$1GeV = 5.39 \times 10^{-19} kg \cdot m \cdot s^{-1}$
velocity	$m \cdot s^{-1}$		$1 = 2.998 \times 10^8 m \cdot s^{-1}$
cross section	m^2	M^{-2}	$1GeV^{-2} = 0.389 \times 10^{-31} m^2$
force	$kg \cdot m \cdot s^{-2}$	M^2	$1GeV^2 = 8.19 \times 10^5$Newton

The traditional unit of length in cosmology is Megaparsec

$$1\ Mpc = 3.1 \times 10^{22} m\ . \tag{1.6}$$

It is interesting to mention the several units of length that are used in astronomy. Besides the metric system in use are the astronomical unit ($a.u.$) which is the average distance from the Earth to the Sun

$$1\ a.u. = 1.5 \times 10^{11} m \tag{1.7}$$

Further, there is a light year, the distance that a photon travels in one year

$$1 \text{ year} = 3.16 \times 10^7 s\ , 1 \text{ light year} = 0.95 \times 10^{16} m \tag{1.8}$$

parsec (pc)-distance from which an object of size $1a.u.$ is seen at angle $1arc$ second

$$1\ pc = 2.1 \cdot 10^5 a.u. = 3.3 \text{ light year} = 3.1 \times 10^{16} m \tag{1.9}$$

It is instructive to give distances of various objects expressed in above units.

$10a.u.$ is the average disance to Saturn, $30a.u.$ is the same for Pluto, $100a.u.$ is the estimate of the maximum distance which can be reached by solar wind (particles emitted by the Sun). The nearest stars-Proxima and Alpha Centauri are at $1.3pc$ from the Sun. The distance to Arcturus and Capella is more than $10pc$, the distances to Canopus and Betelgeuse are about $100pc$ and $200pc$ respectively. Crab Nebula-the remnant of supernova is $2kpc$ away from us.

The next point on the scale of distance is $8kpc$. This is the distance from the Sun to the center of our Galaxy. Our Galaxy is of spiral type, the diameter of its disc is about $30kpc$ and the thickness of the disc is about $250pc$. The distance to the nearest dwarf galaxies that are satelites of our Galaxy is about $30kpc$. Fifteen of these satellites are known, the largest of them are Large and Small Magellanic Clouds are about $50kpc$ away. It is also interesting to note that only eight Milky Way satellites were known by 1994.

The mass density of the usual matter in usual (not dwarf) galaxies is about 10^5 higher than the average over Universe.

The nearest usual galaxy-the spiral galaxy $M31$ in Andromeda constellation- is $800kpc$ away from the Milky Way. Another nearby galaxy is in Triangulum constellation. Our Galaxy together with Andromeda and Triangulum galaxies , their satelites and other 35 smaller galaxies constitute the Local Group which is the gravitationally bound object consisting of about 50 galaxies.

The next scale is the size of clusters of galaxies which is $1-3Mpc$ Rich clusters contain thounsands of galaxies. The mass density in clusters exceeds the average density over the Universe by a factor of a hundred and even sometimes a thousand. The distance to the center of the nearest cluster, which is the Vigo constellation, is about $15Mpc$. Clusters of galaxies are the largest gravitationally bound systems in the Universe.

1.2 Gravitational Field Equations

As we know in General Relativity (GR) the metric tensor is dynamical field and the equations of GR arise as extremum conditions for the action functional. The princip equivalence means that all equations has to have the same form in all reference frames. In other words we require that the action function has to be same in all reference frames which means that the action is scalar. Since the action is given as the integral over time of the Lagrangian we find also that the Lagrangian has to be given as the integral over space section of the spacetime. In summary we postulate thath the gravity action has the form

$$S_{gr} = \int d^4x\sqrt{-g}\mathcal{L}_{gr} \ , \tag{1.10}$$

where the Lagrangian density $\mathcal{L}_{gr}(x)$ transforms as under coordinate transformations $x'^{\mu} = x^{\mu}(x)$

$$\mathcal{L}'(x') = \mathcal{L}(x) \tag{1.11}$$

and due to the fact that $d^4x'\sqrt{-g'(x')} = d^4x\sqrt{-g(x)}$ we really see that S_{gr} does not change under diffeomorphism transformations.

The simplest possibility is to take the Lagrangian density to be equal to constant $\mathcal{L} = -\Lambda$ so that

$$S_\Lambda = -\Lambda\int d^4x\sqrt{-g} \ . \tag{1.12}$$

However this action does not contain the time derivatives of the metric and hence the dynamics that would follow from this action is trivial. For that reason we should search more complicated form of the Lagrangian density.

The Lagrange density is a tensor density, which can be written as $\sqrt{-g}$ times a scalar that is function of the metric and its derivatives. The question is the form of given scalar. Since we know that the metric can be set equal to its canonical form and its first derivatives set to zero at any one point, any nontrivial scalar must involve at least second derivatives of the metric. The Riemann tensor is of course made from second derivatives of the metric, and we argued earlier that the only independent scalar we could construct from the Riemann tensor was the Ricci scalar R. What we did not show, but is nevertheless true, is that any nontrivial tensor made from the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor. Therefore, the only independent scalar constructed from the metric, which is no higher than second order in its derivatives, is the Ricci scalar. Hilbert figured that this was therefore the simplest possible choice for a Lagrangian, and proposed

$$\mathcal{L}_H = \sqrt{-g}R \ . \tag{1.13}$$

The equations of motion should come from varying the action with respect to the metric. In fact let us consider variations with respect to the inverse metric $g^{\mu\nu}$, which are slightly easier but give an equivalent set of equations. Using $R = g^{\mu\nu}R_{\mu\nu}$, in general we will have

$$\begin{aligned} \delta S &= \int d^n x \left[\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + R\delta\sqrt{-g}\right] \\ &= (\delta S)_1 + (\delta S)_2 + (\delta S)_3 \ . \end{aligned} \tag{1.14}$$

The second term $(\delta S)_2$ is already in the form of some expression times $\delta g^{\mu\nu}$; let's examine the others more closely.

Recall that the Ricci tensor is the contraction of the Riemann tensor, which is given by

$$R^\rho{}_{\mu\lambda\nu} = \partial\lambda\Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu) \ . \tag{1.15}$$

We perform the variation of the Riemann tensor in such a way that we firstly perform variation of the connection coefficients and then we substitute into this expression. In fact, after some calculations we find the variation of the Riemann tensor in the form

$$\delta R^\rho{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\lambda\mu}) \ . \tag{1.16}$$

Therefore, the contribution of the first term in (1.14) to δS can be written

$$\begin{aligned} (\delta S)_1 &= \int d^4x\sqrt{-g}\ g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\right] \\ &= \int d^4x\sqrt{-g}\ \nabla_\sigma\left[g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu}) - g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu})\right] \ , \end{aligned} \tag{1.17}$$

where we have used metric compatibility. However the integral above is an integral with respect to the natural volume element of the covariant divergence of a vector; by Stokes's theorem, this is equal to a boundary contribution at infinity which we can set to zero by making the variation vanish at infinity. Therefore this term does not contribute to the total variation.

In order to calculate the $(\delta S)_3$ term we have to use the variation

$$\delta(g^{-1}) = \frac{1}{g}g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{1.18}$$

and consequently

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{1.19}$$

If we now return back to (1.14), and remembering that $(\delta S)_1$ does not contribute, we find

$$\delta S = \int d^4x\sqrt{-g} \ \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} \ . \tag{1.20}$$

However this should vanish for arbitrary variations and consequently we derive Einstein's equations in vacuum:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \ . \tag{1.21}$$

However we would like to get the non-vacuum field equations as well. In other words we consider an action of the form

$$S = \frac{1}{8\pi G}S_H + S_M \ , \tag{1.22}$$

where S_M is the action for matter, and we have presciently normalized the gravitational action (although the proper normalization is somewhat convention-dependent). Following through the same procedure as above leads to

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \ , \tag{1.23}$$

and we recover Einstein's equations if we set

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \ . \tag{1.24}$$

In fact (1.24) turns out to be the best way to define a symmetric energy-momentum tensor.

Einstein's equations may be thought of as second-order differential equations for the metric tensor field $g_{\mu\nu}$. There are ten independent equations (since both sides are symmetric two-index tensors), which seems to be exactly right for the ten unknown functions of the metric components. However, the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ which we prove below represents four constraints on the functions $R_{\mu\nu}$, so there are only six truly independent equations. In fact this is appropriate, since if a metric is a solution to Einstein's equation in one coordinate system x^μ it should also be a solution in any other coordinate system $x^{\mu'}$. This means that there are four unphysical degrees of freedom in $g_{\mu\nu}$ (represented by the four functions $x^{\mu'}(x^\mu)$), and we should expect that Einstein's equations only constrain the six coordinate-independent degrees of freedom.

It is important to stress that as differential equations, these are extremely complicated; the Ricci scalar and tensor are contractions of the Riemann tensor, which involves derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor $T_{\mu\nu}$ will generally involve the metric as well. The equations are also nonlinear, that implies that two known solutions cannot be superposed to find a third. It is therefore very difficult to solve Einstein's equations in any sort of generality. Then in order to solve them we have to perform some simplifying assumptions. The most popular sort of simplifying assumption is that the metric has a significant degree of symmetry, and we will talk later on about how symmetries of the metric make life easier.

We are mainly interested in the existence of solutions to Einstein's equations in the presence of "realistic" sources of energy and momentum. The most common property that is demanded of $T_{\mu\nu}$ is that it represent positive energy densities — no negative masses are allowed. In a locally inertial frame this requirement can be written as $\rho = T_{00} \geq 0$. We write it in the coordinate-independent notation as

$$T_{\mu\nu}V^\mu V^\nu \geq 0 \ , \qquad \text{for all timelike vectors } V^\mu \ . \tag{1.25}$$

This is known as the **Weak Energy Condition**, or WEC. It seems like a reasonable requirement however it is very restrictive. Indeed it is straightforward to show that there are many examples of the classical field theories which violate the WEC, and almost impossible to invent a quantum field theory which obeys it. Nevertheless, it is legitimate to assume that the WEC holds in most cases and it is violated in some extreme conditions. (There are also stronger energy conditions, but they are even less true than the WEC, and we won't dwell on them.)

An important property of the energy momentum tensor is that it is conserved. In the flat background the conservation equation takes the form

$$\partial_\mu T^{\mu\nu} = 0 \ , \tag{1.26}$$

where the first equation $\partial_\mu T^{\mu i} = 0$ expresses the conservation of the energy density while the remaining three equations $\partial_\mu T^{\mu i} = 0$ defines the conservation of the momentum density. In general relativity the conservation equation takes the form

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{1.27}$$

This equation can be proved using the equation of motion for the metric when we apply the covariant derivative on both sides of this equation

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 8\pi G \nabla^\mu T_{\mu\nu} \ . \tag{1.28}$$

We show that the left side of this equation is *identically zero.* Note that generally the matter fields do not have to be on shell since this equation follows from the variation of the action with respect to the metric. To see this we recall the Bianchi identity for the Riemann tensor

$$\nabla_\rho R^\lambda{}_{\sigma\mu\nu} + \nabla_\nu R^\lambda{}_{\sigma\rho\mu} + \nabla_\mu R^\lambda{}_{\sigma\nu\rho} = 0 \ . \tag{1.29}$$

Now we contract λ and μ indices and by definition $R^\mu{}_{\sigma\mu\nu} = R_{\sigma\nu}$ we obtain the identity

$$\nabla_\rho R_{\sigma\nu} - \nabla_\nu R_{\rho\sigma} + \nabla_\lambda R^\lambda{}_{\sigma\nu\rho} = 0 \ . \tag{1.30}$$

Then we contract this equation with $g^{\rho\sigma}$ and we obtain

$$0 = \nabla_\rho R^\rho{}_\nu - \nabla_\nu R + \nabla^\lambda R_{\lambda\nu} = 2\nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0 \ . \tag{1.31}$$

which implies that the covariant conservation law of the stress energy-tensor is a necessary condition for the consistency of the Einstein equation.

On the other hand the stress energy tensor is determined by the matter action. Clearly when we search the extremum of the action we perform the variation of the action with respect to the matter fields so that the energy momentum tensor should be conserved as the consequence of the matter equations of motions as well. Alternatively, we can presume the evolution of the matter fields on the fixed background and in this case the energy-momentum tensor should be conserved as well.

To proceed note that the matter action is diffeomorphism invariant so that the conservation of the energy momentum tensor should follow from the invariance of the action under general diffeomorphism transformation. In fact, under transformation

$$x'^\mu = x^\mu + \xi^\mu \ . \tag{1.32}$$

Then

$$\begin{aligned} g'^{\mu\nu}(x') &= g^{\rho\sigma} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \Rightarrow \\ g'^{\mu\nu}(x') &= g^{\mu\nu}(x) + g^{\nu\lambda}(x)\partial_\lambda \xi^\mu + \partial_\lambda x^\mu g^{\lambda\nu}(x) \end{aligned} \tag{1.33}$$

If we expand

$$g'^{\mu\nu}(x') = g'^{\mu\nu}(x+\xi) = g'^{\mu\nu}(x) + \partial_\lambda g'^{\mu\nu} \xi^\lambda = g'^{\mu\nu}(x) + \partial_\lambda g^{\mu\nu} \xi^\lambda \tag{1.34}$$

we find the variation $g^{\mu\nu}$ as

$$\delta g^{\mu\nu}(x) = g'^{\mu\nu}(x) - g^{\mu\nu}(x) = -\partial_\lambda g^{\mu\nu}(x) \xi^\lambda + g^{\mu\lambda} \partial_\lambda \xi^\nu + \partial_\lambda \xi^\mu g^{\lambda\nu} \ . \tag{1.35}$$

Now we proceed to the transformation property of the matter fields. Their form depends on the character of these fields, whether they are scalars, vectors,..... For example, in case of the scalar field we find

$$\phi'(x') = \phi(x) \Rightarrow \phi'(x) - \phi(x) = -\partial_\lambda \phi \xi^\lambda \tag{1.36}$$

Since the action is invariant under the diffeomorphism invariance we obtain

$$\delta_\xi S_m = \frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu) + \int d^4x\sqrt{-g}\frac{\delta\mathcal{L}_m}{\delta\psi}\delta\psi_\xi = 0 \ , \tag{1.37}$$

where we also used the fact that the variation of the metric can be written as

$$g'^{\mu\nu} - g^{\mu\nu} = \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu \tag{1.38}$$

Note that the equation (1.37) has to be zero of shell. Let us now presume that the matter field equations are satisfied which implies that the second term in (1.37) vanishes. Then using integration by parts we can rewrite (1.37) into the form

$$\delta_\xi S_m(on\ shell) = -\int d^4x\sqrt{-g}\xi^\mu\nabla^\mu T_{\mu\nu} = 0 \tag{1.39}$$

that using the fact that ξ^μ is arbitrary implies the conservation of the stress energy tensor.

We continue with the study of the Einstein equations where we now discuss the possibility of the introduction of a cosmological constant. In order to introduce it we add it to the conventional Hilbert action. We therefore consider an action given by

$$S = \int d^4x\sqrt{-g}(R - 2\Lambda) \ , \tag{1.40}$$

where Λ is some constant. The resulting field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ , \tag{1.41}$$

and of course there would be an energy-momentum tensor on the right hand side if we had included an action for matter. Λ is the cosmological constant. In order to find its meaning it is convenient to move the additional term in (1.41) to the right hand side, and think of it as a kind of energy-momentum tensor, with $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ (it is automatically conserved by metric compatibility). Then Λ can be interpreted as the "energy density of the vacuum," a source of energy and momentum that is present even in the absence of matter fields. This interpretation is important because quantum field theory predicts that the vacuum should have some sort of energy and momentum. In ordinary quantum mechanics, an harmonic oscillator with frequency ω and minimum classical energy $E_0 = 0$ upon quantization has a ground state with energy $E_0 = \frac{1}{2}\hbar\omega$. A quantized field can be thought of as a collection of an infinite number of harmonic oscillators, and each mode contributes to the ground state energy. The result is of course infinite, and must be appropriately regularized, for example by introducing a cutoff at high frequencies. The final vacuum energy, which is the regularized sum of the energies of the ground state oscillations of all the fields of the theory, has no good reason to be zero and in fact would be expected to have a natural scale

$$\Lambda \sim m_P^4 \ , \tag{1.42}$$

where the Planck mass m_P is approximately 10^{19} GeV, or 10^{-5} grams. Observations of the universe on large scales allow us to constrain the actual value of Λ, which turns out to be smaller than (1.42) by at least a factor of 10^{120}. This is the largest known discrepancy between theoretical estimate and observational constraint in physics, and convinces many people that the "cosmological constant problem" is one of the most important unsolved problems today. On the other hand the observations do not tell us that Λ is strictly zero, and in fact allow values that can have important consequences for the evolution of the universe.

1.3 Basic principles of Cosmology

In this section we review basic facts about classical cosmology, following mainly [3]. There are many reviews available on hep-th, see for example [4, 5, 6] [1]. Contemporary cosmological modes are based on the idea that the Universe is pretty much the same everywhere-the idea known as **Copernican principle**. It is clear that this principle can be applied on the large scales only where local variations of density is averaged over. In other words, the Universe is spatially homogeneous and isotropic on the largest scales. Since these claims need more explanation let us pause in our explanation of cosmology and give some more precise definition of mathematical claims given above.

1.4 Map of Manifolds

Since we do not have enough time with explanation of the notion of manifold we presume that reader has enough knowledge regarding this point.

Let M and N be manifolds (generally with different dimensions) and let $\phi : M \to N$ be a map. In a natural manner, ϕ "pulls back" a function $f : N \to R$ on N to the function $f \circ \phi \to M \to R$ that is derived by composing f with ϕ. Similarly, in a natural way, ϕ maps tangent vectors at $p \in M$ to tangent vectors at $\phi(p) \in N$. In other words it defines ma $\phi^* : V_p \to V_{f(p)}$ in following way: For $V \int V_p$ we define $\phi^*(v)$ by

$$(\phi^*(v))(f) = v(f \circ \phi) \tag{1.43}$$

for all smooth $f : N \to R$. It is easy to see that $\phi^* v$ satisfies the properties of tangent vector at $\phi(p)$. Further, in the coordinate bases of a coordinate system (x^ν) at p and a coordinate system (y^μ) at $\phi(p)$ the upper expression takes the form

$$\begin{aligned} & w^\mu(y)\frac{\partial}{\partial y^\mu} f(y) = v^\nu(x)\frac{\partial}{\partial x^\nu} f((\phi(x))) = v^\nu(x)\frac{\partial f(y)}{\partial y^\mu}\frac{\partial y^\mu}{\partial x^\nu} \Rightarrow \\ & w^\mu(\phi(x)) = v^\nu(x)\frac{\partial y^\mu}{\partial x^\nu} \ , \quad (\phi^* v)^\mu \equiv w^\mu \ . \end{aligned} \tag{1.44}$$

[1] Our metric signature is $- + ++$. We use units $\hbar = c = 1$ and define the reduced Planck mass by $M_p = (8\pi G)^{-1/2} \approx 10^{18} GeV$.

In the same way we can use ϕ to "pull back" one forms at $\phi(p)$ to one forms at p. We define the map ("pull back") $\phi_* : V^*_{\phi(p)} \to V^*_p$ by requiring that for $v \in V_p$

$$(\phi_*\omega)_\mu v^\mu = \omega_\nu(\phi^* v)^\nu \ , \tag{1.45}$$

where we used tensor notation. Using the definition of the map ϕ^* given in (1.44) we easily get

$$(\phi_*\omega)_\mu = \omega_\nu \frac{y^\nu}{\partial x^\mu} \ . \tag{1.46}$$

We can easily extend the action of ϕ_* to map tensors of type $(0,l)$ at $\phi(p)$ to tensors of type $(0,l)$ at p by

$$(\phi_* T)_{\mu_1 \dots \mu_l} v_1^{\mu_1} \dots v_l^{\mu_l} = T_{\mu_1 \dots \mu_l} (\phi^* v_1)^{\mu_1} \dots (\phi^* v_l)^{\mu_l} \ . \tag{1.47}$$

In the same way we can extend the action of ϕ^* to map tensors of type $(k,0)$ at p to tensors of type $(k,0)$ at $\phi(p)$ by

$$(\phi^* T)^{\mu_1 \dots \mu_k} (\omega_1)_{\mu_1} \dots (\omega_k)_{\mu_k} = T^{\mu_1 \dots \mu_k} (\phi_* \omega_1)_{\mu_1} \dots (\phi_* \omega_k)_{\mu_k} \tag{1.48}$$

If $\phi : M \to M$ is diffeomorphism and T is a tensor field on M we can compare T with $\phi^* T$. If $\phi^* T = T$ then even though we have moved T via ϕ it is still the same. In other words ϕ is a symmetry transformation for the tensor field T. In the case of the metric $g_{\mu\nu}$ a symmetry transformation-a diffeomorphism ϕ such that

$$(\phi^* g)_{\mu\nu} = g_{\mu\nu}$$

is called an isometry.

Let us now return to our explanation of basic principles of cosmology. Our first task is to formulate precisely the mathematical meaning of this assumption. The evidence comes from the smoothness of the temperature of the cosmic microwave background. In other words, given any two points p and q there is an isometry which takes p into q. We must mention that there is no necessary relationship between homogeneity and isotropy; a manifold can be homogeneous but nowhere isotropic (such as $R \times S^2$ in the usual metric) or it can be isotropic around a point without being homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous). On the other hand, if a space is isotropic *everywhere* then it is homogeneous. On the other hand it should be pointed that, in general, at each point, at most one observer can see the universe as isotropic. For example, if ordinary matter fills the universe, any observer in motion relative to the matter must see an anisotropic velocity distribution of the matter. With this fact in mind we have to give precise formulation of the notion of isotropy than the clam that *Isotropy is the claim that the Universe looks the same in all directions.*: A spacetime is said to be (spatially) isotropic at teach point if there exists a congruence of time-like curves (observes) with tangent vectors denoted u^μ filling the spacetime and satisfying the following property. Given any point p and any two unit spatial tangent vectors $s_1^\mu, s_2^\mu \in V_p$ (In other words vector that are orthogonal to u^μ) there exists an isometry of $g_{\mu\nu}$ that leaves p and u^μ at p fixed but rotates s_1^μ into s_2^μ. Thus, in an isotropic universe it is impossible to construct a geometrically preferred tangent vector orthogonal to u^μ. Then we can see that in the case of a homogeneous and isotropic spacetime the surface Σ_t of homogeneity must be orthogonal to the tangents u^μ to the world-lines of the isotropic observers. Now the space-time metric $g_{\mu\nu}$ induces a Riemannian metric $h_{\mu\nu}(t)$ on each Σ_t by restricting the action of $g_{\mu\nu}$ at each $p \in \Sigma_t$ to vectors tangent to Σ_t. The induced spatial geometry of the surfaces Σ_t is greatly restricted by the following requirements:

- Due to the homogeneity, there must be isometries of $h_{\mu\nu}$ that carry any $p \in \Sigma_t$ into any $q \in \Sigma_t$.

- Due to the isometry it must be impossible to construct any geometrically preferred vectors on Σ_t.

Since there is observation evidence for isotropy and the Copernican principle says that we are not the center of the Universe and therefore observers elsewhere should also observe an isotropy all cosmological models are based on the existence of homogeneity and isotropy of manifold. However it is important to stress that this claim is not certainly true. The Universe is apparently not static, but changing in time. Therefore the cosmological models are based on the idea that the Universe is homogeneous and isotropic in space but not in time. This means that the Universe can be foliated into space-like surfaces such that each slice is homogeneous and isotropic. Then it is natural to consider our space-time to be $R \times \Sigma$ where R represents the time direction and Σ is a homogeneous and isotropic three-manifold. Since we may think of isotropy as invariance under rotation and homogeneity as invariance under translation we get that Σ must be a maximally symmetric space. More precisely, the homogeneity and isotropy imply that the space has its maximum possible number of Killing vectors. Therefore we can write the metric in the form

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x)dx^i dx^j \ . \tag{1.49}$$

Here t is time-like coordinate and (x^1, x^2, x^3) are the coordinates on Σ where γ_{ij} is the maximally symmetric metric on Σ. The function $a(t)$ is known as **scale factor** that tells us how big the space-like slice Σ is at the moment t. The coordinates used here in which the metric is free of cross terms $dtdx^i$ and the space-like components are proportional to a single function of t are known as **comoving coordinates** and an observer who stays at constant x^i is also called as "comoving". Only comoving observer will think that the Universe looks isotropic.

It is important to stress that these observers, that are at rest to this frame are *geodesic* which means that they are free. Note that for these particles (observers) we have $ds^2 = -dt^2$ as follows from the fact that $dx^i = 0$ which implies that t has the meaning of the proper time for particles at rest.

We show that the world-line $x^i = \text{const}$ obeys the geodesic equation in the metric (1.49). Note that the geodesic equation takes the form

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0 \ , \tag{1.50}$$

where u^μ is 4− velocity

$$\frac{dx^\mu}{d\lambda} \tag{1.51}$$

and where λ is the parameter along the world-line of the particle. To begin with we calculate the Christoffel symbols

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}) \ . \tag{1.52}$$

For the metric (1.49) we have following non-zero components

$$g_{00} = -1 \ , \quad g_{ij} = a^2(t) \gamma_{ij} \tag{1.53}$$

with the inverse components

$$g^{00} = -1 \ , \quad g^{ij} = \frac{1}{a^2(t)} \gamma^{ij} \ , \tag{1.54}$$

where

$$\gamma^{ij} \gamma_{jk} = \delta^i_{\ k} \ . \tag{1.55}$$

It can be shown that the only non-zero components of $\Gamma^\mu_{\nu\lambda}$ are

$$\Gamma^i_{0j} = \frac{1}{2} g^{ik} \partial_0 g_{jk} = \frac{\dot{a}}{a} \delta^i_{\ j} \ , \Gamma^0_{ij} = -a\dot{a}\gamma_{ij} \ , \Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk} \ , \tag{1.56}$$

where ${}^{(3)}\Gamma^i_{jk}$ are the Christoffel symbols for metric γ_{ij}.

Let us now again consider the equation (1.50). The only non-zero component of the 4−velocity $u^\mu = \frac{dx^\mu}{d\lambda}$ of the particle at rest is

$$u^0 = \frac{dx^0}{d\lambda} \tag{1.57}$$

Now the on-shell condition implies

$$u^\mu u^\nu g_{\mu\nu} = -1 \Rightarrow \frac{dx^0}{d\lambda} = 1 \ . \tag{1.58}$$

Then clearly (1.50) is obviously satisfied since $\frac{du^0}{d\lambda} = 0$ and Γ^μ_{00} for all μ. In other words the world-lines of particles which are at rest in our reference frame are indeed geodesic.

As we have shown in introduction the maximally symmetric Euclidean three-metric γ_{ij} obey

$$R^{(3)}_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \ , \tag{1.59}$$

where k is some constant and the superscript on the Riemann tensor reminds to us that it is associated with the three metric γ_{ij} not to the metric of entire space-time. Then the Ricci tensor is

$$R^{(3)}_{jl} = \gamma^{ik}R^{(3)}_{ijkl} = 2k\gamma_{jl} \ . \tag{1.60}$$

Since the space is maximally symmetric then it will certainly be spherically symmetric as well. For such a space-time the metric can be put in the form

$$d\sigma^2 = \gamma_{ij}dx^i dx^j = e^{2\beta}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1.61}$$

The Ricci tensor for the metric given above has components

$$\begin{aligned}
R^{(3)}_{11} &= \frac{2}{r}\partial_r\beta \ , \\
R^{(3)}_{22} &= e^{-2\beta}(r\partial_r\beta - 1) + 1 \\
R^{(3)}_{33} &= [e^{-2\beta}(r\partial_r\beta - 1) + 1]\sin^2\theta \ .
\end{aligned}$$

If we compare these expressions to (1.60) we can solve for $\beta(r)$:

$$\begin{aligned}
\frac{2}{r}\partial_r\beta \ &= \ 2ke^{2\beta} \Rightarrow 2d\beta e^{-2\beta} = 2kr \Rightarrow \beta = -\frac{1}{2}\ln(C - kr^2) \ , \\
& e^{-2\beta}(r\partial_1\beta - 1) + 1 = 2kr^2 \Rightarrow e^{-2\beta}(r^2ke^{2\beta} - 1) + 1 = 2kr^2 \Rightarrow \\
& \Rightarrow -e^{-2\beta} + 1 = kr^2 \Rightarrow C = 1
\end{aligned} \tag{1.62}$$

and the third equation is identically solved. Then we obtain following metric on space-time:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \ . \tag{1.63}$$

This form of metric is known as **Friedman-Robertson-Walker metric** (FRW). Then the Einstein equations will determine the behavior of the scale factor $a(t)$. We can also easily see that the metric is invariant under the scaling transformations:

$$\begin{aligned}
k &\rightarrow \frac{k}{|k|} \ , \\
r &\rightarrow \sqrt{|k|}r \ , \\
a &\rightarrow \frac{a}{\sqrt{|k|}} \ .
\end{aligned} \tag{1.64}$$

Therefore it is clear that the only relevant parameter is $k/|k|$ and there are three cases of interest: $k = -1$, $k = 0$ and $k = 1$. The case $k = -1$ corresponds to constant negative curvature on Σ and is called **open**, the case $k = 0$ corresponds no curvature on Σ and is called **flat** ; the case $k = 1$ corresponds to positive curvature on Σ and is called **closed**. Now we will examine these possibilities in more details:

- For $k = 0$ the metric on Σ is

$$d\sigma^2 = dx_i dx^i \ , i = 1, 2, 3 \tag{1.65}$$

  that is simply the Euclidean space. Globally, it could describe R^3 or more complicated manifold, as for example three torus $S^1 \times S^1 \times S^1$.

- For $k = 1$ we define

$$r = \sin\xi \ , dr = \cos\xi d\xi \tag{1.66}$$

  and hence the metric on Σ can be written as

$$d\sigma^2 = d\xi^2 + \sin^2\xi d\Omega^2 \tag{1.67}$$

  which is the metric of three sphere. In this case the only possible global structure is actually three sphere.

- The case $k = -1$ we can write

$$r = \sinh\psi \tag{1.68}$$

  and the metric on Σ is

$$d\sigma^2 = d\psi^2 + \sinh^2\psi d\Omega^2 \tag{1.69}$$

  which is the metric of three dimensional space of constant negative curvature. Globally such a space can extend forever but it can also describe a non-simply connected compact space.

In order to solve the Einstein's equations of motion we have to calculate the Christoffel's symbols for the metric ansatz (1.63). If we denote $\dot{a} \equiv \frac{da}{dt}$ then these symbols are given by

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2} \ , \quad \Gamma^0_{22} = a\dot{a}r^2 \ , \quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta \ , \\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a} \ , \\
\Gamma^1_{22} &= -r(1-kr^2) \ , \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta \ , \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \ , \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \ , \Gamma^3_{23} = \Gamma^3_{32} = \sin\theta \ .
\end{aligned} \tag{1.70}$$

After simple calculations we can find following nonzero components of the Ricci tensor

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \ , \\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \ , \\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \ , \\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin\theta \ .
\end{aligned} \tag{1.71}$$

Then the Ricci scalar is equal to

$$R = g^{\mu\nu}R_{\nu\mu} = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \ . \tag{1.72}$$

Since Universe is not empty we are not interested in the vacuum Einstein equations. Rather we must study the solutions of the Einstein's equations that contain the nontrivial right hand side. The standard model with we begin is the Universe filled by a perfect fluid that is defined as fluids that are isotropic in their rest frame. The energy momentum tensor for a perfect fluid can be written

$$T_{\mu\nu} = (p + \rho)U_\mu U_\nu + pg_{\mu\nu} \ , \tag{1.73}$$

where p and ρ are energy density and pressure as measured in the rest frame and U_μ is the four-velocity of the fluid. It is clear that if a fluid which is isotropic in some frame leads to a metric which is isotropic in some frame, the two frames will coincide, that is the fluid will be in rest frame in comoving coordinates. The four-velocity is then

$$U^\mu = (1, 0, 0, 0) \ , \tag{1.74}$$

and the energy tensor is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix} \ . \tag{1.75}$$

If we raise its index we obtain

$$T^\mu_{\ \nu} = g^{\mu\kappa}T_{\kappa\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{1.76}$$

and note that the trace is equal to

$$T \equiv T^\mu_{\ \mu} = -\rho + 3p \ . \tag{1.77}$$

For letter purposes it is also instructive to consider the zero component of the conservation of the stress energy tensor

$$\begin{aligned} 0 &= \nabla_\mu T^\mu_{\ 0} = \partial_\mu T^\mu_{\ 0} + \Gamma^\mu_{\mu 0} T^0_{\ 0} - \Gamma^\lambda_{\mu 0} T^\mu_{\ \lambda} = \\ &= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \ . \end{aligned} \tag{1.78}$$

To proceed it is necessary to choose the equation of state, the relation between ρ and p. It appears that all perfect fluids relevant to cosmology obey the simple equation of state

$$p = w\rho \ , \tag{1.79}$$

where w is constant independent on time. Then the conservation of energy becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{1.80}$$

that can be integrated and we obtain

$$\rho = a^{-3(1+w)} \ . \tag{1.81}$$

The most interesting examples of cosmological are **dust** and **radiation**. Dust is characterized with $w = 0$. Examples include ordinary stars and galaxies where the pressure is negligible in comparison with the energy density. Dust is also known as matter and Universes whose energy is mostly due to dust are known as **matter-dominated**. The energy density in matter falls as

$$\rho \sim a^{-3} \tag{1.82}$$

that can be interpreted as the decrease in the number density of particles as the Universe expands. (For dust the energy density is dominated by the rest energy that is proportional to the number density.)

The second form of the fluid, **Radiation** may be used to describe either actual electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light so that they become indistinguishable from photons. The stress energy tensor of the radiation can be expressed in terms of the field strength as

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^\nu_{\ \lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right) . \tag{1.83}$$

Then the trace of this stress energy tensor is

$$T = T^{\mu\nu}g_{\nu\mu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F_{\mu\lambda} - \frac{(4)}{4}F^{\lambda\sigma}F_{\lambda\sigma}\right] = 0 \tag{1.84}$$

Since this should be also equal to (1.77) we get that

$$p = \frac{1}{3}\rho \ . \tag{1.85}$$

An Universe in which most of the energy density is in the form of radiation is known as **radiation-dominated**. The energy density in radiation then falls off as

$$\rho \sim a^{-4} \ . \tag{1.86}$$

This result implies that the energy density of radiation falls of faster than that in matter. It is believed that today the energy density of the Universe is dominated by matter with $\rho_{mat}/\rho_{rad} \sim 10^6$. However in the past the Universe was much smaller and the energy density in radiation would have dominated at very early times.

There is also one important form of energy density that is sometimes considered, namely that of the vacuum itself. Introducing energy into the vacuum is equivalent to introducing a cosmological constant so that Einstein's equations with cosmological constant are

$$E_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{1.87}$$

that is clearly the same form as the equations with no cosmological constant but an energy-momentum tensor for the vacuum

$$T^{vac}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \ . \tag{1.88}$$

This has form of the perfect fluid with

$$\rho = -p = \frac{\Lambda}{8\pi G} \tag{1.89}$$

that implies that $w = -1$ and from (1.81) we see that the energy density is independent on a. Since the energy density of matter and the radiation decreases as the Universe expands, if there is nonzero vacuum energy it tends to wind over the long term. If this happens we say that the Universe became **vacuum-dominated**.

Now we turn to the Einstein's equations. Recall that they can be written in the form

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \ . \tag{1.90}$$

The $\mu\nu = 00$ components is

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) \ , \tag{1.91}$$

and the $\mu\nu = ij$ equations give

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \ . \tag{1.92}$$

Using (1.91) we simplify (1.92) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ . \tag{1.93}$$

(1.93) together with (1.91) are known as **Friedmann equations**.

Now we introduce some terminology considering cosmological parameters. The rate of expansion is characterized by the **Hubble parameter**

$$H = \frac{\dot{a}}{a} \ . \tag{1.94}$$

The value of the Hubble parameter at present epoch is the Hubble constant, H_0. There is also the **deceleration parameter**

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{1.95}$$

that measures the rate of change of the rate of expanding. Another useful parameter is the **density parameter**

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}} \ , \tag{1.96}$$

where the critical density is defined by

$$\rho_{crit} = \frac{3H^2}{8\pi G} \ . \tag{1.97}$$

This quantity, that is generally time dependent, is called critical density because the Friedmann equation (1.93) can be written as

$$\Omega - 1 = \frac{k}{H^2 a^2} \ , \tag{1.98}$$

where generally H is time dependent. The sign of k is therefore determined by whether Ω is greater than, equal to, or less than one. In other words, we have

$$\begin{aligned} &\rho < \rho_{crit} \Rightarrow \Omega < 1 \Rightarrow k = -1 \rightarrow \text{open} \ , \\ &\rho = \rho_{crit} \Rightarrow \Omega = 1 \Rightarrow k = 0 \rightarrow \text{flat} \ , \\ &\rho > \rho_{crit} \Rightarrow \Omega > 1 \Rightarrow k = 1 \rightarrow \text{closed} \ . \end{aligned} \tag{1.99}$$

It is useful to know the qualitative behavior of various possibilities of the solutions of the Friedman equations. Let us for the moment set $\Lambda = 0$ and consider the behavior of Universe filled with fluids of positive energy $\rho > 0$ and nonnegative pressure $p > 0$. Then (1.91) implies that $\ddot{a} < 0$. Since we know from observation that the Universe is expanding ($\dot{a} > 0$) this means that the Universe is decelerating which could be intuitively expected since the gravitation attraction of the matter in the Universe works against the expanding. The fact that the Universe is decelerating means that it must have been expanding even faster in the past; if we trace the evolution backward in time, we reach the singularity at $a = 0$. Notice that if $\ddot{a}$ were exactly zero, $a(t)$ would be straight line $a(t) = Ct$ (we have chosen the integration constant that at $t = 0, a(0) = 0$ and hence $H(t) = \frac{\dot{a}}{a} = \frac{1}{t}$ so that H_0^{-1} would determine the age of the Universe.

The singularity at $a = 0$ is known as **Big Bang**. It represents the creation of Universe from a singular space, not explosion of matter into a pre-existing space-time. Since for $a \to 0$ the energy density becomes arbitrary high we do not expect classical general relativity to give a correct description of nature in this regime.

The future evolution is different for different k. For the open and flat cases $k = -1, 0$ the (1.93) implies

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k| \ . \tag{1.100}$$

Since the right hand side is strictly positive so $\dot{a}$ never passes through zero. Since $\dot{a} > 0$ today it follows that $\dot{a} > 0$ for all time. Thus open and flat Universes expand forever-they are temporally and spatially open. It is however important to keep in mind that this works on the presumption of nonzero positive energy density. Negative energy density Universes do not have to expand forever, even if they are open.

The question is how fast these Universes keep expanding? Let us now consider the quantity ρa^3 (recall that this is constant in matter dominated Universe). Using the conservation of energy (1.78) we get

$$\frac{d}{dt}(a^3\rho) = a^3(3\frac{\dot{a}}{a}\rho + \dot{\rho}) = -3pa^2\dot{a} \tag{1.101}$$

that implies that

$$\frac{d}{dt}(a^3\rho) < 0 \ . \tag{1.102}$$

This result implies that $a^2\rho$ must go to zero in an ever-expanding Universe where $a \to \infty$ [2] Then (1.100) implies that

$$\dot{a}^2 \to |k| \ . \tag{1.103}$$

(We must stress that it holds for $k = -1, 0$. Thus for $k = -1$ an expanding approaches the limiting value $\dot{a} \to 1$ while for $k = 0$ the Universe keeps expanding but more and more slowly.

For the closed Universe ($k = 1$) (1.93) implies

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1 \ . \tag{1.104}$$

[2] For example, when $a(t) \sim t$ we should have $\rho \sim t^{-4}$ at least and hence $a^2\rho \sim t^{-2} \to 0$ for $t \to \infty$.

It is clear that the argument that $\rho a^2 \to 0$ as $a \to \infty$ still holds. In this case the right hand side of the upper equation becomes negative which clearly cannot happen. Therefore the Universe does not expand indefinitely, a posses an upper bound a_{max}. As a approaches a_{max} the equation (1.91) implies

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a_{max} < 0 \tag{1.105}$$

and hence $\ddot{a}$ is finite and negative at this point, so a reaches a_{max} and starts decreasing. Since $\ddot{a} < 0$ it will inevitably continue to contract to zero- the Big Crunch. Thus, the closed Universe (on presumption of positive ρ and non negative p) is closed in time as well as space.

We will now list some of the exact solutions corresponding to only one type of energy density. For dust-only Universe ($p = 0$) it is convenient to define a **development angle** $\phi(t)$, rather than using t as a parameter directly. The solutions are then, for open Universes;

$$a = \frac{C}{2}(\cosh\phi - 1) \ , \quad t = \frac{C}{2}(\sinh\phi - \phi) \ , \quad k = -1 \ , \tag{1.106}$$

for flat Universes

$$a = \left(\frac{9C}{4}\right)^{1/3} t^{2/3} \ , \quad k = 0 \ , \tag{1.107}$$

and for closed Universes

$$a = \frac{C}{2}(1 - \cos\phi) \ , t = \frac{C}{2}(\phi - \sin\phi) \ , \quad k = +1 \ , \tag{1.108}$$

where we have defined

$$C = \frac{8\pi G}{3}\rho a^3 = \text{constant} \ . \tag{1.109}$$

For Universes filled with nothing but radiation, $p = \frac{1}{3}\rho$, we have once again open Universes,

$$a = \sqrt{C'}\left[\left(1 + \frac{t}{\sqrt{C'}}\right)^2 - 1\right]^{1/2} \ , \quad k = -1 \tag{1.110}$$

flat Universes,

$$a = (4C')^{1/4} t^{1/2} \ , \quad k = 0 \tag{1.111}$$

and closed Universes,

$$a = \sqrt{C'}\left[1 - \left(1 - \frac{t}{\sqrt{C'}}\right)^2\right]^{1/2} \ , \quad k = +1 \tag{1.112}$$

where we have defined

$$C' = \frac{8\pi G}{3}\rho a^4 = \text{constant} \ . \tag{1.113}$$

Let us now consider the case of nonzero cosmological constant. We start with $\Lambda < 0$. In this case Ω is negative and we get that $k = -1$. The solution in this case is

$$a = \sqrt{\frac{-3}{\Lambda}} \sin\left(\sqrt{\frac{-\Lambda}{3}} t\right) \ . \tag{1.114}$$

There is also an open ($k = -1$) solution for $\Lambda > 0$ given by

$$a = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}} t\right) \ . \tag{1.115}$$

A flat vacuum-dominated Universe must have $\Lambda > 0$ and the solution is

$$a \sim \exp\left(\pm\sqrt{\frac{\Lambda}{3}} t\right) \tag{1.116}$$

while the closed Universe must also have $\Lambda > 0$ and satisfies

$$a = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}} t\right) \ . \tag{1.117}$$

These solutions are a little misleading. In fact the three solutions for $\Lambda > 0$ - (1.115),(1.116),(1.117)-all represent the same space-time, just in different coordinates. This space-time, known as **de Sitter space** is maximally symmetric as a space-time. The $\Lambda < 0$ solution is also maximally symmetric and is known as **anti-de Sitter space**

Before we conclude this section we spend some time with the discussion of the situation when the matter sector in Universe constitutes more general form of matter. For example, we can presume that all components of the matter are present. Then the total density parameter takes the form

$$\Omega = \sum_i \Omega_i \tag{1.118}$$

and the Friedman equation can be written as

$$\Omega - 1 = \frac{k}{H^2 a^2} \ . \tag{1.119}$$

As in the particular previous example we obtain that the sign of k is determined whether Ω is greater than, equal to, or less than one. Explicitly, we have

$$\begin{aligned} \rho < \rho_{crit} \Rightarrow \Omega < 1 \rightarrow k = -1 \ , \text{open} \ , \\ \rho = \rho_{crit} \Rightarrow \Omega = 1 \rightarrow k = 0 \ , \text{flat} \ , \\ \rho > \rho_{crit} \Rightarrow \Omega > 1 \rightarrow k = 1 \ , \text{closed} \ . \end{aligned} \tag{1.120}$$

Since $\rho_i \sim a^{-n_i}$ we have

$$\frac{\rho_i}{\rho_j} = \frac{\Omega_i}{\Omega_j} = a^{-(n_i - n_j)} \tag{1.121}$$

so that relative amount of energy in different components changes as the Universe evolves.

1.5 Motion of the probe in the FRW Universe

In order to understand properties of given background it is common strategy to study the dynamics of the probe in given background. Let us then consider the motion of particle in the FRW Universe.

Let us consider the action for the massive particle

$$S = -\int d\lambda \sqrt{-\hat{g}_{\mu\nu} u^\mu u^\nu} \ , u^\mu = \frac{dx^\mu}{d\lambda} \ . \tag{1.122}$$

where λ is parameter that labels the world-line. We introduce einbain $e(\tau)$ so that the action takes the form

$$S = \frac{1}{2} \int d\lambda [\frac{1}{\epsilon} \hat{g}_{\mu\nu} u^\mu u^\nu - m^2 \epsilon] \ , \tag{1.123}$$

To see the equivalence between these two formulations we perform the variation with respect to ϵ that gives

$$-\frac{1}{\epsilon^2} \hat{g}_{\mu\nu} u^\mu u^\nu - m^2 = 0 \Rightarrow \epsilon = \frac{1}{m} \sqrt{-\hat{g}_{\mu\nu} u^\mu u^\nu} \tag{1.124}$$

that inserting back to the action we obtain the original action. Further, the equation of motion with respect to x^μ gives

$$-2 \frac{d}{d\lambda} (\frac{1}{\epsilon} \hat{g}_{\mu\nu} u^\nu) + \frac{1}{\epsilon} \partial_\mu \hat{g}_{\rho\sigma} u^\rho u^\sigma = 0 \tag{1.125}$$

It is important to stress that the action is invariant under $\tau' = f(\tau)$ so that $d\tau' = \frac{df}{d\tau} d\tau$. We can fix the gauge by imposing $\epsilon = \frac{1}{m}$ so that we obtain on-shell condition

$$\hat{g}_{\mu\nu} u^\mu u^\nu = -1 \tag{1.126}$$

Note that this relation allows us to write (when $\hat{g}_{0u} = 0$)

$$-1 = (-\hat{g}_{00} + \hat{g}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt})(\frac{dt}{d\lambda})^2 \Rightarrow \frac{dt}{d\lambda} = \frac{1}{\sqrt{\hat{g}_{00} - \hat{g}_{ij} v^i v^j}} \ , v^i \equiv \frac{dx^i}{dt} \tag{1.127}$$

Then the equation of motion for x^μ takes the form

$$\begin{aligned}
&\frac{du^\mu}{d\lambda} + \hat{g}^{\mu\nu} \partial_\rho \hat{g}_{\nu\sigma} u^\sigma u^\rho - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\nu \hat{g}_{\rho\sigma} u^\rho u^\sigma \Longrightarrow \\
&\frac{du^\mu}{d\lambda} + \frac{1}{2} \hat{g}^{\mu\nu} (\partial_\rho \hat{g}_{\nu\sigma} + \partial_\sigma \hat{g}_{\nu\rho} - \frac{1}{2} \partial_\nu \hat{g}_{\rho\sigma}) u^\rho u^\sigma = 0 \Rightarrow \\
&\frac{d^2 x^\mu}{d^2\lambda} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \ .
\end{aligned} \tag{1.128}$$

It is also interesting to insert the solution of the equation of motion ϵ into the action so that it takes the form

$$\begin{aligned} S &= \frac{1}{2}\int d\lambda[\frac{1}{\epsilon}\hat{g}_{\mu\nu}u^\mu u^\nu - m^2\epsilon] = \\ &= \frac{m}{2}\int dt\sqrt{\hat{g}_{00}-\hat{g}_{ij}v^iv^j}[-(\hat{g}_{00}-\hat{g}_{ij}v^iv^j)(\frac{dt}{d\lambda})^2 - 1] = \\ &= -m\int dt\sqrt{\hat{g}_{00}-\hat{g}_{ij}v^iv^j} \ . \end{aligned} \tag{1.129}$$

It is also interesting to analyze the equation of motion that follows from the original action

$$S = -m\int d\lambda\sqrt{-g_{MN}\partial_\tau X^M\partial_\sigma X^N} \tag{1.130}$$

The equations of motion have the form

$$-\frac{\partial_K g_{MN}\dot{X}^M\dot{X}^N}{2\sqrt{-g_{MN}\dot{X}^M\dot{X}^N}} + \partial_\tau\left(\frac{g_{KN}\dot{X}^N}{\sqrt{-g_{MN}\dot{X}^M\dot{X}^N}}\right) = 0 \ . \tag{1.131}$$

This is equation of motion for X. Let us denote the variation of the action with respect to X as $\frac{\delta S}{\delta X}$. If we multiply given expression with $\dot{X}^K$ we obtain

$$\begin{aligned} \frac{\delta S}{\delta X^K}\dot{X}^K &= -\frac{\partial_\tau g_{MN}\dot{X}^M\dot{X}^N}{2\sqrt{-g_{MN}\dot{X}^M\dot{X}^N}} + \partial_\tau(\frac{1}{\sqrt{-g_{MN}\dot{X}^M\dot{X}^N}})\dot{X}^K g_{KM}\dot{X}^N + \\ &+\frac{1}{\sqrt{-g_{MN}\dot{X}^M\dot{X}^N}}\dot{X}^K\partial_\tau g_{KM}\dot{X}^N + \frac{1}{\sqrt{-g_{MN}\dot{X}^M\dot{X}^N}}\dot{X}^K g_{KM}\ddot{X}^N = 0 \end{aligned} \tag{1.132}$$

Note that it holds as identity and not as a consequence of the equations of motion.

Let us now consider the flat FRW background

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ . \tag{1.133}$$

so that the action takes the form

$$S = -m\int dt\sqrt{1-a^2\delta_{ij}\dot{x}^i\dot{x}^j} \ . \tag{1.134}$$

It is interesting to determine the Hamiltonian formulation of this system

$$p_i = \frac{\delta L}{\delta\dot{x}^i} = a^2 m\frac{\delta_{ij}\dot{x}^j}{\sqrt{1-a^2\delta_{ij}\dot{x}^i\dot{x}^j}} \ . \tag{1.135}$$

Then we find

$$\mathcal{H} = p_i\dot{x}^i - L = \frac{ma^2}{\sqrt{1 - a^2\delta_{ij}\dot{x}^i\dot{x}^j}} = a^2\sqrt{\frac{1}{a^2}p_i\delta^{ij}p_j + m^2} \tag{1.136}$$

using

$$a^2\dot{x}^i\delta_{ij}\dot{x}^j = \frac{p_i\delta^{ij}p_j}{m^2a^2 + p_i\delta^{ij}p_j} \tag{1.137}$$

Now the equation of motion takes the form

$$\begin{aligned} \dot{x}^i = \left\{x^i, H\right\} = \frac{\delta^{ij}p_j}{\sqrt{a^{-2}p_i\delta^{ij}p_j + m^2}} \ , \\ \dot{p}_i = \{p_i, H\} = 0 \Rightarrow p_i = k_i \ . \end{aligned} \tag{1.138}$$

We see that the momentum p_i is constant. On the other hand the norm of state slows since the norm is given as $p_i g^{ij} p_j = \frac{1}{a^2}k_i\delta^{ij}k_j$.

On the other hand let us introduce following variable

$$X^i = ax^i \ , \dot{x}^i = \frac{1}{a}(\dot{X}^i - HX^i) \tag{1.139}$$

Using these variables we find the action in the form

$$S = -m\int dt\sqrt{1 - (\dot{X}^i - HX^i)\delta_{ij}(\dot{X}^j - HX^j)} \ . \tag{1.140}$$

The meaning of the variables X^i can be found when we take the non-relativistic limit where we replace $\sqrt{1-A} = 1 - \frac{1}{2}A^2$ so that the action

$$\begin{aligned} S_{nonrel} &= -m\int dt + \int dt\frac{m}{2}(\dot{X}^i - HX^i)\delta_{ij}(\dot{X}^j - HX^j) = \\ &= \int dt\frac{m}{2}\dot{X}^i\dot{X}_i + \dots \ , \end{aligned} \tag{1.141}$$

where we neglected the remaining terms. Comparing this expression with the standard form of the non-relativistic Lagrangian we can interpret $X^i = a(t)x^i$ as the physical variable even if we mean that both variables are physical.

Now from (1.140) we determine the momenta conjugate to X^i

$$P_i = \frac{\delta L}{\delta\dot{X}^i} = m\frac{\delta_{ij}(\dot{X}^j - HX^j)}{\sqrt{(\dots)}} \tag{1.142}$$

and hence the Hamiltonian takes the form

$$\mathcal{H} = \dot{X}^i P_i - L = \frac{m}{\sqrt{(\dots)}} + P_i X^i H = \sqrt{m^2 + P_i P^i} + P_i X^i H \tag{1.143}$$

Using this Hamiltonian we derive the equation of motion

$$\begin{aligned} \dot{X}^i = \left\{X^i, H\right\} = \frac{P^i}{\sqrt{m^2 + P_i P^i}} + X^i H \ , \\ \dot{P}_i = \{P_i, H\} = -P_i H \end{aligned} \tag{1.144}$$

The last equation can be integrated as

$$dP_i = -P_i \frac{da}{a} \Rightarrow \ln P_i = -\ln a + \ln K_i \Rightarrow P_i = \frac{K_i}{a} \ . \tag{1.145}$$

We see that the "physical" momentum P_i is red shifted as the universe expands. Note that we can also find the time dependence of X^i by integrating the first equation since it takes generally the dorm

$$\dot{X}^i = F^i(t) + G(t) X^i \tag{1.146}$$

so that we search the solution of the homogeneous equation

$$\dot{X}^i = G(t) X^i \Rightarrow X^i = C^i \exp(\int dt G(t)) \tag{1.147}$$

Note that we have

$$\int dt G(t) = \int \frac{da}{dt} \frac{1}{a} dt = \int \frac{da}{a} = \ln a \Rightarrow e^{\int dt G(t)} = e^{\ln a} = a \ . \tag{1.148}$$

Then we say that C^i depends on time so we obtain that it has to obey the equation

$$\frac{dC^i}{dt} = e^{-\int dt' G(t')} F(t) \Rightarrow \frac{dC^i}{dt} = \frac{K^i}{a\sqrt{m^2 a^2 + K_i K^i}} \tag{1.149}$$

that can be in principle integrated if we know the time dependence of a. There is a particulary simple solution corresponding to the particle with zero physical momentum when $K_i = 0$. From upper equation we immediately find that $C^i = \mathcal{C}^i = \mathrm{const}$ and hence

$$X^i = \mathcal{C}^i a \tag{1.150}$$

that is an expected result. The physical interpretation of this result is that particle slows down with respect to comoving coordinates as the Universe expands (since $a \to \infty$). In fact this is an actual slowing down, in the sense that a gas of particles with initially high relative velocities will cool down as the Universe expands.

Very interesting is the case of the particle with null mass which is photon. In principle we could use the the action for the massive particle written without the square root and then take the limit $m \to 0$ however we will be more conservative and consider the standard treatment of the electromagnetic wave in curved background.

We consider the action of free electromagnetic field

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\nu\sigma} \ , F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.151)$$

Consider now the propagation of a photon in the homogeneous isotropic Universe. Since the photon wavelength is small compared to the spatial curvature radius even if the Universe is open or closed. Then we can consider the metric that is spatially flat with the metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ . \quad (1.152)$$

Let us introduce conformal time η instead of t that is defined as

$$dt = ad\eta \quad (1.153)$$

or equivalently

$$\eta = \int \frac{dt}{a(t)} \ . \quad (1.154)$$

This result can be generally integrated so that we have $\eta = \eta(t)$ and we presume that this relation can be inverted so that $t = t(\eta)$ and consequently $a = a(\eta)$. Now the metric has the form

$$ds^2 = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j] \quad (1.155)$$

and we see that the metric element in FRW spacetime is conformally flat in the sense that

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \ . \quad (1.156)$$

where the Minkowski metric is spanned by coordinates (η, x^i). Then we clearly have

$$g^{\mu\nu} = a^{-2}\eta^{\mu\nu} \ , \sqrt{g} = a^4 \quad (1.157)$$

and we find that in η, x^i coordinates the action of the electromagnetic field has the form

$$S = -\frac{1}{4}\int d^4x\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\rho}F_{\nu\sigma} \ . \quad (1.158)$$

Now it is clear that the solution of the equation of motion for the free electromagnetic field in the Universe is given as the superposition of the plane waves

$$A_\mu^{(\alpha)} = e_\mu^{(\alpha)}e^{ik\eta - i\mathbf{k}\mathbf{x}} \quad (1.159)$$

where $\mathbf{k}$ is constant vector, $|\mathbf{k}| = k$ and $e_\mu^{(\alpha)}$ is the standard polarization vector of photons with $\alpha = 1, 2$. Note that k is not the physical frequency as follows from following arguments. The quantity $\triangle x = \frac{2\pi}{k}$ is the coordinate wavelength of a photon while the physical wavelength at time t is

$$\lambda(t) = a(t)\triangle x = 2\pi \frac{a(t)}{k} \ . \tag{1.160}$$

In the same way we define period $\triangle\eta = \frac{2\pi}{k}$ of electromagnetic wave in conformal time while the period of the physical time is

$$T = a(t)\triangle\eta = 2\pi \frac{a(t)}{k} \ . \tag{1.161}$$

Then we see that the frequency is equal to

$$\omega(t) = \frac{2\pi}{T} = \frac{k}{a(t)} \tag{1.162}$$

and since we know that the frequency is equal to the magnitude of the physical momentum of photon we obtain that the physical momentum depends on time as in case of the massive particle namely

$$\mathbf{p} = \frac{\mathbf{k}}{a(t)} \tag{1.163}$$

We see that in the expanding universe the scale factor $a(t)$ is growing and hence the physical wavelength grows. On the other hand the physical momentum is decreasing function of time. The phenomena when the wavelength is growing during the expansion of the Universe is named as the redshift. Explicitly, if the photon was emitted at time t_i with physical wave length λ_i in the physical process as for example when the electron in the excited state in the atom drops to the ground state which is certainly physical process. Now we know that the state propagates freely as in (1.159) and then it is again detected in time t_0 where t_0 we means the present time in the reversed physical process when its physical wave length now is

$$\lambda(t_0) = a(t_0)\frac{2\pi}{k} \tag{1.164}$$

Now expressing $\frac{2\pi}{k}$ using the physical wave length at time of emission we find the famous relation

$$\lambda(t_0) = \frac{a(t_0)}{a(t_i)}\lambda_i \equiv \lambda_i(1 + z(t_i)) \ . \tag{1.165}$$

The quantity

$$z(t_i) = \frac{a(t_0)}{a(t_i)} - 1 \tag{1.166}$$

is called *redshift*. The earlier the object emits the photon then this photon has to travel longer and consequently $a(t_i)$ is smaller and hence object at larger distances have the larger redshifts.

Note that these formulas are valid in general for all z. Let us now consider objects that are not in large distance. Then the difference $t - t_0$ is not very large and we can expand

$$a(t_i) = a(t_0) - \dot{a}(t_0)(t_0 - t_i) \tag{1.167}$$

Using the present value of the Hubble parameter $H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \equiv \frac{\dot{a}_0}{a_0}$ we can write

$$a(t_i) = a_0[1 - H_0(t_0 - t_i)] \tag{1.168}$$

so that to the linear order we find following expression for the redshift

$$z(t_i) = \frac{1}{1 - H_0(t_0 - t_i)} - 1 \simeq H_0(t_0 - t_i) \ . \tag{1.169}$$

Finally the travel time is equal to

$$\begin{aligned} 0 = -dt^2 + a(t)^2 dr^2 = -dt^2 + (a_0 - \dot{a}_0(t_0 - t))^2 dr^2 \approx \\ -dt^2 + a_0^2 dr^2 \Rightarrow (t_0 - t) = a_0(r_i - r_0) \equiv R \end{aligned} \tag{1.170}$$

where R is the physical distance of the object from the our observer. Inserting this expression into (1.169) we derive famous Hubble law

$$z = H_0 R \ , z \ll 1 \ . \tag{1.171}$$

The redshift is something that can be measured, we know the rest-frame wavelengths of various spectral lines in the radiation of distant galaxies, so that we can determine how much their wavelengths have changed along the path from time t_i when they were emitted to time t_0 when they were observed. We therefore know the ratio of the scale factors at these two times however we do not know the times themselves.

1.6 Horizons

One of the most crucial concepts of the FRW Universe is the existence of *horizons*.

Suppose a emitter, **e** sends a light signal to an observer **o**, who is at $r = 0$. Restricting to the radial geodetic (that means that $d\phi = d\theta = 0$ we obtain from the vanishing of the metric elements the equation for null geodetics in the form

$$ds^2 = 0 = a^2(\eta)(-d\eta^2 + dr^2) \Rightarrow \eta = \pm r + r_0 \ , \tag{1.172}$$

where η is conformal time. Let us presume that the light hits the observer at time η_0 that is larger that η_e where η_e is time when this signal was emitted. Since for $\eta = \eta_o$ we have $r = 0$ we get $\eta_o = r_0$ and consequently $\eta - \eta_o = \pm r$. Since also for η_e this equation implies

$$\eta_o - \eta_e = \mp r_e$$

and we obtain that we should choose the positive sign in front of r since $\eta_o - \eta_e > 0$ and r is positive. Finally we get the relation

$$\eta_o - \eta_e = r_e \ . \tag{1.173}$$

Let us now presume that η_e is bounded from below by $\tilde{\eta}_e$; for example $\tilde{\eta}_e$ might represent the Big Bang singularity. Then there exist a maximum distance to which the observer can see, known as a *particle horizon distance* given by

$$r_{ph}(\eta_o) = \eta_o - \tilde{\eta}_e \tag{1.174}$$

Similarly, suppose that η_o is bounded from above by $\tilde{\eta}_o$. Then there exists a limit to space-time events which can be influenced by the emitter. This limit is known as the *event horizon distance* given by

$$r_{eh}(\eta_o) = \tilde{\eta}_o - \eta_e \tag{1.175}$$

These horizon distance may be converted to *proper horizon distances* at cosmic time t. For example, we have an emitter at time $\tilde{\eta}_e$ at $r_e = 0$. Then at time η. Then from the equation for geodetics we obtain

$$\eta - \tilde{\eta}_c = r(\tau) \tag{1.176}$$

since $d\eta = \frac{dt}{a(t)}$ we obtain

$$\eta - \tilde{\eta} = \int_{t_e}^{t} \frac{dt'}{a(t')} \tag{1.177}$$

using also the fact that the proper distance at time t is given by multiplication with $a(t)$ we get the *proper horizon distance* as

$$dh = a(t) \int_{t_e}^{t} \frac{dt'}{a(t')} \ . \tag{1.178}$$

2. Our Universe Today

In this section we will discuss the remarkable properties that have been discovered in past few years. Most remarkable among them is the fact that the universe is dominated by a uniformly- distributed and slowly varying source of "dark energy" which may be a vacuum energy (cosmological constant), a dynamical field or something completely different.

2.1 Matter

The inventory of constituencies comprising actual Universe is complicated by the fact that they are not at all equally visible. In the years before we knew the dark energy was an important constituent of the Universe and before observations of galaxy and distributions and CMB anisotrophies observational cosmology measured two numbers: The Hubble constant H_0 and the matter density parameter Ω_M. Measuring the extragalactic distances is very difficult, but most current measurement of the Hubble constant performed Planck experiment in 2013 gives the value of the cosmological constant to be equal to

$$H_0 = 67.80 \pm 0.77 \ km/sec/Mpc \ , \tag{2.1}$$

where

$$1Mpc = 10^6 \text{ parsec} = 3 \times 10^{24} cm \ . \tag{2.2}$$

We see that the Hubble parameter in fact has the dimension $[t^{-1}]$ so that it has the value

$$H_0^{-1} = h^{-1} \cdot 3 \cdot 10^7 s = h^{-1} \cdot 10^{10} yrs \approx 1.4 \cdot 10^{10} yrs \ , \tag{2.3}$$

where h is a dimensionless parameter

$$h = 0.678 \ . \tag{2.4}$$

In particle physics units ($\hbar = c = 1$) this is equal to

$$H_0 \sim 10^{-33} eV \ . \tag{2.5}$$

It is convenient to express the Hubble constant as

$$H_0 = 100 \ h \ km/sec/Mpc \ . \tag{2.6}$$

It turns out that the scale H_0^{-1} gives order of magnitude of the age of the Universe and the distance scale H_0^{-1} is roughly the size of the observable part equal to

$$H_0^{-1} \approx h^{-1} \cdot 3000 Mpc \approx 4.3 \cdot 10^3 Mpc \ . \tag{2.7}$$

Note that since $\rho_i = 3H_0^2 \Omega_i / 8\pi G$ measurement of ρ_i is often expressed as measurement of $\Omega_i h^2$. The Hubble constant provides the rough measure of the scale of the Universe since in the matter or radiation dominated Universe is $t_0 \sim H_0^{-1}$.

For years, determinations of Ω_M based on dynamics of galaxies and clusters have leaded to values of Ω_M between 0.1 and 0.4. Alteratively, the determination of Ω_M is the same as the determination of the **baryons**. Recent measurements suggest that baryons contribute to Ω as

$$\Omega_B = 0.05 \ . \tag{2.8}$$

In other words baryons constitute rather small fraction of the present energy density in the Universe. It is also important to stress that the most of the baryons in our Universe are dark: direct measurements of th mass density of stars give an estimate

$$\Omega_{stars} \sim 0.005 \tag{2.9}$$

that is about an order of magnitude smaller than Ω_B. The fact that most of the baryons are dark follows from the dynamics of individual galaxies implies that there is even matter there. The implied existence this celebrated **dark matter** is confirmed by applying the viral theorem to clusters of galaxies, by looking at the temperature profiles of clusters, by "weighing" clusters by gravitational lensing and by large-scale motions of clusters between galaxies. On the other hand there is nothing dramatic about this observation: baryons may hide in dust and neutral gas clouds, brown dwarfs etc.

The next form of matter are **Photons**. They however contribute even smaller fraction

$$\Omega_\gamma \approx 6 \cdot 10^{-4} \ . \tag{2.10}$$

From electric neutrality the number density of **electrons** is about the same [3] as that of baryons, but then due to their very small mass their contribution to the total mass fraction is negligible.

The remaining known stable particles are **neutrinos**. As we will sketch bellow their number density is calculable in Hot Big Ban theory and these calculations are confirmed by Big Bang Nucleosynthesis. The number density of each type of neutrinos is

$$n_{\nu_a} = 115 \frac{1}{cm^3} \ , \tag{2.11}$$

where $\nu_a = \nu_e, \nu_\mu, \nu_\tau$. Direct limit on the mass of electron neutrino $m_{\nu_e} < 2.6 \ eV$ together with the observations of neutrino oscillations suggests that every type of neutrino has mass smaller than 2.6 eV. Then the estimation of the energy density of neutrinos is

$$\rho_{\nu,total} = \sum_\alpha m_{\nu_\alpha} n_{\nu_\alpha} < 8 \cdot 10^{-7} \frac{GeV}{cm^3} \tag{2.12}$$

that implies

$$\Omega_{\nu,total} < 0.16 \ . \tag{2.13}$$

However this estimate does not make use any cosmological date. In fact cosmological observations give stronger bound

$$\Omega_{\nu,total} < 0.01 \ . \tag{2.14}$$

In terms of the neutrino masses this bound reads

$$\sum m_{\nu_a} < 0.42 eV \tag{2.15}$$

so that every neutrino has to be lighter than $0.14eV$. On the other hand atmospheric neutrino data and further experiments tell that the mass of at least one neutrino must be larger than $0.02eV$. These results suggest that there is window for measuring neutrino masses by cosmological observations.

[3] There are also neutrons whose number is somewhat smaller than the number of protons.

We see that most of the energy density in the present Universe is not in the form of known particles, most energy in the present Universe has to be in something "unknown". In fact essentially every known particle in he Standard Model of particle physics has been ruled out as a candidate for this "unknown" matter. Moreover, there is a strong evidence that this "something unknown" has two components: **clustered dark energy** and **unclustered dark energy**.

It is believed that **Clustered dark matter** consists of new stable massive particles. These make clumps of energy density that encounter for much of the mass of galaxies and most of the mass of galactic clusters. There are number of ways of estimating the contribution of non-baryonic dark matter into the total density of the Universe:

- Composition of the Universe affects the angular anisotropy of cosmic microwave background (CMB). The present measurements of the CMB anisotropy enable to estimate the total mass density of dark matter.
- The density of non-baryonic dark matter is crucial for structure formation of the Universe. If we compare the results of numerical simulations of structure formation with observational data gives reliable estimate of the mass density of non-baryonic clustered dark matter.

One of the few things we know about the dark matter is that it must be "cold"- not only is it non-relativistic today, but it must have been that way for a very long time. The other thing we know about cold dark matter (CDM) is that it should interact very weakly with ordinary matter, so as to have escaped detection thus far. In summary the non-baryonic cold dark matter has

$$\Omega_{CDM} \approx 0.25 \ . \tag{2.16}$$

There is a direct evidence that dark matter exists in the largest gravitationally bound objects-clusters of galaxies. There are various methods to determine the gravitating mass of a cluster and even mass distribution in a cluster, which give consistent results, for example:

- We measure velocities of galaxies in galactic clusters and make use of the gravitational virial theorem

  $$\text{Kinetic energy of a gravity} = \frac{1}{2}\ \text{Potential energy} \ .$$

  In this way we obtain the gravitational potential and thus the distribution of the total mass in a cluster.

- The second example of the measurement of masses of clusters use the notion of intra-cluster gas. Its temperature that is determined from $X-$ray measurements is also related to the gravitational potential through the virial theorem.

- The third example of measurement is based on observation of gravitational lensing of background galaxies by clusters.

Finally, dark matter exists also in galaxies. Its distribution is measured by the observations of rotation velocities of distant stars and gas clouds around a galaxy.

At present there are many hypotheses considering candidates for this form of dark matter. One such an idea is that the natural candidates are particles which participate in weak interactions that of course needs more detailed justification.

Unclustered dark energy

Non-baryonic clustered dark matter is not the whole story. If we use the above estimates we obtain an estimate for the energy density of all particles

$$\Omega_\gamma + \Omega_B + \Omega_{\mu_{total}} + \Omega_{CDM} \approx 0.3 \ . \tag{2.17}$$

Since the observation that $\Omega_T \approx 1$ implies that 70 percent of the energy density is unclustered.

In fact this result nicely fits recent observations. Indeed, it can be shown that neither relativistic nor non-relativistic matter can lead to the accelerated expansion of the Universe [4]. In other words the accelerated expansion requires energy stored in something dramatically different from conventional particles and it has to have negative pressure. In fact the analysis of the entire set of cosmological date in terms of dark energy with phenomenological equation of state

$$p = w\rho \ , w = const \tag{2.18}$$

gives

$$\Omega_\Lambda = 0.72 \pm 0.02 \tag{2.19}$$

(here subscript Λ refers to dark energy) and

$$-1.2 < w < -0.8 \ . \tag{2.20}$$

It is worth noting that the vacuum value, $w = -1$ is right in the middle of the allowed region that corresponds to a vacuum energy density

$$\rho_\Lambda \sim (10^{-3} eV)^4 \ . \tag{2.21}$$

Given the significance of these results it is natural to ask what level of confidence we should have in them. There are potential sources of systematic error and these were discussed in the original papers [1, 2]. On the other hand the recent measurements of the cosmic microwave background confirmed the picture outlined above with the matter density and nonzero cosmological constant.

[4] We will discuss this problem in the next subsection.

In summary, the composition of the present Universe is fairly complex. It is challenging for future physics that most of the energy density comes from species which particle physicists are unfamiliar with: vacuum or vacuum-like dark energy and non-baryonic clumped dark matter. This poses serious problems for both fundamental physics and cosmology:

- **What are the particles of non-baryonic dark matter?**

  Currently popular option is the lightest supersymmetric particle that is stable in many supersymmetric extensions of the Standard model. Of course there are many other options, such as axions, gravitinos and so on. In any case experimental discovery of the dark matter particle would be great achievement of both particle physics and cosmology.

- **Why there are baryons and no anti-baryons in our Universe?**

  Alliteratively,what is the origin of matter-antimatter asymmetry of the Universe? We will discuss this issue later and here we notice only that the solution of this problem is based on extension of the Standard Model.

- **Why the mass density of the non-baryonic dark matter is so similar to the mass density of baryons?**

  Both these densities scale as $a^{-3}(t)$ so their ratio stays constant during most of the evolution of the Universe. Then it is possible that mechanism which create baryons and dark matter particles in the early Universe are related to each other so that the approximate equality of the mass densities is not a mere coincidence. On the other hand it is difficult to construct corresponding particle model.

- **What is the origin of dark energy? If this is vacuum,why vacuum has non-zero energy density, which, however, is very small by particle physics standard?**

  This is one of the most fundamental problems of the microscopic physics. In natural units the vacuum density is about

  $$\rho_c \sim 10^{-46} GeV^4 \ . \tag{2.22}$$

  On the other hand we would expect on the basis of the dimensional grounds that the vacuum energy takes value $1GeV^4$ (QCD-scale) or $10^8 GeV^4$ (electroweak scale). It is great challenge to explain this enormous discrepancy but despite numerous attempts it remains an open problem.

- **Why now?**

The energy density of non-relativistic dark matter and dark energy scales differently: The non-relativistic dark matter scales as $a^{-3}(t)$ while the latter stays approximately constant. Hence at early times (small $a(t)$) the energy density of non-relativistic matter exceeded by far the dark energy density. Conversely, future expansion of the Universe will be dominated by dark energy. On the other hand these energy densities are of the same order of magnitude today. The question is why is this the case? What is special about the present epoch of the evolution of the Universe?

2.2 Supernovae and the Accelerating Universe

The first hint that the matter does not dominate the Universe came from the studies of the Type Ia supernovae that are commonly recognized as "standard candles". The special property of Supernovae Type Ia is that it has nearly uniform intrinsic luminosity (absolute magnitude $M \sim -19.5$). It turns out that they can be detected at high redshifts ($z \sim 1$) that allows in principle a good handle on cosmological effects.

The importance of the supernovae measurements began to be clear from the works of two independent groups that observed distant supernovae in order to measure cosmological parameters: the High-Z Supernova Team and the Supernova Cosmology Project.These groups obtained the dependence of the redshift on apparent magnitude. These date are much better fit by a universe dominated by a cosmological constant than by a flat matter-dominated model. In fact, the supernova results alone allow huge range of possible values of Ω_M and Ω_Λ. On the other hand if we presume that we know something about one of these parameters the second one will be tightly constrained and in particular they imply (2.19).

Since these observations are very fundamental one has to ask the question about the level of confidence of them. In fact there are number of potential sources of systematic error that have been considered by these two research teams. In summary these results are commonly accepted with their significant predictions considering the vacuum energy of the Universe.

2.3 Dark Energy

It appears that the most difficult problem to solve is the origin of the dark energy. The most disappointing possibility would be that the carrier of dark energy is **vacuum**: The difficulties with this option will be discussed below.

Another option, more promising from the observational viewpoint is that dark energy is due to some light field. In fact, there are good reasons to consider the this dynamical dark matter as an alternative to cosmological constant. Firstly, the dynamical energy density can evolve slowly to zero so that we can solve the cosmological constant problem .

The simplest possibility how to describe dark matter is the same kind of source that is involved in models of inflation in the very early Universe; a scalar field ϕ rolling slowly in a potential, something known as *quintessence*.

As an example, consider a homogeneous scalar field $\phi(t)$ in an expanding Universe. The action of the scalar field is

$$S = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right) \ , \tag{2.23}$$

where $V(\phi)$ is potential. The equations of motions that follow from the action above have the form

$$\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi] - \sqrt{-g}\frac{\delta V}{\delta\phi} = 0 \tag{2.24}$$

that for homogeneous field in an expanding Universe takes the form

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \ . \tag{2.25}$$

In order to take the back-reaction of this scalar field on the Einstein equations into account we have to determine the components of the stress energy tensor. In field theory the stress energy tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{2.26}$$

that for the action of the form $S = -\int d^4x\sqrt{-g}\mathcal{L}$ takes the form

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}, \tag{2.27}$$

where we have used

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \ . \tag{2.28}$$

More precisely, for the action (2.23) the stress energy tensor takes the form

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}(\nabla_\alpha\phi)(\nabla_\beta\phi) + V(\phi)\right] \ . \tag{2.29}$$

Let us now restrict to the homogeneous case in which all quantities depend only on cosmological time t and we also set $k = 0$. A homogeneous real scalar field behaves as a perfect fluid with

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi) \ . \tag{2.30}$$

The other components of the stress energy tensor take the form

$$T_{ij} = -g_{ij}(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V) + \partial_i\phi\partial_j\phi \ . \tag{2.31}$$

If we define pressure as

$$p = \frac{1}{3}\sum_{i=1}^{3} T_{ii} \tag{2.32}$$

we get

$$p = \frac{\dot{\phi}^2}{2} - V(\phi)\ . \tag{2.33}$$

Thus any state which is dominated by the potential energy of a scalar field will have negative pressure.

If the slope of the potential V is quite flat we will have solutions for which ϕ is nearly constant and only evolving very gradually with time, the energy density in such a configuration is

$$\rho_\phi \approx V(\phi) \approx const. \tag{2.34}$$

Thus we see that slowly-rolling scalar field is an appropriate candidate for dark energy with the vacuum equation of state

$$p_\phi = -\rho_\phi \tag{2.35}$$

but the energy density ρ_ϕ slowly decreases in time. But this proposal raises several questions: why the genuine vacuum energy density is zero (constant part of the potential V_0) so that it does not contribute to dark energy density? What is the physics behind the field ϕ? Where does the small energy scale, $V(\phi) \sim 10^{-46} GeV$ today, come from? All these questions remain unanswered [5].

In fact, it is important to stress that introducing dynamics opens up the possibility to bring new problems that depend on form and specific kind of model being considered. Most quintessence models feature scalar fields ϕ with masses of order the current Hubble scale

$$m_\phi \sim H_0 \sim 10^{-33} eV\ . \tag{2.36}$$

In quantum field theory the light scalar fields are unnatural, renormalization effects tend to drive scalar masses up to the scale of new physics. It is then very difficult to understand the origin of masses of such a small value when we know that the scale of new physics is approximately $10^{11} eV$. Moreover, light scalar fields give rise to long-range forces and time-dependent coupling constant that should be observable. Therefore we have to invoke additional fine-tunings to explain why the quintessence field has not already been experimentally detected.

Another possibility, how to explain today acceleration of Universe, is that there is nothing special about the present era; rather acceleration is just something that happens from time to time. This can be enforced by oscillating dark energy. In these models the potential takes the form of a decaying exponential with small perturbations

$$V(\phi) = e^{-\phi}[1 + \alpha \cos\phi]\ . \tag{2.37}$$

[5] For certain scalar potentials the fourth question can be explained.

Another models of quintessence are *k-essence models* that are based on presumption that the scalar field ϕ has the form

$$K = f(\phi)g(\dot{\phi}^2) \ , \tag{2.38}$$

where f, g are functions specified by the model. Unfortunately, in neither the k-essence models nor the oscillating models do we have a compelling particle-physics motivation for the chosen dynamics and in both cases the behavior still depends sensitively on the precise form of parameters and interactions chosen.

Given the challenge of the problem it is worthwhile considering the possibility that cosmic acceleration is not due to some kind of stuff but rather arise from new gravitational physics.

As a first attempt, consider the simplest correction to the Einstein-Hilbert action,

$$S = \frac{M_p^2}{2} \int d^4x\sqrt{-g}\left(R - \frac{\mu^4}{R}\right) + \int d^4x\sqrt{-g}\mathcal{L}_M \ , \tag{2.39}$$

where μ is a new parameter with units of $[mass]$ and $\mathcal{L}_M$ is the Lagrangian density for matter. The equations arising from this action are complicated and it is difficult to solve them. It is convenient to transform from the action used in (2.39) which we call the *matter frame* to the *Einstein frame* where the gravitational Lagrangian takes the Einstein-Hilbert form and the additional degrees of freedom ($\ddot{H}$ and $\dot{H}$) are represented by a fictitious scalar field ϕ. In terms of the new metric $g_{\mu\nu}$ the theory is that of a scalar field $\phi(x)$ minimally coupled to Einstein gravity and non-minimally coupled to matter with the potential

$$V(\phi) = \mu^2 M_p^2 \exp\left(-2\sqrt{\frac{2}{3}}\frac{\phi}{M_p}\right)\sqrt{\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_p}\right) - 1} \ . \tag{2.40}$$

Yet another option for the explaining the accelerated expansion of our Universe is that gravity deviates from General Relativity at cosmological distances and time scales so that the Friedmann equation is not valid at present epoch. Finally, any modification of the Einstein-Hilbert action must, of course, be consistent with the classic solar system tests of gravity theory as well as numerous other astrophysical dynamical tests. In known Lorentz-Invariant examples of such a theory there either exist ghosts (fields with negative energy unbounded from below) or gravity becomes strongly coupled at quantum level. A consistent theory of this sort would probably require "gravitational Higgs mechanism" and violation of Lorentz-invariance but even this-rather exotic idea- has not yet lead to a consistent model that would be able to explain the accelerated expansion of the Universe.

In summary, there are many models whose aim is to explain current acceleration area. All of these models have many problems however it is certainly very important to study them.

2.4 Observational Evidence for Dark Energy

In this section we briefly review facts considering observational evidence for dark energy. The first one is based on so named *Luminosity distance*

2.4.1 Luminosity distance

In 1998 the accelerated expansion of the Universe was reported on the observations of Type Ia Supernova (SN Ia).This observations are based on the existence of redshift in the expanding Universe that is related to the fact that the light emitted by a stellar object becomes red-shifted due the expanding of the Universe. The wavelength λ increases proportionality to the scale factor a according to the formula

$$1+z=\frac{\lambda_0}{\lambda}=\frac{a_0}{a} \ , \tag{2.41}$$

where z is named as redshift and where the subscript zero denotes the quantities given at present epoch.

Another important concept that is related to the observational tools in an expanding background is the definition of the distance. In fact there are many ways how to define distance in expanding Universe. For example, we can consider comoving distance as a distance measured in comoving variables. It turns out that this distance does not change during the evolution of the Universe. On the other hand we can define physical distance that scales proportionally to the scale factor. An alternative way of defining of distance is through the luminosity distance that plays a very important role in astronomy, including supernova observations.

Let us consider for a moment Minkowski space-time and define an absolute luminosity L_s of source that is related to the energy flux $\mathcal{F}$ at the distance d from the source by the formula

$$\mathcal{F}=\frac{L_s}{4\pi d^2} \ . \tag{2.42}$$

We can generalize this relation to the expanding Universe and define the luminosity distance d_L as

$$d_L^2 \equiv \frac{L_s}{4\pi \mathcal{F}} \ . \tag{2.43}$$

Let us consider an object with an absolute luminosity L_s located at coordinate distance χ [6] from an observer located at $\chi=0$. The energy of object that is emitted

[6] Recall that the metric has following form:

$$ds^2=-dt^2+a^2(t)[d\chi^2+f_K^2(\chi)(d\theta^2+\sin^2\theta d\phi^2)] \ , \tag{2.44}$$

where

$$\begin{aligned} f_K &= \sin\chi \ , k=1 \ , \\ f_K &= \chi \ , k=0 \ , \\ f_K &= \sinh\chi \ , k=-1 \ . \end{aligned}$$

in time interval $\triangle t_1$ let is denoted as $\triangle E_1$ while the energy that reaches the sphere at radius χ is written as $\triangle E_0$. From the basic principles it is clear that $\triangle E_1$ and $\triangle E_0$ are proportional to the frequencies of light at $\chi = \chi_s$ and $\chi = 0$ respectively. In other words, $\triangle E_1 \sim \nu_1$, $\triangle E_0 \sim \nu_0$. We also define the luminosity L_s and L_0 through the relations

$$L_s = \frac{\triangle E_1}{\triangle t_1} \ , \quad L_0 = \frac{\triangle E_0}{\triangle t_0} \ . \tag{2.46}$$

The speed of light is given by $c = \nu_1 \lambda_1 = \nu_0 \lambda_0$ where λ_1, λ_0 are wavelengths at $\chi = \chi_s$ and $\chi = 0$. Then (2.41) implies

$$\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0} = \frac{\triangle E_1}{\triangle E_0} = \frac{\triangle t_0}{\triangle t_1} = 1 + z \ , \tag{2.47}$$

using also the fact that $\nu_0 \triangle t_0 = \nu_1 \triangle t_1$. If we now combine (2.47) and (2.46) we obtain

$$\frac{L_s}{L_0} = \frac{\triangle E_1}{\triangle E_0} \frac{\triangle t_0}{\triangle t_1} = (1+z)^2 \ . \tag{2.48}$$

The light travailing along χ direction satisfies the geodetic motion $ds^2 = -dt^2 + a^2(t) d\chi^2 = 0$ that implies

$$\chi_s = \int_0^{\chi_s} d\xi = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{h(z')} \ , h(z) = \frac{H(z)}{H_0} \ , \tag{2.49}$$

where we have take t_0 as the time at present epoch and consequently $\chi_0 = 0$. We have also used the fact that

$$1 + z = \frac{a_0}{a} \Rightarrow \frac{dz}{dt} = -\frac{a_0}{\dot{a}} \Rightarrow dt = -\frac{dz \dot{a}}{a_0} \ . \tag{2.50}$$

Now the form of the metric (2.44) implies that the area of two sphere at $t = t_0$ is given by $S = 4\pi (a_0 f_K(\chi_s))^2$, where χ_s corresponds to the fact that we observe signal from the distance χ_s. Hence the observed energy flux is

$$\mathcal{F} = \frac{L_0}{4\pi (a_0 f_K(\chi_s))^2} \ . \tag{2.51}$$

Using these results we obtain

$$d_{L_s}^2 = \frac{L_s}{4\pi \mathcal{F}} = \frac{L_s 4\pi (a_0 f_K(\chi_s))^2}{4\pi L_0} = a_0^2 f_K(\chi_s)^2 (1+z)^2 \ . \tag{2.52}$$

If we combine (2.49) with (2.52) and use the fact that in FRW background $f_K(\chi) = \chi$ we obtain

$$d_L = \frac{1+z}{H_0} \int_0^z \frac{dz'}{h(z')} \ . \tag{2.53}$$

(2.45)

.

We can invert this result and express $H(z)$ as function of $d_L(z)$ and z

$$H(z) = \left(\frac{d}{dz}\left[\frac{d_L(z)}{1+z}\right]\right)^{-1} . \tag{2.54}$$

If we measure the luminosity distance observationally we can determine the expanding rate of the Universe.

As we know the energy density on the right hand side of the Friedmann equations includes all components that are presented in Universe, namely non-relativistic particles, relativistic particles, cosmological constant:

$$\rho = \sum_i \rho_i^{(0)}(a/a_0)^{-3(1+w_i)} = \sum_i (1+z)^{3(1+w_i)} , \tag{2.55}$$

where we have used (2.41). Here w_i and $\rho_i^{(0)}$ correspond to the equation of state and the present energy density of each component.

Then the Friedmann equation takes standard form

$$H^2 = H_0^2 \sum_i \Omega_i^{(0)}(1+z)^{3(1+w_i)} , \quad \Omega_i^{(0)} = \frac{8\pi G\rho_i^{(0)}}{3H_0^2} = \frac{\rho_i^{(0)}}{\rho_c^{(0)}} . \tag{2.56}$$

Hence the luminosity distance in a flat geometry is given by

$$d_L = \frac{(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\sum_i \Omega_i^{(0)}(1+z')^{3(1+w_i)}}} . \tag{2.57}$$

The formula above is the basic theoretical ingredient for the direct evidence of the current acceleration of the Universe that is related to the observation of luminosity distances of high redshift supernovae.

The Type Ia supernova (SN Ia) can be observed when the white dwarf starts exceed the mass of the Chandrasekhar limit and explode. The common belief is that SN Ia are formed in the same way irrespective of where they are in the Universe that means that they have a common absolute magnitude M independent of the redshift z. This implies that they can be treated as an ideal standard candle. We do not go to these details but it is important that using these methods the luminosity distance of the SN Ia supernovae that was observed is

$$H_0 d_L \simeq 1.16 , \quad \text{for } z = 0.83 . \tag{2.58}$$

On the other hand the theoretical estimate that follows from (2.57) is

$$\begin{aligned} &H_0 d_L \simeq 0,95 , \quad \Omega_m^{(0)} \simeq 1 , \\ &H_0 d_L \simeq 1.23 , \quad \Omega_m^{(0)} \simeq 0.3 , \quad \Omega_\Lambda^{(0)} \simeq 0.7 . \end{aligned} \tag{2.59}$$

for two-component form of matter. There are of course lot of literature considering the fitting the estimate date and the form of the matter that is present in Universe. The conclusion is that the present experimental date suggests the form of the matter given above.

2.5 The age of the Universe and the cosmological constant

Another important evidence for the existence of the cosmological constant emerges when we compare the age of the Universe t_0 to the age of the oldest stellar populations t_s. It is clear that the consistency demands that $t_0 > t_s$. On the other hand it is difficult to satisfy this condition for a flat cosmological model with normal form of matter. On the other hand the presence of cosmological constant can resolve this problem.

To begin with we review the estimates of the oldest stellar objects. It was estimated that the age of the oldest objects lay in the interval $11-13$ Gyr. Consequently the age of the Universe needs to satisfy the lower bound $t_0 > 11 - 12$ Gyr. Let us calculate the age of the Universe from the Friedmann equations where we consider three contributions to the matter: radiation ($w_r = 1/3$), pressure-less dust ($w_m = 0$) and cosmological constant $w_\Lambda = -1$.

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = H_0^2[\Omega_r^0 \left(\frac{a}{a_0}\right)^{-4} + \Omega_m^{(0)} \left(\frac{a}{a_0}\right)^{-3} + \\ +\Omega_\Lambda^{(0)} - k_0 \left(\frac{a}{a_0}\right)^{-2}] \, , k_0 = \frac{k}{a_0^2 H_0^2} \, . \tag{2.60}$$

Then using the fact that $1 + z = \frac{a_0}{a}$ we can determine the age of the Universe as

$$t_0 = \int_0^{t_0} dt' = \int_0^{a_0} \frac{da}{Ha} = (-dz = \frac{a_0 da}{a^2}) = \\ = \int_0^\infty \frac{dz}{H(1+z)} = \int_0^\infty \frac{dz}{H_0 x[\Omega_r^0 x^4 + \Omega_m^{(0)} x^3 + \Omega_\Lambda^{(0)} - k_0 x^2]^{1/2}} \, , \tag{2.61}$$

where $x = 1 + z$. Since the radiation dominated period is much shorter than the total age of the Universe it is a natural to neglect its contribution to the formula above. In other words the integral coming from the region $z \geq 1000$ does not affect too strongly the integral (2.61). Hence we set $\Omega_r^{(0)} = 0$ when we evaluate t_0.

Let us start with the case when the cosmological constant is absent ($\Omega_\Lambda^{(0)} = 0$). Since $k_0 = \Omega_m^{(0)} - 1$ the integral (2.61) is equal to

$$t_0 = \int_0^\infty \frac{dz}{H_0 x \sqrt{\Omega_m^{(0)} x^3 - k_0 x^2}} = \int_0^\infty \frac{dz}{H_0 (1+z)^2 \sqrt{1 + \Omega_m^{(0)} z}} \, . \tag{2.62}$$

For a flat Universe that is characterized with $k_0 = 0$ and $\Omega_m^0 = 1$ we obtain

$$t_0 = \frac{2}{3H_0} \, . \tag{2.63}$$

As we know the present Hubble parameter is constrained to be

$$H_0^{-1} = 9.776 h^{-1} \text{ Gyr} \, , \; 0.64 < h < 0.8 \, . \tag{2.64}$$

Then (2.63) gives

$$t_0 = 8 - 10 \text{ Gyr} \ . \tag{2.65}$$

However this does not satisfy the stellar age bound

$$t_0 < 11 - 12 \text{ Gyr} \ .$$

In other words the flat Universe without a cosmological constant suffers from a serious age problem.

For arbitrary $\Omega_m^{(0)}$ the equation (2.61) can be integrated and we obtain

$$H_0 t_0 = \frac{1}{1-\Omega_m^{(0)}} - \frac{\Omega_m^{(0)}}{2(1-\Omega_m^{(0)})^{3/2}} \ln\left(\frac{1-\sqrt{1-\Omega_m^{(0)}}}{1+\sqrt{1-\Omega_m^{(0)}}}\right) \tag{2.66}$$

that is of course valid for $\Omega_m^{(0)} < 1$ only. Let us consider various limits of the equation above. For $\Omega_m^{(0)} \to 0$ we obtain $H_0 t_0 \to 1$ while for $\Omega_m^{(0)} \to 1$ we obtain $t_0 H_0 \to 2/3$. As we know the observation of the CMB constraints the curvature of the Universe to be close to be flat $|k_0| = |\Omega_m^{(0)} - 1| \ll 1$. However since then $\Omega_m^{(0)} \approx 1$ in this case we again obtain

$$t_0 = \frac{2}{3H_0} \simeq 8 - 10 \text{ Gyr} \tag{2.67}$$

that is again consistent with the time of the stellar age bound.

On the other hand the age problem can be easily solved in a flat Universe ($k_0 = 0$) with a cosmological constant $\Omega_\Lambda \neq 0$). In this case the equation (2.61) gives

$$\begin{aligned} H_0 t_0 &= \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_m^{(0)}(1+z)^3 + \Omega_\Lambda^{(0)}}} = \\ &= \frac{2}{3\sqrt{\Omega_\Lambda^{(0)}}} \ln\left(\frac{1+\sqrt{\Omega_\Lambda^{(0)}}}{\sqrt{\Omega_m^{(0)}}}\right) \ , \end{aligned} \tag{2.68}$$

where $\Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$. We see that $H_0 t_0 \to \infty$ for $\Omega_m^{(0)} \to 0$ and $H_0 t_0 \to 2/3$ for $\Omega_m^{(0)} \to 1$. When $\Omega_m^{(0)} = 0.3$ and $\Omega_\Lambda^{(0)} = 0.7$ one has

$$t_0 = 0.964 \ H_0^{-1} = 13.1 \text{ Gyr} \ , \ \text{for } h = 0.72 \ . \tag{2.69}$$

Hence this easily satisfies the constraint $t_0 > 11 - 12$ Gyr that arises from the observation the oldest stellar populations. Thus the presence of Λ solves the age-crisis problem.

2.6 The Cosmological Constant Problem

In classical general relativity the cosmological constant Λ is a completely free parameter. Let us determine corresponding dimension of given constant. Note that it appears in the action in the form

$$\frac{1}{8\pi G}\int d^4x\sqrt{-g}\Lambda \ . \tag{2.70}$$

Since the dimension of G is $[G] = M^{-2}$ where M is mass scale and since $[d^4x] = M^{-4}$ we find from the requirement that the action is dimensionless that the dimension of Λ is given by the equation

$$[\Lambda] = M^2 \tag{2.71}$$

while

$$\rho_\Lambda = \frac{1}{8\pi G}\Lambda \tag{2.72}$$

has dimension $[\rho_\Lambda] = M^4$ as it is expected for the energy density. In fact, Λ is completely free and its value should be determined by experiment.

The introduction of quantum mechanics changes the situation in some way. Firstly, the Planck's constant allows us to define the reduced Planck mass $M_P \sim 10^{18}GeV$, as well as reduced Planck length

$$L_P = (8\pi G)^{1/2} \sim 10^{-32}cm \ . \tag{2.73}$$

Hence the natural guess for the value of the cosmological constant is

$$\Lambda_P^{guess} \sim L_P^{-2} \ , \tag{2.74}$$

or as an energy density

$$\rho_{vac}^{gusss} \sim M_P^4 = (10^{18}GeV)^4 \ . \tag{2.75}$$

We can find support for this guess by thinking about the quantum fluctuation of vacuum. As we know any quantum field can be considered as collection of infinite number of harmonic oscillators. From quantum mechanics we know that harmonic oscillator with frequency ω has the vacuum energy $\frac{1}{2}\hbar\omega$. Since each mode of the quantum field contributes to the vacuum energy and the net result should be an integral over all of these modes. Usually we perform an integration over infinite interval and hence this integral diverges so that the vacuum energy appears to be infinite. However, the infinity arises from contribution of modes with very small wavelengths, it is possible to be mistake to include such a modes since we do not know what happens at these scales. In other words we do not have any justification whether the quantum field theory approach can be applied in these small scales as well. To account for our ignorance we should include the cut-off energy above which we ignore any potential contributions and hope that some more complete theory could justify this approach. If the cut-off is at the Planck scale we get the value given above.

However, we claim to have measured the vacuum energy. The observed value is different from the theoretical estimate:

$$\rho_{vac}^{obser} \sim 10^{-120} \rho_{vac}^{guess} \ . \tag{2.76}$$

In other words, we can express the vacuum energy in terms of the mass scale

$$\rho_{vac} = M_{vac}^4 \tag{2.77}$$

so that the observed result is

$$M_{vac}^{obs} \sim 10^{-3} eV. \tag{2.78}$$

The discrepancy is thus

$$M_{vac}^{obs} \sim 10^{-30} M_{vac}^{guess} \ . \tag{2.79}$$

In addition to the fact that it is very small to its natural value the vacuum energy at present posses an additional puzzle. The coincidence between observed vacuum energy and current matter density. It can be shown that the ratio of vacuum energy to matter density depends on time as follows from

$$\frac{\Omega_\Lambda}{\Omega_M} = \frac{\rho_\Lambda}{\rho_M} \sim a^3 \quad . \tag{2.80}$$

As a consequence, at early times the vacuum energy was negligible with respect in comparison to matter and radiation while at late times matter and radiation are negligible.

To date the value of the cosmological constant is one of the most mysterious problems in current physics, perhaps it could be compared with the mysterious radiation of the black body at the end of 19' century. On the other hand it is instructive to consider an example of supersymmetry which relates to the cosmological constant problem in interesting way. The main idea of supersymmetry is that for each fermionic degree of freedom there is corresponding bosonic degree of freedom and vice-versa. For example, for spin 1/2 electron there should be spin 0 electron of the same mass and charge. The good news is that while bosons contribute positively to the vacuum energy the fermion contributions is negative. Hence, if the degrees of freedom exactly match the vacuum energy is zero.

We do not, however, live in supersymmetric state. If supersymmetry exists, then it must be broken at some scale M_{susy}. In other words, for physical processes where the characteristic energy is much smaller than M_{susy} we do not see any supersymmetry and this is the case how our word looks like. On the other hand when we probe physics with energy scale higher with M_{susy} we can expect that supersymmetry is restored. More precisely, we can explain this situation as follows. We expect that SUSY is broken in nature, for example spontaneously broken which means that there is one ground state. The fluctuation above states gain masses and one expect that super-partners of known particles, get masses of order M_{susy}. Then for energies much smaller than M_{susy} these particles are not visible, on the other hand for energies larger than M_{susy} we can neglect their masses and these particles look like massless again. Then we say that supersymmetry is restored at higher energies. This has an consequence for the vacuum energy. Recall that the vacuum energy was defined as sum over infinite number of oscillators. For modes with energy much larger that M_{susy} these modes find their super-partners and hence their contribution to the vacuum energy vanishes. This is of course does not happen for modes with energy smaller than M_{susy}. In other words we can expect that the vacuum energy will be equal to

$$\rho_{vac} \sim M_{susy}^4 \ . \tag{2.81}$$

The question is how high M_{susy} should be. Nice property of SUSY is that it helps us to understand *hierarchy problem*- why scale of electroweak symmetry breaking is much smaller than the scales of quantum gravity or grand unification. For SUSY to be relevant to the hierarchy problem we need the SUSY breaking scale to be just above the electroweak breaking scale

$$M_{susy} \sim 10^3 \ GeV \ . \tag{2.82}$$

Since this is very close to the experimental bound it is now common belief that SUSY should be discovered soon at Fermilab or CERN, if it is connected to electroweak physics. However considering relation between SUSY and cosmological constant we again see that we are in discrepancy with observation:

$$M_{vac}^{(obs)} \sim 10^{-15} \ M_{susy} \ \text{(Experiment)}. \tag{2.83}$$

Of course there exists a possibility that our estimate $M_{vac} \sim M_{susy}$ is incorrect. For example let us guess following formula

$$M_{vac} \sim \left(\frac{M_{susy}}{M_P}\right) M_{susy} \ . \tag{2.84}$$

Interestingly, since M_P is fifteen orders of magnitude larger than M_{susy} and M_{susy} is fifteen orders of magnitude larger than M_{vac} this guess gives up the correct answer. Unfortunately this is simple numerology, we do not know how this formula should come from.

Another possibility how to explain the value of the cosmological constant is the presumption that it is simply feature of our local environment. This is the idea commonly known as **anthropic principle**.

In order to give this idea concrete meaning let us presume that there are many different regions of the Universe in which the vacuum energy takes different values.

Then we can expect that we find ourselves in a region which was suitable for our own existence. Larger value of cosmological constant than we presently observe would either have led to a rapid re collapse of the universe (if ρ_{vac} were negative) or an inability to form galaxies (if ρ_{vac} were positive).

The idea environmental selection is based on certain special conditions and we do not understand whether these conditions hold in our Universe. In particular we have to show that there can be a huge number of different domains with slightly different values of the vacuum energy and that these domains are big enough that our entire observable Universe is in a single domain. Further we also have to show that the possible variation of other physical quantities from domain to domain is consistent with observations.

Recent work in string theory whose pure essence is the currently very popular idea of **String Landscape** supports the idea that there are huge number of possible vacuum states rather than a unique one. Unfortunately the detailed discussion of this idea is beyond the scope of this introduction review.

To conclude, at present, unfortunately,t here is not any theory that could explain the mysterious facts considering cosmological constant. To find such a theory is one of the most prominent goals of physical community.

2.7 The Cosmic Microwave Background

Most of the radiation we observe in Universe today is in the form of the almost isotropic black body spectrum with temperature approximately $2.7K$ known as *Cosmic Microwave Background (CMB)*. The small angular fluctuations in temperature of the CMB reveal a great deal about the constituents of the Universe.

We have seen previously that the radiation gas evolves and sources the evolution of the expanding Universe. Since the radiation and dusts have different evolution laws that as we approach earlier and earlier times in the Universe with smaller and smaller scale factors the ratio of the energy density in radiation to that in matter grows proportionally to $1/a(t)$. Furthermore, even particles which are now massive and contribute to matter used to be hotter, at sufficiently early times were relativistic and thus contributed to radiation. In summary, we say that the early Universe was dominated by radiation. More precisely, at early times the CMB photons were easily energetic enough to ionize hydrogen atoms and therefore the Universe was filled with a charged plasma. This phase lasted until the photons red shifted enough to allow protons and electrons to combine during the era of *recombination*. Shortly after this time the photons decoupled from the now neutral plasma and free streamed through the Universe.

More precisely, the concept of an expanding Universe provides us with a clear explanation of the origin of the CMB. Black body radiation is emitted by bodies in thermal equilibrium. The present Universe is certainly not in this state, and so without an evolving space-time we should have no explanation for the origin of this radiation. However, at early times, the density and energy densities in the Universe were high enough that matter was in approximate thermal equilibrium at each point in space, yielding a blackbody spectrum at early times. Then there is crucial thermodynamic fact about the CMB. A blackbody distribution, such as that generated at early Universe, is such that at temperature T, the energy flux in the frequency range $[\nu, \nu + d\nu]$ is given by Planck distribution

$$P(\nu, T)d\nu = 8\pi h \left(\frac{\nu}{c}\right)^3 \frac{1}{e^{h\nu/kT} - 1} d\nu \ , \tag{2.85}$$

where h is Planck's constant and k is the Boltzmann constant. Under recalling $\nu \to \lambda\nu$, with $\lambda = constant$ the shape of the spectrum is unaltered if $T \to T/\lambda$. We know that the wave length are stretched with the cosmic expansion and therefore the frequencies will scale inversely due to the same effect. We then see that the effect of cosmic expanding on an initial blackbody spectrum is to retain its blackbody nature, but just at lower and lower temperatures

$$T \sim \frac{1}{a} \ . \tag{2.86}$$

This is what we mean when we say that the Universe is cooling as it expands.

It is also well known that CMB is not a perfectly isotropic radiation bath. Deviations from isotropy at the level of one part in 10^5 have developed over the last decade into one of our most precise observation tool in cosmology.The small temperature anisotrophies on the sky are usually analyzed by decomposing the signal into spherical harmonics via

$$\frac{\triangle T}{T} = \sum_{l,m} a_{lm} Y_{lm}(\phi, \theta) \ , \tag{2.87}$$

where a_{lm} are expansion coefficients and θ and ϕ are spherical polar angles on the sky. Next we define the power spectrum as

$$C_l = \left\langle |a_{lm}|^2 \right\rangle \ . \tag{2.88}$$

The fluctuations in the CMB spectrum are useful for the study of cosmology from many reasons. To understand why, we should show at the first place why they arise. Matter today in the Universe exists in the form of clusters of starts, galaxies, and clusters and super-clusters of galaxies. Our understanding how large scale structures developed is that initially small density perturbations in the otherwise homogeneous Universe grew through the gravitational instability to the objects we observe today. Such picture requires that from place to place there were small variations in the density of matter at the time when CMB firstly decoupled from the photon-baryon plasma. Then CMB photons propagated freely through the Universe nearly unaffected by anything except the cosmic expanding itself. However it the time of their decoupling different photons were released from regions of space with slightly different gravitational potentials. Since the gravitational potential affects the photon redshift, photons from some regions redshift slightly more than those from other regions , giving rise to a small temperature anisotropy in the CMB observed today. In this sense CMB reflects the initial conditions that ultimately gave rise to structure in the Universe.

It is important that CMB fluctuations give us the value of Ω_{total}. In fact, careful analysis of all of the features of the CMB power spectrum provide constraints on essentially all of the cosmological parameters. For example, let us consider recent result from $WMAP$. For total density of the Universe they find

$$0.98 \leq \Omega_{total} \leq 1.08 \tag{2.89}$$

at 0.95 confidence which is a strong evidence for a flat Universe. Nevertheless, much tighter constraints on the remaining values can be derived by assuming either an exactly a flat Universe or a reasonable value of Hubble constant. When for example we presume a flat Universe, we can derive values for the Hubble constant, matter density (which then implies the vacuum density from $\Omega_{total} = 1$) and baryon density:

$$\begin{aligned} h &= 0.72 \pm 0.05 \ , \\ \Omega_M &= 1 - \Omega_\Lambda = 0.29 \pm 0.07 \ , \\ \Omega_B &= 0.047 \pm 0.006 \ . \end{aligned} \tag{2.90}$$

If we instead assume that the Hubble constant is given by the value determined by HST project

$$H_0 = 100 \text{ h km sec}^{-1} \text{ Mpc}^{-1} \ , \quad \text{h} = 0.71 \pm 0.06 \tag{2.91}$$

we can derive separate tight constraints on Ω_M and Ω_Λ.

In summary, taking all of the data together we obtain a remarkably consistent picture of the current constituents of our Universe:

$$\begin{aligned} \Omega_B &= 0.04 \ , \\ \Omega_{DM} &= 0.26 \ , \\ \Omega_\Lambda &= 0.7 \ . \end{aligned} \tag{2.92}$$

There are many mysterious things considering these values. Firstly, the baryon density is mysterious due to the asymmetry between baryons and antibaryons. Secondly, the problem with dark matter is that we have never detected it directly and only have promising ideas as to what it might be. However the biggest mystery is the vacuum energy, we now try to explain why it is mysterious and what kinds of mechanism might be responsible for its value.

3. Early Times in the Standard Cosmology

Early times at the in the Standard Cosmology are characterized by very high temperatures and densities with many particle species kept in (approximate) thermal equilibrium by rapid interactions. Our goal is then to develop some tools of the thermodynamics in expanding Universe. In fact, up the mild-1960 it was not clear whether the early Universe had been hot or cold. This situation changed with the Pensias and Wilson's 1964-1965 discovery of 2.7 K microwave background radiation arriving from the farthest reaches of the Universe since the existence of the microwave background has been predicted by the hot Universe theory.

3.1 Review of the building blocks of the standard cosmology and matter

For reader's convenience we review some basics facts considering the standard models of cosmology.

- **The Classical general relativity:**

  The classical general relativity provides good description of the geometry of space-time for scales $l \gg l_P = M_P^{-1} = 10^{-33} cm$ or equivalently for energy scales below the Planck scale M_P.

- Physical scales are stretched by the scale factor $a(t)$ with respect to the comoving scales

$$l_{phys}(t) = a(t) l_{com} \ . \tag{3.1}$$

  A physical wavelength redshifts proportional to the scale factor where its time derivative obeys the Hubble law

$$\frac{dl_{phys}(t)}{dt} = \frac{\dot{a}}{a} a l_{com} = H(t) l_{phys}(t) = \frac{l_{phys}}{d_H(t)} \ . \tag{3.2}$$

- The equilibrium temperature decreases as the Universe expands as

$$T(t) = \frac{T_0}{a(t)} \ . \tag{3.3}$$

- **The Standard Model of Particle Physics:**

  The current standard model of particle physics that is experientially tested with remarkable precision describes the theory of strong (QCD), weak and electroweak interactions (EW) as a gauge theory based on the gauge group

$$SU(3)_c \otimes SU(2) \otimes U(1)_Y \ . \tag{3.4}$$

  The particle content is: three generations of quarks and leptons:

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} ; \begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \tag{3.5}$$

vector Bosons: 8 gluons (massless) that mediate the strong interactions in QCD, $Z^0, W^{\pm}$ that are massive with masses $M_Z = 91.18 \pm 0.02\ GeV$ and $M_W = 80.4 \pm 0.06\ GeV$ that mediate the electroweak interactions, the photon (massless)-the mediator of electromagnetic interaction and the scalar Higgs that was discovered in 2011 at LHC with the mass $M_H = 125.09\ GeV$.

- It is known that the couplings associated with strong, weak and electrodynamics interactions depend on the mass scale that characterize given process. The current theoretical ideas propose that these couplings are unified in a grand unified theory (GUT) at the scale

  $$M_{GUT} \sim 10^{16}\ GeV\ .$$

  Further, the UV scale where the Gravity is eventually unified with the rest of particle physics is the Planck scale

  $$M_P \sim 10^{19}\ GeV\ .$$

  On the other hand the physics of the Standard Model describes phenomena at energy scales below M_S where

  $$M_S \sim 100\ GeV\ .$$

- The connection between the Standard model of particle physics and early Universe cosmology is through Einstein's equations that couple the space-time geometry to the matter-energy content. We study gravity semi-classically at energy scales well below the Planck scale. The Standard model of particle physics is a **quantum field theory** (QFT) thus the space-time is classical but with sources that are quantum fields. Semi classical gravity is defined by the Einstein equations with the expectation value of the energy-momentum tensor $\hat{T}^{\mu\nu}$ as sources

  $$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{\left\langle \hat{T}^{\mu\nu} \right\rangle}{M_P^2}\ , \tag{3.6}$$

  where the expectation value $\left\langle \hat{T}^{\mu\nu} \right\rangle$ is taken in given quantum state or density matrix that is compatible with homogeneity and isotropy so that it has to be translational and rotational invariant. The ground state of the quantum field theory is usually the state that solves the classical equations of motion or the equations of motion with the quantum correction. In this case the vacuum expectation value of the stress energy tensor corresponds to the classical one. The general formula above has important in case we study the properties of the fluctuations above given classical solutions.

As the next step we review basic facts about the Energy scales, time scales and phase transitions

Energy scales,time scales and phase transitions

In this section we give a brief overview of the main cosmological epochs by focusing on the energy scales of particle, nuclear and atomic physics.

Energy scales:

- **Total Unification**

  It is expected that Gravitational, strong and electroweak interactions become unified and described by a single quantum theory at the Planck scale $M_P \sim 10^{19}\ GeV$. The most promising approach to this unification is in terms of string theory however their theoretical consistency is still studied and experimental confirmation is not available.

- **Grand Unification:**

  Strong and electroweak interactions are expected to become unified at an energy scale
  $$M_{GUT} \sim 10^{16}\ \text{GeV}\ , T_{GUT} \sim 10^{29} K$$
  under large gauge group G, for example $SU(5), SO(10)$ that breaks spontaneously
  $$G \to SU(3)_c \otimes SU(2) \otimes U(1)_Y$$
  at scale below unification. Main arguments for the existence of GUT theories follow from merging of the running coupling constants of the strong, electromagnetic and weak interactions for the minimal supersymmetric model and also the explanation of the small neutrino masses via see-saw mechanism.

- **Electroweak:**

  Weak and electromagnetic interactions are unified in the electroweak theory based on the gauge group
  $$SU(2) \otimes U(1)_Y\ .$$
  The weak interactions become short ranged after symmetry breaking phase transition
  $$SU(2) \otimes U(1)_Y \to U(1)_{em}$$
  at the energy scale of the order of the mass of the $Z^0, W^{\pm}$ vector bosons corresponding to temperature
  $$T_{EW} \sim 100\ GeV \sim 10^{15}\ K\ .$$
  More precisely, at temperature $T > T_{EW}$ the symmetry is restored as a consequence of the fact that the effective potential of the theory depends on the

temperature as well. For temperature $T > T_{EW}$ the stable minimum of the potential corresponds to the symmetric phase where all vector bosons are massless and hence the symmetry is restored. On the other hand for $T < T_{EW}$ the stable minimum of the potential corresponds to the situation when the vector bosons $W^{\pm}, Z^0$ become massive through Higgs mechanism while photon remains massless corresponding unbroken $U(1)$ abelian symmetry of quantum electrodynamics. The temperature T_{EW} determines the temperature scale of the electroweak phase transition in the early Universe.

- **QCD**

  The strong interaction has a typical energy scale

$$\Lambda_{QCD} \sim 200 \ MeV \ .$$

  At this coupling the coupling constant becomes strong $\alpha_s \sim O(1)$ that corresponds to the temperature scale

$$T_{QCD} \sim 10^{12} \ K$$

  QCD is asymptomatically free theory that means that the coupling between quarks and gluons becomes smaller at large energies but diverges at the scale Λ_{QCD}. For energies below Λ_{QCD} the quantum chromodynamics is strongly interacting theory and quarks and gluons are bound into mesons and baryons. This phenomenon is interpreted in terms of a phase transition at an energy scale Λ_{QCD} or T_{QCD}. For $T > T_{QCD}$ the relevant degrees of freedom are weakly interacting quarks and gluons, while below are hadrons. In the limit when we can presume that up and down quarks are massless, QCD possesses new $SU(2)_L \otimes SU(2)_R$ chiral symmetry that is spontaneously broken at about the same temperature scale as the scale of QCD transition. Pions are the Goldstone bosons that emerge in the breakdown of the chiral symmetry

$$SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_{R+L} \ .$$

  The high temperature phase above T_{QCD} where the quarks and gluons are almost free (because the coupling is small by asymptotic freedom) is a *quark-gluon plasma.*

- **Nuclear Physics**

  The low energy scales that are relevant in cosmology are determined by the binding energy of light elements. For example, the binding energy of deuterium is $\sim 2 \ MeV$ that corresponds to the temperature $T \sim 10^{10} \ K$. This is the energy scale that determines the origin of the primordial nucleosynthesis. The first step in the system of the nuclear reactions that yields the primordial elements is the formation of deuteron in the reaction

$$n + p \leftrightarrow d + \gamma \ .$$

These nuclear reactions continue and all neutrons end up in nuclei, mainly helium.

- **Atomic physics**

  A further important low energy scale relevant for cosmology is the binding energy of hydrogen $\sim 10 \ eV$. This is the energy scale at which free protons and electrons combine into neutral hydrogen. The large number of photons per baryons implies that recombination actually takes place at an energy scare of order $0.3 \ eV$, at about 400000 years after the beginning of the Universe. At this time when the neutral hydrogen is formed the Universe becomes transparent since then photons no longer scatter and travel freely. These are the photons measured by CMB experiments today.

Time Scales:

- **Inflation epoch**

  This is (according to current cosmological scenario) the earliest period in the life of Universe where the scale factor grows exponentially as

  $$a(t) = e^{Ht} \ .$$

  Current experiments put upper bound on the energy scale of inflation as

  $$H \le 10^{13} \ GeV \ .$$

  In order to solve the entropy and horizon problems the inflationary stage hast to last a time interval δt so that

  $$\delta t H \sim 60 \Rightarrow \delta t \sim 10^{-34} \ sec \ .$$

- **Radiation dominated era**

  The inflationary stage is followed by a radiation dominated era after a short period of reheating during which the energy stored in the field that drives inflation decays into quanta of many other fields. These fields reach the state of thermal equilibrium through the scattering processes.

  After the thermal equilibrium is reached we obtain a detailed picture of the thermal history of the Universe. This description is based on the combination of the statistical mechanics with the basic principles of QFT: During the first 1000 years of the Universe and after the inflation stage that lasted $\sim 10^{-34}$ sec the Universe was radiation dominated. Universe also expands and cools almost adiabatically. The electroweak transition occurred at the energy scale $T \sim 100\ GeV$ that corresponds to the time

$$t_{EW} \sim 10^{-12}\ sec\ .$$

The QCD transition occurs at

$$t_{QCD} \sim 10^{-5}\ sec\ .$$

Local Thermal Equilibrium (LTE) and Non equilibrium

Weather some process occurs in or out of a local thermodynamics equilibrium depends on the comparison of two time scales-the expanding rate and the reaction rate. To have a contact with standard thermodynamics note that we can formulate the same problem as the problem of comparing of the cooling rate (the rate how temperature decreases) and the rate of reaction. In fact the rate of cooling is related to the rate of the expanding through the formula

$$\frac{\dot{T}}{T} = -\frac{1}{Ta^2}\dot{a} = -H(t) \tag{3.7}$$

as follows from the fact that $T \sim \frac{1}{a}$. On the other hand collisions as well as non-collisional processes contribute to establish the equilibrium with a rate Γ. The local thermodynamic equilibrium is established when

$$\Gamma > H(t) \tag{3.8}$$

In this case the evolution is adiabatic in the sense that the thermodynamics functions depend slowly on time through the temperature. On the other hand when the expanding is too fast

$$H(t) \gg \Gamma$$

local thermodynamics equilibrium cannot be established, the temperature drops too fast for the system to have time to relax.

While a detailed understanding of the relaxation dynamics requires an analysis of the quantum Boltzmann equations a simple order of magnitude estimate for a collision rate is given as follows.

The collision rate can be calculated in the standard statistical physics as

$$\Gamma \sim < \sigma n v > , \tag{3.9}$$

where < ... > means statistical ensemble average and where σ is a scattering cross section, n is the density of particles that scatter and v is velocity of given particles. For electromagnetic scattering a typical cross section is of order

$$\sigma_{em} \sim \frac{\alpha^2}{Q^2} ,$$

where Q^2 is transferred momentum and α is the electromagnetic coupling constant. At high temperature single photon exchange implies the estimate (the transferred momentum is proportional to the momenta of one photon that is proportional to the temperature)

$$\sigma_{em} \sim \frac{\alpha}{T^2} .$$

The density of relativistic degrees of freedom is $n \sim T^3$ and for $v \sim 1$ (This estimate follows from the fact that particles are ultra-relativistic) we obtain

$$\Gamma_{em} \sim \alpha^2 T .$$

In QCD that in the high temperature regime can be treated perturbatively the estimate of the single gluon exchange can be performed in the similar way and we get

$$\Gamma_{QCD} \sim \alpha_s^2 T ,$$

where α_s is corresponding coupling constant. We have to compare these estimates with H. However $H^2 \sim \rho$ that in the case of the radiation dominated era we show that $\rho \sim T^4$.Then in this case we find that $H \sim T^2/M_{pl}$ so that

$$\frac{\Gamma_{QCD}}{H} = \frac{\alpha_s^2 M_{pl}}{T} > 1 \tag{3.10}$$

and we obtain that the strong interactions are in LTE for

$$T \leq 10^{16}\ GeV$$

In the same way we obtain that electromagnetic interactions are in LTE for

$$T \leq 10^{14}\ GeV .$$

It is important to stress for $T \leq \alpha^2 M_{pl} \sim 10^{16} GeV$ all perturbative interactions should be frozen out and are not effective in maintaining thermal equilibrium. In other words all known interactions together with any new interactions that arise from grand unification are not sufficient for maintaining the thermal equilibrium in the Universe at temperatures greater than 10^{16} GeV that corresponds to the time earlier than $10^{-38}s$. In other words Universe is not in thermal equilibrium at its earliest epoch.

The estimate in case of weak interaction is slightly more involved: a typical scattering process with an energy transfer $E \ll M_W$ has a scattering cross section

$$\sigma \sim G_F^2 E^2 \ , \quad E \ll M_W$$

whereas if $E \gg M_W$ we have

$$\sigma \sim \frac{g^4}{E^2} \ , \quad E \gg M_W \ .$$

Then in thermal medium with $E \sim T$ and with a density of relativistic particles $n \sim T^3$ a typical weak reaction rate is

$$\Gamma_{EW} \sim g^4 T \ , \quad T \gg M_W$$

and

$$\Gamma_{EW} \sim G_F^2 T^5$$

for $T \ll M_W$. In this latter temperature regime the ratio

$$\frac{\Gamma_E}{H} \sim \left(\frac{T}{MeV}\right)^3$$

and hence the weak interactions fall out of LTE for $T \leq 1 \ MeV$.

Even if this analysis provides an intuitive estimate for the relaxation time scales this analysis neglected several important aspects that however have to be studied on a case-by-case basis. One such an example of subtle effects are *Screening and infrared phenomena:* The relaxation rates Γ were calculated on presumption of an exchange of a vector boson of relativistic degrees of freedom. In a medium at a high temperature and a density there are important screening effects that can change these estimates.

3.2 Hot Big Bang

We begin this section with the description of the evolution of the Universe in its hot stage.

The basic presumption is that it is plausible to extrapolate the evolution of the Universe back in time using the known microscopic physics (electrodynamics, nuclear physics, QCD and electroweak theory) and General Relativity. This theory is called as **Hot Big Bang Theory**. According to this theory the Universe was hotter at earlier stages (equivalently, at smaller values of $a(t)$) and the temperature scales as $a(t)^{-3}$ both for non-relativistic and relativistic particles. At high enough temperatures the Universe was in the phase that is completely different from what we observe today. Instead of the almost empty space with galaxies here and there was dense, hot and almost homogeneous plasma that fills the whole Universe. This is the area whose physical laws are governed by microscopic physics. Note that gravity plays the role of the spectators of the theory and it is considered as classical. Of course we consider back-reaction of this matter on the time evolution of the Universe using the Friedmann equations.

More precisely, the hot Universe theory is based on the phenomena of the phase transitions and the symmetry breaking. Let us consider for example the simplest GUT model based on the gauge group $SU(5)$. For temperature $T \geq 10^{15} GeV$ there was no difference between weak, strong and electroweak interactions. The matter in the Universe was in the form of the dense plasma containing quarks, photons, gluons etc. Then there was no problem in the transformation of quarks to leptons. In other words it does not make sense to speak about baryon conservation. At $t_1 \sim 10^{-35} sec$ when the temperature has dropped to $T \sim T_{c_1} \sim 10^{14} - 10^{15} GeV$ the fist symmetry breaking phase transition takes place: $SU(5)$ breaks to $SU(3) \times SU(2) \times U(1)$ where $SU(3)$ is gauge symmetry of the QCD, theory of the strong interactions. In other words string interactions were separated from electroweak and leptons. Then at $t_2 \sim 10^{-10}$ sec when the temperature dropped to $T_{c_2} \sim 10^2$ GeV there was a second phase transition that broke the symmetry between weak and electromagnetic interactions $SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)$. As the temperature reduces further to $T_{c_3} \sim 10^2$ MeV there was another phase transition with the formation of baryons and mesons from quarks.

3.3 Review of the study of the expansion of the Universe

Let us again analyze the evolution of the Universe. As we have argued before at early times the Universe was radiation dominated, then matter dominated and presently dark energy dominated while the curvature term $\frac{k}{a^2}$ was never important.

Deceleration to Acceleration

Since the dark energy dominates at present the Universe accelerates. On the other hand when matter was dominating the Universe was decelerating. In order to see when the change in regime occurred we write the Friedmann equations as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 = \frac{8\pi G}{3} a^2(\rho_M + \rho_\Lambda) \quad , \tag{3.11}$$

where we have neglected spatial curvature and also ultra-relativistic matter for the moment. The reason for this simplification is that the relativistic matter dominates an expanding of the Universe at much earlier stage. The time derivative of the equation above implies

$$\begin{aligned} 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(\dot{\rho}_M a^2 + 2(\rho_M + \rho_\Lambda)\dot{a}a\right) = \\ = \frac{8\pi G a}{3}\left(-\dot{a}a\rho_M + 2\dot{a}a\rho_\Lambda\right) \end{aligned} \tag{3.12}$$

where we used $\dot{\rho}_M = -3\frac{\dot{a}}{a}\rho_M$. The expression above is zero when (This event defines the turning point between decelerating and accelerating phase)

$$\frac{2\rho_\Lambda}{\rho_M} = 1 \tag{3.13}$$

or equivalently

$$\frac{a_0^3}{a^3} \equiv (1+z)^3 = \frac{2\Omega_\Lambda}{\Omega_M} \ , \tag{3.14}$$

where or course Ω_M is time-dependent. For expected values $\Omega_\Lambda = 0.7, \Omega_M = 0.3$ we have

$$\text{deceleration} \rightarrow \text{acceleration: } z \approx 0.7$$

In other words, the Universe was decelerating until fairly recently. Before $z \approx 0.7$ the expansion was dominated by the non-relativistic matter.

Radiation domination to matter domination

As we know the energy density of ultra-relativistic matter (radiation) scales as a^{-4} while the energy density of non-relativistic matter scales as a^{-3}. Then it follows that the dominant contribution to the energy density of the Universe at very small a (small t) came from ultra-relativistic matter. Now we estimate z_{eq} at which the equilibrium between matter and radiation occurred. In other words we would like estimate z_{eq} when the expansion regime changed from the dominance of ultra-relativistic particles to the dominance of non-relativistic matter, we write

$$\frac{\rho_M(t)}{\rho_{rad}(t)} = \frac{\rho_{M0}a_0^3 a^{-3}(t)}{\rho_{rad0}a_0^4 a^{-4}(t)} = \left(\frac{\rho_M}{\rho_{rad}}\right)_0 \frac{a(t)}{a_0} \ , \tag{3.15}$$

where again the subscript 0 refers to present values. Equilibrium occurs at

$$\frac{\rho_M(t_{eq})}{\rho_{rad}(t_{eq})} \approx 1 \tag{3.16}$$

that gives

$$\frac{a_0}{a(t_{eq})} \equiv 1 + z_{eq} \approx \left(\frac{\rho_M}{\rho_{rad}}\right)_0 = \frac{\Omega_M}{\Omega_{rad}} \ . \tag{3.17}$$

Since $\Omega_{rad} \approx 10^{-4}$, $\Omega_M \approx 0.3$ we obtain

$$\text{radiation domination} \rightarrow \text{matter domination : } z_{eq} \approx 3000 \ .$$

The corresponding temperature is

$$T_{eq} = T_0(1 + z_{eq}) \approx 10^4 K \approx 1eV \ . \tag{3.18}$$

At higher temperatures the expansion of the Universe was dominated by ultra-relativistic matter. We must to stress that it is important for structure formation that the most of the part of the lifetime of the Universe is dominated by non-relativistic matter. This follows from the fact that the expanding rate at both radiation dominated and vacuum dominated eras is such that gravitational perturbations grow slowly and only during the matter dominated stage their growth is fast enough so that the existing structures of the Universe can arise.

3.4 Epochs of the early Universe

There are two important epochs in the evolution of the Universe:**Recombination epoch** that is the transition from plasma to neutral gas. This occurs at temperature $T \sim 3000K, t \sim 3 \cdot 10^5 years$ and **nucleosynthesis epoch** that occurs at temperatures $T = 1MeV$ to a few $\cdot 10keV$. Another event is **neutrino decoupling**. Briefly, at high temperatures the neutrino was in thermal equilibrium with the rest of cosmic plasma. The plasma became transparent for neutrinos at temperature about $1MeV$. This decoupling of neutrinos is very important for nucleosynthesis since it affects the neutron-proton ratio just before nucleosynthesis (Since neutrinos decouples the reaction that transfers proton into neutrons simply cannot occur) and hence it leads to the abundances of light elements that need neutrinos for their formations. Further, the fact that neutrinos decoupled much earlier than photons implies that the present neutrino-to-photon ration is less than one. This is consequence of the fact that photons are additionally heated, after neutrino decoupling, due to the annihilations of e^+ with e^-.

If we move further back in time we obtain that the cosmic plasma has more and more components. At temperatures roughly $0.5MeV$ there are many electrons and positrons that are frequently pair created and annihilate: at $T > 100MeV$ the plasma contains muons and pions. This plasma remains in thermal equilibrium except possibly for **phase transitions**

- **QCD phase transition**

  At temperatures above $100MeV$ (QCD scale) strongly interacting particles are dissolved into quarks and gluons. This quark-gluon plasma converts into hadronic matter (mostly pions) during the quark-hadron phase transitions. Theoretical estimates suggest that the temperature of this phase transition is about $170MeV$.

- **Electroweak transition**

  Briefly, at temperatures well above $100GeV$ electroweak symmetry is unbroken. The consequence of this fact is that W and Z bosons are massless. At $T \sim 100GeV$ the phase transition of the electroweak symmetry breaking takes place.

- **GUT transition**

  It is slightly uncertain when we extrapolate back further (equivalently, we go to higher temperatures), but if we do so we come to the Grand Unification epoch. The temperature of this epoch is set by GUT scale, $T_{GUT} \sim 10^{16} GeV$. We expect that at this temperature the Grand Unified phase transition occurs. On the other hand many models of inflation suggest that the Universe never had such a high temperature after inflation.

Expansion rate and life-time at radiation domination

Now we will discuss in more details the expansion of the Universe in radiation dominated stage where we will presume thermal equilibrium of all ultra-relativistic species [7]. In the very early stages of its evolution was filled with an ultra-relativistic gas of photons, electrons, positrons, etc. At that time the excess of baryons over antibaryons small fraction (at most 10^{-19}) of the total number of particles. The matter could be considered as a gas of free particles where their rest masses are small compared to temperature. In other words the energy density and entropy density corresponds to the massless species

$$\rho = 3p = \frac{\pi^2}{30} g_*(T) T^4 \ , \quad s = \frac{2\pi^2}{45} g_*(T) T^3 \ . \tag{3.19}$$

where the effective number of particle species $g_*(T)$ is $g_*(T) = g_B(T) + \frac{7}{8} g_F(T)$ where g_B and g_F are the number of boson and fermions species degrees of freedom with masses $m \ll T$. For example, for photons $g_B = 2$, $g_F = 2$ for neutrinos and $g_F = 4$ for electrons (Let us sketch the way how to derive the dependence of ρ on T. By definition

$$\rho = \int d^3k e(k) f(\frac{e}{T})$$

where $f(\frac{e}{T})$ is distribution functions and $e(k)$ is an energy. For particles with $m \ll T$ we can neglect their rest masses so that $e = k$. After substitution $\frac{k}{T} = m$ we obtain $\rho = T^4 \int d^3m e(m) f(m) \sim T^4$.)

Generally $g_*(T)$ increases with increasing T but rather slowly. This follows from the fact that at higher temperatures more species are ultra-relativistic (say, electrons contribute at $T > 0.5 MeV$ and do not contribute at lower temperatures.)

Let us now list some time scales that are relevant for the early stage of the evolution of the Universe:

- **Nucleosynthesis**

  The temperature relevant for nucleosynthesis rages from a few MeV to about $70 keV$. This era begins at

$$t \sim 1 \ s \ . \tag{3.20}$$

[7] This presumption is not however valid for neutrinos at temperatures below $1 MeV$.

and ends at

$$t \sim 200\ s \sim 3\ min\ . \tag{3.21}$$

After this brief introduction we will discuss the properties of the early Universe in brief details.

3.5 Describing Matter

We try to describe matter a perfect fluid described by an energy-momentum tensor

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}\ , \tag{3.22}$$

where U_μ is the fluid four-velocity, ρ is the energy density at rest frame of the fluid and p is the pressure in that same frame. By definition the stress energy tensor is covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0\ . \tag{3.23}$$

In more complicated examples a fluid will be characterized by quantities in addition to the energy and pressure. Many fluids have a conserved quantity associated with them and so we will also introduce a *number flux density* N^μ which is also conserved

$$\nabla_\mu N^\mu = 0\ . \tag{3.24}$$

For non-tachyonic matter N^μ is a time-like 4-vector and therefore we can write

$$N^\mu = n U^\mu\ . \tag{3.25}$$

In the same way we can introduce an *entropy flux density* S^μ. This quantity is not conserved but rather obeys a covariant version of the second law of thermodynamics

$$\nabla_\mu S^\mu \geq 0\ . \tag{3.26}$$

It is useful to resolve S^μ into components parallel and perpendicular to the fluid 4-velocity

$$S^\mu = s U^\mu + s^\mu\ , \tag{3.27}$$

where $s_\mu U^\mu = 0$. The scalar s is the rest-frame entropy density that can be written as

$$s = \frac{\rho + p}{T}\ . \tag{3.28}$$

We must also specify an equation of state. Typically we do this in such a way as to treat n and s as independent variables.

For adiabatic expanding Universe $sa^3 \approx const$ eq. (3.19) implies

$$T(t) \sim \frac{1}{a(t)}\ . \tag{3.29}$$

We see that the temperature cools during the expansion of the Universe. The background radiation is a result of the cooling of the hot photon gas during the expansion of the Universe.

3.6 Particles in Equilibrium

The various particles inhabiting the early Universe can be characterized according to three criteria: in equilibrium vs. out of equilibrium (decoupled), bosonic vs fermionic and relativistic (velocities near 1) vs. non-relativistic. In this subsection we will consider species which are in equilibrium with surrounding thermal bath.

Now we must discuss the conditions under which a particle is in equilibrium with the surrounding thermal plasma. The particles will be in thermal equilibrium as long as its interaction rate is larger then the expansion rate of the Universe. In other words, particles have enough time to share the energy among themselves or equivalently, equilibrium requires that it should be possible for the products of a given reaction have the opportunity to recombine in the reverse reaction. If the expanding of the Universe is rapid enough this will not happen. A particle species for which the interaction rates have fallen below the expanding rate of the Universe is said to have *frozen out or decoupled*. The interaction rate of some particle with the background plasma is Γ where Γ is inverse of the mean time between the reaction of given particle species with the thermal background. Now the particle will be decoupled from the thermal bath when the particle has not time enough to react with thermal bath if

$$\Gamma \ll H \ , \tag{3.30}$$

where the Hubble constant H sets the cosmological timescale.

At the early Universe the particles are in thermal equilibrium (unless they are very weakly coupled). This can be seen from Friedmann equation when the energy density is dominated by plasma with $\rho \sim T^4$ and we have

$$H^2 \sim \rho \Rightarrow H \sim \sqrt{\rho} \sim \left(\frac{T}{M_P}\right) T \tag{3.31}$$

so that the Hubble parameter is suppressed with respect to the temperature by a factor of T/M_P. At extremely early times (near the Planck era) the Universe may be expanding so quickly so that no species are in equilibrium but as the expansion rate slows the equilibrium becomes possible.

At extremely early times near the Planck era, the Universe may be expanding so quickly that no species are in equilibrium; as the expansion rate slows, equilibrium becomes possible. On the other hand the interaction rate Γ for a particle with cross section σ is typically of the form

$$\Gamma = n \left\langle \sigma v \right\rangle \ , \tag{3.32}$$

where n is the number density and v is typical particle velocity. Since $n \sim a^{-3}$ the density of particles will reduce so that the equilibrium can once again no longer be maintained. In our current Universe no species are in equilibrium with the background plasma (represented by CMB photons).

Now we review some facts about particles at equilibrium. For a gas of weakly-interacting particles we can describe the state in terms of a *distribution function* $f(\mathbf{p})$ where the three momentum $\mathbf{p}$ satisfies

$$E(\mathbf{p})^2 = m^2 + |\mathbf{p}|^2 \ . \tag{3.33}$$

The distribution function characterizes the density of particles of given momentum. The number density, energy density and pressure of some species labeled i are given by

$$\begin{aligned} n_i^{eq}(T) &= \frac{g_i}{(2\pi)^3}\int f_i(\mathbf{p})d^3p = \frac{g_i}{2\pi^2}T^3 I_i^{11}(\mp) \ , \\ \rho_i^{eq}(T) &= \frac{g_i}{(2\pi)^3}\int E(\mathbf{p})f_i(\mathbf{p})d^3p = \frac{g_i}{2\pi^2}T^4 I_i^{21}(\mp) \ , \\ p_i^{eq}(T) &= \frac{g_i}{(2\pi)^3}\int \frac{|\mathbf{p}|^2}{3E(\mathbf{p})}f_i(\mathbf{p})d^3p = \frac{g_i}{6\pi^2}T^4 I_i^{03}(\mp) \ , \end{aligned} \tag{3.34}$$

where

$$I_i^{mn}(\mp) = \int_{x_i}^{\infty} y^m (y^2 - x_i^2)^{n/2}(e^y \mp 1)^{-1}dy \ , \quad x_i = \frac{m_i}{T} \ , \tag{3.35}$$

and where g_i is number of spin states of the particles (massless photons, $g_\gamma = 2$, massive vector bosons Z , $g_Z = 3$.) Further, $-/+$ refers as before to bosons/fermions. As usual, particles and antiparticles are treated as separate, for spin 1/2 electrons and positrons we have $g_{e^-} = g_{e^+} = 2$. In thermal equilibrium at a temperature T the particles will be in either Fermi-Dirac or Bose-Einstein distributions

$$f(\mathbf{p}) = \frac{1}{e^{E(\mathbf{p})/T} \pm 1} \ , \tag{3.36}$$

where the plus sign is for fermions while the minus sign for bosons.

We can do the integrals over the distribution functions in two opposite limits, particles which are relativistic $T \gg m$ and highly non-relativistic $T \ll m$. For relativistic (R) particles that are characterized by condition $x_i = \frac{m_i}{T} \ll 1$ the integrals in (3.35) are

$$\begin{aligned} \text{bosons}: \quad & I_R^{11}(-) = 2\zeta(3) \ , \quad I_R^{21}(-) = I_R^{03}(-) = \frac{\pi^4}{15} \ , \\ \text{fermions}: \quad & I_R^{11}(+) = \frac{3\zeta(3)}{2} \ , \quad I_R^{21}(+) = I_R^{03}(+) = \frac{7\pi^4}{120} \ , \end{aligned} \tag{3.37}$$

where ζ is Riemann Zeta function and $\zeta(3) = 1.202$. Then we obtain, for relativistic bosons, following results:

$$n_i^{eq} = \frac{\zeta(3)}{\pi^2} g_i T^3 \ ,$$

$$\rho_i^{eq} = \frac{\pi^2}{30} g_i T^4 \ ,$$
$$p_i^{eq} = \frac{1}{3}\rho_i \tag{3.38}$$

and for relativistic fermions

$$n_i^{eq} = \left(\frac{3}{4}\right)\frac{\zeta(3)}{\pi^2} g_i T^3 \ ,$$
$$\rho_i^{eq} = \left(\frac{7}{8}\right)\frac{\pi^2}{30} g_i T^4 \ ,$$
$$p_i^{eq} = \frac{1}{3}\rho_i \ . \tag{3.39}$$

On the other hand non-relativistic (NR) limit, where we have $x \gg 1$ is the same for bosons and fermions and we recover the Boltzmann distribution

$$n_i^{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}$$
$$\rho_i^{eq} = m_i n_i \ ,$$
$$p_i^{eq} = n_i^{eq} T \ll \rho_i^{eq} \ . \tag{3.40}$$

independently of whether the particle is bosons or fermions. The results given above imply several interesting facts. For example, since the densities of relativistic particles are roughly the same, the relativistic particles remain approximately equal abundances in equilibrium. We also see that once the particles become non-relativistic, they become exponentially suppressed with respect to the relativistic species. This is a result of the fact that it becomes harder for massive particle-antiparticle pairs to be produced in a plasma with $T \ll m$.

We would like also mention that although matter is much more dominant than radiation in the Universe today, since their energy densities scale differently, the early Universe was radiation dominated. We can write the ratio of the density parameters in matter and radiation as

$$\frac{\Omega_M}{\Omega_R} = \frac{\Omega_{M0}}{\Omega_{R0}}\frac{a}{a_0} = \frac{\Omega_{M0}}{\Omega_{R0}}(1+z)^{-1} \ . \tag{3.41}$$

In the same way as we did above we can determine the redshift of the matter-radiation equality as

$$1 + z_{eq} = \frac{\Omega_{M0}}{\Omega_{R0}} \approx 3 \times 10^3 \ . \tag{3.42}$$

From the form of the expression above where we compare the densities that scale as a^{-3} for matter and a^{-4} for radiation it is clear that we have made an assumption that particles that are non-relativistic today were also non-relativistic at z_{eq}. It can be shown that this presumption is safe.

At any given time not all particles will, be in fact in equilibrium at a common temperature T. A particle will be in kinetic equilibrium with the background thermal plasma, i.e when $T_i = T$ only while it is interacting. In other words as long as the scattering rate

$$\Gamma = n < \sigma v > > H \ . \tag{3.43}$$

Here $< \sigma v >$ is the velocity averaged cross-section for $2 \to 2$ processes such as

$$i\gamma \to i\gamma \ , il^{\pm} \to il^{\pm} \tag{3.44}$$

that maintain good thermal contact between i-particles and the particles (that has the particle density n) that constitute the background plasma (γ-fotons, $l^{\pm}$-refers to electrons which are abundant down to $T \sim m_e$ and remain strongly coupled to photons through the Compton scattering through the entire Radiation dominate era so that $T_e = T$ always.) We say that i-particle decouple at the temperature T_i when the condition

$$\Gamma(T_i) \approx H(T_i) \tag{3.45}$$

is satisfied. Of course no particle is ever truly decoupled since there are always some residual interactions. On the other hand it can be shown that their effects are generally negligible.

If the particle is relativistic at this time ($m_i < T_i$) then it will also be in the chemical equilibrium with the thermal plasma that is characteristic with the condition for chemical potentials of the particles i μ_i , their anti-particles $\mu_{\bar{i}}$ and the chemical potential of photons μ_γ

$$\mu_i + \mu_{\bar{i}} = \mu_{l^+} + \mu_{l^-} = \mu_\gamma = 0 \tag{3.46}$$

through processes such as

$$i\bar{i} \leftrightarrow \gamma\gamma \ , i\bar{i} \leftrightarrow l^+ l^- \tag{3.47}$$

Then its abundance at decoupling will be just the equilibrium value at the temperature of decoupling

$$n_i^{eq}(T_i) = \left(\frac{g_i}{2}\right) n_\gamma(T_i) f_{B.F} \ , \tag{3.48}$$

where $f_B = 1$ if i is boson and $f_F = \frac{3}{4}$ if i is fermion.

Then the decoupled particles i will expand freely without interactions so that their number in a comoving volume is conserved as $n_i a^3 = const$ and their pressure and energy density are functions of the scale factor a alone. Even if these particles do not interact their phase space distribution will retain their equilibrium form (3.36) with T_i. As long as the particles remain relativistic, E_i and T_i scale as a^{-1}. Initially the temperature T_i will track the photon temperature T. However as the Universe cools below to some mass thresholds (in other words temperature is less than some mass of particles), these massive particles will become non-relativistic and annihilate. The annihilation will produce additional photons and other interacting particles that has an effect of the heating of them. On the other hand T_i is not affected and hence T_i will drop below T and consequently the faction n_i/n_γ will decrease below its value at decoupling.

It can be shown that decoupled photons maintain a thermal distribution even if they are not in thermal equilibrium. This follows from the fact that the thermal distribution function redshifts into similar distribution function with lower temperature proportional $1/a$. Then we can speak about an effective temperature of relativistic species that freezes out at a temperature T_f and a scale factor a_f so that

$$a_f T_f = aT(a) \Rightarrow T^{rel}(a) = T_f \left(\frac{a_f}{a}\right) \ . \tag{3.49}$$

For example, neutrinos decouple at $T \approx 1MeV$, shortly thereafter electrons and positrons annihilate into photons and hence transfer energy and entropy into plasma leaving neutrinos decoupled. Consequently we expect a neutrino background and current Universe with a temperature of approximately $2K$ while the photon temperature (that arise from the annihilation of electrons and positrons after decoupling of neutrinos) is about $3K$.

Similar effect occurs for particles which are non-relativistic at decoupling however there is one important difference. For non-relativistic particles the temperature is proportional to $\frac{1}{2}mv^2$ that has the redshift as $1/a^2$ and we therefore have

$$T_i^{non-rel}(a) = T_f \left(\frac{a_f}{a}\right)^2 \ . \tag{3.50}$$

The whole picture is as follows: We imagine that the species freeze out while relativistic or non-relativistic and stay this way afterwards.

Now the notion of the effective temperature allows us to define a corresponding notion of an effective number of relativistic degrees of freedom that can be defined as

$$g_* = \sum_{bosons} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{fermions} g_i \left(\frac{T_i}{T}\right)^4 \ , \tag{3.51}$$

where the temperature T is actual temperature of the background plasma assumed to be in equilibrium while we have taken into account that different species i could have a thermal distribution with a different temperature that of the photons. Then the total energy density in all relativistic species comes from adding the contribution of each species and we obtain a simple formula

$$\rho = \frac{\pi^2}{30} g_* T^4 \ . \tag{3.52}$$

We can do the same thing for the entropy density. Since the entropy density of relativistic particles goes as T^3 rather T^4, we define the effective number of relativistic degrees of freedom for entropy as

$$g_{*S} = \sum_{bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{fermions} g_i \left(\frac{T_i}{T}\right)^3 \tag{3.53}$$

so that the entropy density of relativistic species is then

$$s = \frac{2\pi}{45} g_{*S} T^3 \ . \tag{3.54}$$

For example, in Standard model, we have

$$g_* \approx g_{*S} \begin{cases} 100 \text{ for } & T > 300\ MeV \\ 10 \text{ for } 300\ MeV > T > 1\ eV \\ 3 \text{ for } & T < 1\ MeV \end{cases} \tag{3.55}$$

The events that change the effective number of relativistic degrees of freedom are the QCD phase transition at 300 MeV where quarks and gluons start to form bound states, and the annihilation of electron-positron pairs at $T \approx 1\ MeV$.

Thanks to the release of the energy into the background plasma when species annihilate it is only approximation that the temperature goes as $1/a$. It is better to say that comoving entropy density is conserved so that

$$s \approx a^{-3} \tag{3.56}$$

which holds in all forms of adiabatic evolutions, entropy is only produced at a process like a first-order phase transition or out-equilibrium decay. It is expected that the entropy production from such processes is very small compared to the total entropy and the adiabatic presumption is excellent approximation for almost the entire early Universe. If we now combine (3.56) with (3.54) we obtain a better expression for the evolution of the temperature

$$T \approx g_{*S}^{-1/3} a^{-1} \ . \tag{3.57}$$

We see the difference with the naive time dependence $T \sim 1/a$. In fact, the temperature will consistently decrease under adiabatic evolution in an expanding Universe but it decreases more slowly when the effective number of relativistic degrees of freedom is diminished.

3.7 Thermal relics

As we know particles typically do not stay in equilibrium forever, they density can be so low that the interactions become infrequent and the particle freeze out. Since essentially all of the particles in our current universe belong to this category it is important to study the relic abundance of decoupled species.

We have seen that relativistic or hot particles have a number density that is proportional to T^3 in equilibrium. Thus a species X that freezes out while still relativistic will have number density at freeze-out T_f given by

$$n_X(T_f) \sim T_f^3 \ . \tag{3.58}$$

Since this is comparable to the number density of photons at that time and since after this freeze-out both photons and species X have densities that dilute by a factor $a(t)^{-3}$ as the Universe expands, we see that the abundance of X particles today should be comparable to the abundance of CMB photons

$$n_{X_0} \sim n_{\gamma_0} \sim 10^2 cm^{-3} \ . \tag{3.59}$$

We express this estimate as 10^2 rather as the precise number since the roughness of this estimate does not warrant such misleading precision. For example, neutrinos that are light ($m_\nu < MeV$) have a number density of $n_\nu = 115 cm^{-3}$ for each species. Then a corresponding contribution to the density parameter (if they are heavy enough to be non-relativistic today)

$$\Omega_{0,\nu} = \left(\frac{m_\nu}{92 \ eV}\right) h^{-2} \ . \tag{3.60}$$

Thus, a neutrino with $m_\nu \sim 10^{-2} \ eV$ would contribute $\Omega_\nu \sim 2 \times 10^{-4}$. We see that this is not large enough to make neutrinos to be dark matter.

Let us now consider species X that is non-relativistic or cold at the time of decoupling. In this case it is much harder to calculate the relic abundance of a cold relic than a hot one simply because the equilibrium abundance of non-relativistic species is changing rapidly with respect to the background plasma. Then we have to be quite precise following the freeze-out process to obtain a reliable answer. The direct calculation typically involves very complicated procedure. We rather give here reasonable approximate expression. If σ_0 is annihilation cross-section of the species X at temperatures $T = m_X$, then the final number density in terms o the photon density can be determined to be equal to

$$n_X(T < T_f) \sim \frac{1}{\sigma_0 m_X M_P} n_\gamma \ . \tag{3.61}$$

Since the particles are non-relativistic at the time of decoupling, they are certainly non-relativistic today and their energy density is

$$\rho_X = m_X n_X \ . \tag{3.62}$$

Then finally we obtain the density parameter

$$\Omega_X = \frac{\rho_X}{\rho_{cr}} \sim \frac{n_\gamma}{\sigma_0 M_P^3 H_0^2} \ . \tag{3.63}$$

Numerically, when $\hbar = c = 1$ we have $1\ cm \sim 2 \times 10^{-14}\ GeV$ so the photon density today is

$$n_\gamma \sim 100\ cm^{-3} \sim 10^{-39}\ GeV^{-3}\ . \tag{3.64}$$

The present value of the Hubble constant is

$$H_0 \sim 10^{-42}\ GeV \tag{3.65}$$

and the Planck mass is

$$M_P \sim 10^{18}\ GeV\ . \tag{3.66}$$

Then finally (3.63) gives

$$\Omega_X \sim \frac{1}{\sigma_0 (10^9\ GeV^2)}\ . \tag{3.67}$$

We see an interesting fact that Ω_X does not depend on m_X but it depends on the annihilation cross-section. Let us elaborate more about this result and consider some weakly interacting massive particle. The annihilation cross-section of these particles, since they are weakly interacting, should be $\sigma_0 \sim \alpha_W^2 G_F$, where α_W is weak coupling constant and G_F is the Fermi constant. Using

$$G_F \sim (3000\ GeV)^2\ , \quad \alpha_W \sim 10^{-2} \tag{3.68}$$

and we obtain

$$\sigma_0 \sim 10^{-9}\ GeV^{-2}\ . \tag{3.69}$$

Then the density parameter of such particles would be

$$\Omega_X \sim 1\ . \tag{3.70}$$

In other words, a stable particle with weak interaction cross section produces relic density of order of the critical density today and hence provides a perfect candidate for cold dark matter.

After this introduction let us present the simplest possible scenario, that, of course, can be refined by more careful calculations.

Let us again assume that there exists a heavy stable particle X and its anti-particle $\overline{X}$. Let us also presume that the dominant process in which these particles can be destroyed or created is their pair-annihilation or creation with annihilation products being the particles of the Standard Model. Let us also presume that there is no asymmetry between X and $\overline{X}$ in the early Universe, in other words the densities X and $\overline{X}$ are equal to each other. However we have to mention that this is actually a strong assumption that is valid in many, but not all, realistic extensions of the Standard Model [8].

[8] In fact, the alternative scenario with the generation of X asymmetry is also interesting since it might be related to baryon asymmetric the density of dark matter.

Let us outline the overall cosmological behavior of these particles. At hight temperatures, $T \gg M_X$, the X- particles are in thermal equilibrium with the rest of cosmic plasma. There are many $X - \overline{X}$ pairs in the plasma that are continuously created and annihilate. As the temperature drops below M_X, the equilibrium number density decreases. At some "freeze-out" temperature T_f the number density becomes so small so that X and $\overline{X}$ can no longer meet each other during the Hubble time and their annihilation terminates. After that the number densities of survived X and $\overline{X}$ decreases as $a^{-3}(t)$ and these relic particles contribute to the mass density of the present Universe. The purpose of the following analysis is to estimate the range of properties of X particles in which their present mass density is of the order of the critical density ρ_c so that X may serve as dark matter candidates.

Let us again assume thermal equilibrium. It is well known that the mean free path $< l >$ of a particle in a gas depends on the lifetime τ_{ann} of a non-relativistic X-particle as

$$\sigma_{ann} \cdot v \cdot \tau_{ann} \cdot n_{\overline{X}} = < l > \ , \tag{3.71}$$

where v is mean velocity of X particle, σ_{ann} is the annihilation cross section at velocity ν and $n_{\overline{X}} = n_X$ is equilibrium number density

$$n_X = g_X \left(\frac{m_X T}{2\pi} \right)^{3/2} e^{-\frac{m_X}{T}} \ . \tag{3.72}$$

In order to find the life-time of the non-relativistic particle X we have to take some reasonable value of $< l >$. It is natural to presume that it is of order 1 in the natural units $< l > \sim 1$. Further, it can be also shown that for non-relativistic velocities the annihilation cross section takes the form

$$\sigma_{ann} = \frac{\sigma_0}{\nu} \ , \tag{3.73}$$

where σ_0 is constant. We will discuss its value later. We should now compare the life-time with the Hubble time, or annihilation rate $\Gamma_{ann} = \tau_{ann}^{-1}$ with the expansion rate H. At $T \sim m_X$ the equilibrium density is of order $n_X \sim T^3$ and $\Gamma_{ann} \ll H$ for not too small σ_0. Conversely, the life-time is much smaller than Hubble time and consequently the annihilation and creation of $X - \overline{X}$ pairs is rapid and hence X-particles are in equilibrium with plasma. On the other hand for very small temperatures $T \ll m_X$ the number density n_X is exponentially small and $\Gamma_{ann} \ll H$ ($\tau_{ann} \gg H^{-1}$). Than it is clear that the thermal equilibrium between X-particles and background plasma is not maintained. In other words the number density n_X gets diluted only because of cosmological expansion.

The freeze-out temperature T_f is determined by the relation

$$\tau_{ann}^{-1} \equiv \Gamma_{ann} \sim H \ , \tag{3.74}$$

where we can still use the equilibrium formula as X particles are in thermal equilibrium (with respect to annihilation and creation) just before freeze-out. Then we find

$$\sigma_0 n_X(T_f) \sim H \sim \frac{T_f^2}{M_P^*} \ , \tag{3.75}$$

where we have introduced the effective Planck mass

$$M_P^* = \frac{M_P}{1.66\sqrt{g_*(t)}} \ , \tag{3.76}$$

and hence the expansion rate is equal to

$$H(t) = \frac{T^2(t)}{M_P^*} \ . \tag{3.77}$$

The solution of the equation (3.75) gives the freeze-out temperature, up to log terms

$$T_f \approx \frac{m_X}{\ln(m_P^* m_X \sigma_0)} \ . \tag{3.78}$$

This temperature is quite bit smaller than m_X which means that X-particles freeze out when they are indeed non-relativistic and hence it is natural to call them as *cold dark matter*.

At the freeze-out temperature we use (3.75) to get

$$n_X(T_f) = \frac{T_f^2}{M_P^* \sigma_0} \ . \tag{3.79}$$

It is interesting to note that this density is inversely proportional to the annihilation cross section. The explanation of this fact is that for higher annihilation cross section the creation-annihilation processes are longer in equilibrium and less X particles survive.

In order to estimate eh present density X-particles, it is convenient to consider ratio n_X/s where s is the entropy density

$$s = \frac{2\pi^2}{45} g_* T^3 \ . \tag{3.80}$$

The point is that during the adiabatic expansion after freeze-out, the entropy density scales as $s \sim a^{-3}$ since in the adiabatic process $sa^3 = const$. In the same way since we are in the freeze-out regime we have that $n_X a^3 = const$ we obtain that n_X scales in the same way $n_X \sim a^{-3}$. Then, up to a factor of order 1, this ratio at freeze-out is

$$\frac{n_X}{s} \sim \frac{1}{g_*(T_f) M_P^* T_f \sigma_0} \ . \tag{3.81}$$

At late times, the entropy density, again up to actor of order 1, is equal to the number density of photons, so the present number density of particles is of order

$$\frac{n_{X,0}}{s_0} = \left(\frac{n_X}{s}\right)_{freeze-out} \Rightarrow$$
$$\Rightarrow n_{X,0} = s_0 \left(\frac{n_X}{s}\right)_{freeze-out} \sim s_{\gamma,0} \left(\frac{n_X}{s}\right)_{freeze-out} \tag{3.82}$$

and the present mass density is

$$\rho_{X,0} = m_X n_{X,0} =\sim n_{\gamma,0} \frac{\ln(M_P^* m_X \sigma_0)}{g_*(T_f) M_P^* \sigma_0} \ , \tag{3.83}$$

where we have also used (3.78). The formula above is very interesting since we see that the present mass density depends mostly on one parameter, the annihilation cross section σ_0. The dependence on the mass of X-particle is through the logarithm and $g_*(T_f)$ is very mild. From this formula we derive the condition that ensure that X-particles are dark energy candidates, i.e. their present mass density is of order ρ_c

$$\sigma_0 \sim \frac{n_{\gamma,0}}{g_*(T_f) M_P^* \rho_c} \ln(M_P^* m_X \sigma_0) \tag{3.84}$$

that leads to the estimate

$$10^{-11} \sigma_0 < 10^{-9} \ GeV^{-2} \ , \tag{3.85}$$

where the uncertainty in the estimate is a consequence of the way we deal with various numerical factors. In any case the estimate given above tells us what the relevant range of mass scales is. To see this note that the annihilation cross section may be parameterized as

$$\sigma_0 = \frac{\alpha^2}{\mathcal{M}^2} \ , \tag{3.86}$$

where α is some coupling constant and $\mathcal{M}$ is the mass scale (In the calculation above $\mathcal{M}^2 = G_F$.). With $\alpha \sim 10^{-2}$ the estimate of the mass scale for $\sigma_0 \sim 10^{-11}$ is roughly

$$\mathcal{M} \sim 1 \ TeV \ . \tag{3.87}$$

In other words, we very mild assumptions we find that the non-baryonic dark energy matter may naturally originate from the TeV-scale physics. Then it follows that one natural candidate for the cold dark matter is neutralino. More precisely, in supersymmetric extensions of the Standard Model the neutralino-that is mixture of super-partners of photon, Z-boson and neutral Higgs bosons- is the lightest supersymmetric particle that is often stable with the suitable value of the annihilation cross section. In fact, the search for both direct and indirect signals from neutralino dark matter is an active area of experimental research.

The mechanism discussed here is of course not the only one mechanism that is able to model cold dark matter. Other dark matter candidates include very heavy relics produced toward the end of inflation, axions, gravitinos, massive gravitons and so on.

3.8 Baryongenesis

The symmetry between particles and antiparticles is firmly established in collider physics. However then we lead to the following question; why the observed Universe is composed almost entirely of matter with little or no primordial antimatter.

Outside the particle accelerators the antimatter can be seen in cosmic rays in the form of a anti protons where the ratio of these andirons to protons is

$$\frac{n_{\overline{p}}}{n_p} \sim 10^{-4} . \tag{3.88}$$

However this ratio is consistent with secondary anti proton productions through accelerator-like processes

$$p + p \rightarrow 3p + \overline{p} \tag{3.89}$$

as the cosmic rays stream toward us. In other words there is no evidence for primordial antimatter in our galaxy. Also let us imagine that we have clusters of matter and antimatter galaxies. Then we could expect that we could detect background of γ-radiation from nucleon anti nucleon annihilations with clusters. This background is not observed and so we conclude that there is negligible antimatter on the scale of clusters.

All these considerations put an experimental upper bound on the amount of antimatter in the Universe.

In order to study this problem in more details let us introduce the *baryon to entropy ratio*

$$\eta \equiv \frac{n_B}{s} = \frac{n_b - n_{\overline{b}}}{s} , \tag{3.90}$$

where n_B is the difference between the number of baryons and anti-baryons per unit volume. The range of η was determined recently as is equal to

$$\eta = 6.1 \times 10^{-10} \pm 0.210^{-10} . \tag{3.91}$$

At early times, at temperatures well above 100 MeV,cosmic plasma contained many quark-anti quark pairs whose number density was of the order of the entropy density

$$n_q + n_{\overline{q}} \sim s , \tag{3.92}$$

while baryon number density was related to densities of quarks and antiquarks as follows (baryon number of quarks equals 1/3)

$$\triangle n_b = \frac{1}{3}(n_q - n_{\overline{q}}) . \tag{3.93}$$

Hence in terms of quantities characterize the very early epoch, the baryon asymmetry may be expressed as

$$\eta \sim \frac{n_q - n_{\overline{q}}}{n_q + n_{\overline{q}}} \ . \tag{3.94}$$

We see that there was one extra one extra quark per about 10 billion quark-antiquark pairs. It is this thiny excess that is responsible for entire baryonic matter in the present Universe. Thus the natural question arises, as the Universe coolled from early times to today, what processes, both particle and cosmological, were responsible for the generation of this very specific baryon assymmetry?

Of course there is no logical contradiction to suppose that this thiny excess of quarks to antiquarks was built in as an initial condition. Of course, this is not very satisfactory for physics. Furthermore, inflationary scenario does not provide such an initial condition for Hot Big Bang, rather, inflation theory predicts that the Universe was baryon-symmetric just after inflation. In other words we would like to explain the baryon asymmetry dynamically.

As pointed by Sakharov, a small baryon asymmetry may have been produced in the early Universe from initially symmetric state if three necessary conditions are satisfied:

- Baryon number (B) violation,
- Violation of C (charge conjugation symmetry) and CP (the composition of parity and C)
- Departure from thermal equilibrium.

The first condition is clear since when we start from a baryon symmetric Universe, baryon number violation must take case in order the Universe to evolve into the state with baryon number violation. In other words, if the baryon number were conserved that this charge would remain constant during time evolution and hence w we would not observe the baryon number asymmetry.

The second Sakharov criterion is required since, when C and CP are exact symmetries it can be shown that the total rate for any processes that produces an excess of baryons is equal to the rate of the complementary process which produces an excess of antibaryons and so no net baryon number can be created. CP violation is present either if there are complex phases in the Lagrangian which cannot be reabsorbed by field redefinition (explicit symmetry breaking) or if some High scalar field acquires an VEV which is not real (spontaneous symmetry breaking).

Finally, in order to explain the third equilibrium let us calculate the thermal equilibrium average of the baryon number operator B at temperature $T = 1/\beta$

$$\langle B \rangle_T = \mathrm{Tr}(e^{-\beta H} B) = \mathrm{Tr}\left((CPT)(CPT)^{-1} e^{-\beta H} B\right) =$$

$$\mathrm{Tr}\left(e^{-\beta H}(CPT)^{-1}B(CPT)\right) = -\mathrm{Tr}(e^{-\beta H}B) \ , \tag{3.95}$$

using the fact that (CPT) commutes with H and cyclicity of the trace. Finally, we have used the fact that B is odd under (PC). Then from the equation above we see that in the thermal equilibrium the baryon number is equal to zero and there is not any generation of baryon number.

The first two Sakharov's conditions may be investigated only within a given particle model, while the third condition the departure from thermal equilibrium may be discussed in a more general way.

3.9 Baryon Number Violation

At present there are two well understood mechanisms of baryon number non-conservation. One emerges in Grand-Unified Theories (GUT). Briefly, these GUT describe the fundamental interactions by means of the unique gauge group G that contains the Standard Model group

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \ .$$

The fundamental idea of GUT is that at energies higher than a certain energy M_{GUT} the group symmetry is G and that, at lower energies, the symmetry is broken down to the SM gauge symmetry, possibly through the chain of symmetry breaking. The motivation for this scenario, whose explanation, however, is beyond the scope of this review, it the fact that in some models, the (running) gauge couplings of the SM unify at the scale $M_{GUT} \simeq 2 \times 10^{16}\ GeV$.

The interesting fact considering GUT is that the baryon number violation emerges very naturally in it. Briefly, the mechanisms of the baryon number violation is due to the exchange of super-massive particles. The scale of these new, baryon number violating interacting is of order $10^{16}\ GeV$.

Another mechanism of the baryon number violation is related to the triangle anomaly in the baryonic current. It exists already in the Standard Model and possibly it operates in all its extensions. The main feature of this mechanism, as applied to the early Universe, is that it is effective over a wide range of temperatures

$$100\ GeV < T < 10^{11}\ GeV \ .$$

In summary, realistic mechanism of baryon number non-conservation are rare, but there are several ways the baryon asymmetry could have been generated. They differ by the characteristic temperature at which the asymmetry is produced.

The GUT mechanisms operates at extremely high temperatures

$$T \sim 10^{15} - 10^{16}\ GeV$$

The most well developed source of the baryon asymmetry in this context are B- and CP- violating decays of ultra-heavy particles. At late times the baryon number is violated by anomalous electroweak processes.

Electroweak baryogenesis is scenario in which the baryon asymmetry is generated entirely due to the anomalous electroweak processes. Its generation would occur at temperature of order 100 GeV which is the energy at which these anomalous processes are switched off. On the other hand the electroweak baryogenesis is still under development.

In summary, the observed asymmetry may be explained by a number of mechanisms all of which, however, exist in *extensions of the Standard Model only.* The problem is that direct proof that any given mechanism is indeed responsible for the baryon asymmetry.

3.10 Departure from the Thermal Equilibrium

In some scenarios, such as GUT baryogenesis, the third Sakharov condition is satisfied due to the presence of superheavy decaying particles in a rapidly expanding Universe. These processes are called as out-of-equilibrium decay mechanisms.

The underlying idea is simple.If the decay rate Γ_X of the superheavy particles X at the time they become non-relativistic (at the temperature $T \sim M_X$) is much smaller than the expansion rate of the Universe, then the X particles cannot decay on the time scale of the expansion and so they remain as abundant as photons for $T \leq M_X$. In other words at some temperature $T > M_X$ the superheavy particles X are so weakly interacting so the they decouple from the thermal bath while they are still relativistic, so that

$$n_X \sim n_\gamma \sim T^3 \tag{3.96}$$

at the time of decoupling.

Then we obtain that at temperature $T \simeq M_X$ they populate the Universe with an abundance which is much larger than the equilibrium one. This abundance is precisely the departure from thermal equilibrium needed to produce a final non-vanishing baryon asymmetry when heavy states X decay in B and CP violating decays.

It can be shown that the out-of-equilibrium condition requires very heavy states

$$M_X \leq (10^{10} - 10^{16})\ GeV\ , \tag{3.97}$$

if these heavy particles decay through renormalizable operators.

A different mechanism of the departure from the thermal equilibrium can be found in the electroweak theory.

A further natural way to depart from equilibrium is provided by the dynamics of the topological defects.

3.11 Neutrino background

As an example of the previous discussion let us consider the fate of neutrinos in the expanding Universe. The dynamics of the neutrinos and their reactions with other components of the matter are governed by the Standard model. Then using the rules of standard quantum field theory one can calculate the reaction rate Γ of the neutrinos with the rest of the matter (Roughly speaking the inverse Γ^{-1} is the average time between collision of the neutrinos with all form of the matter). When Γ^{-1} is larger than H^{-1} (conversely, when Γ is less than H) there cannot occur the reactions between the neutrinos and the rest of the matter. We say that in this case neutrinos effectively decouple from the rest of matter. It can be shown that the relevant ration is given by

$$\frac{\Gamma}{H} \approx \left(\frac{T}{1.4MeV}\right)^3 = \left(\frac{T}{1.6 \times 10^{10} K}\right)^3 . \tag{3.98}$$

This formula implies that for $T \leq 1.6 \times 10^{10}$ the neutrinos decouple from the rest of the matter. On the other hand electrons and positrons can still annihilate at slightly lower temperature. This process increases the number of the photons. As a result the photon temperature goes up with respect to neutrino temperature (Remember that it is natural to speak about two different temperatures for two different species of particles since they have already decoupled.). We can calculate this increase of temperature as follows. The increase of T is due to the change of degree of freedom g and is given by

$$\frac{(aT_\gamma)^3_{after}}{(aT_\gamma)^3_{before}} = \frac{g_{before}}{g_{after}} = \frac{\frac{7}{8}(2+2)+2}{2} = \frac{11}{4} . \tag{3.99}$$

Let us explain factors given above. In the numerator, one 2 is for electron, one 2 is for positron and the factor 7/8 arises because of fermions. The remaining 2 in numerator is for photon. In denominator 2 is for photon since they remain after the annihilation of positrons with electrons. Using the relation above we obtain

$$\begin{aligned}(aT_\gamma)_{after} &= \left(\frac{11}{4}\right)^{1/3}(aT_\gamma)_{before} = \left(\frac{11}{4}\right)^{1/3}(aT_\nu)_{before} = \\ &= \left(\frac{11}{4}\right)^{1/3}(aT_\nu)_{after} = 1.4(aT_\nu)_{after} .\end{aligned} \tag{3.100}$$

The first equality is from (3.100), the second follows from the fact that the photons and neutrinos had the same temperature originally. The third equality follows from the fact that for decoupled neutrinos aT_ν are constant. The final result leads to the prediction that at present the Universe will contain a bath of neutrinos that has temperature that is lower than of CMBR.

3.12 Primordial Nucleosynthesis

Theory of Big Bang Nucleosynthesis and observations of primordial abundances of light elements probe the earliest epoch of the evolution of the Universe that is accessible to observation today. This epoch corresponds to temperatures ranging from 1 MeV to a few 10 keV and age of the Universe from 1 s to 200 s.

Let us briefly review the properties of the matter at this early epoch of the Universe.

At temperatures above 1 MeV there is a thermal equilibrium with respect to reactions

$$p + e \leftrightarrow n + \nu_e \ . \tag{3.101}$$

As the Universe cools down below $T \approx 1\ MeV$ neutrons are no longer produced or destroyed, they concentration (relative to protons) "freezes out". Alternatively saying, the weak interactions are frozen out and neutrons and protons cannot interconvert. The equilibrium abundance of neutrinos at this temperature is about 1/6 the abundance of neutrons due to the slightly larger neutron mass.

When we reach a temperature somewhat below 100 keV the Bing-Bang Nucleosynthesis (BBN) begins [9]. At that point the neutron/proton ration is about 1/7. Since it is energetically favorable for nucleons to form He the most part of the free neutrinos are converted into He. For every two neutrons and fourteen protons we end up with one helium nucleus and twelve protons. In other words 25 % of the baryons are converted to helium. There are also trace amounts of deuterium and lithium. Heavier elements are not synthesized in the Big Bang but require supernova explosions in the later universe. These elements remain in the Universe so their primordial abundance is measurable today.

It is important to stress that Big Bang Nucleosynthesis serves also as a source of constraints on particle physics. The fact that the temperature of the Universe reached at least 1 MeV or so and that the expansion was described by know physics at this stage constrain significantly some extensions of the Standard models.

The most amazing fact about nucleosynthesis is that, given the Universe is radiation dominated during the relevant epoch, the relative abundances of the light elements depend essentially on one parameter, the *baryon to entropy ratio*

$$\eta \equiv \frac{n_B}{s} = \frac{n_b - n_{\bar{b}}}{s} \ , \tag{3.102}$$

where n_B is the difference between the number of baryons and anti-baryons per unit volume. The range of η was determined recently as is equal to

$$\eta = 6.1 \times 10^{-10} \pm 0.2 10^{-10} \ . \tag{3.103}$$

[9]Note that the nuclear binding energy per nucleon is typically of order 1 MeV so that one could expect that BBN would occur earlier. However the large number of photons per nucleons at that time prevent BBG to occur until the temperature drops below 100 keV.

Let us be now more specific. We know that at present the Universe is expanding and filled with radiation that is very cold today ($T_0 = 2.73K$). If we trace the evolution of the Universe back in time to earlier epochs that were hotter and denser, the early Universe is a Primordial Nuclear Reactor during its first 20 minutes (≈ 1000). In fact,when the temperature of the Universe is higher than the binding energy of nuclei ($\sim MeV$) none of the heavy elements (helium and metals) could have existed in the Universe. The binding energy of the first four light nuclei, H^2, H^3, He^3 and He^4 are $2.22MeV, 6.92MeV, 7.72MeV$ and $28.3MeV$ respectively. Since the average energy in the thermal ansamble is proportional to the temperature we obtain that these nuclei could be formed when the temperature of the Universe is in the range $(1-30)MeV$. Surprisingly, the actual synthesis takes place at much lower temperature $T_{nuc} = T_n \approx 0.1MeV$. The reason for this delay is the high entropy of the Universe that implies that the ration of photons to baryons, η^{-1} is high. Numerically

$$\eta = \frac{n_B}{n_\gamma} = 5.5 \times 10^{-10} \left(\frac{\Omega_B h^2}{0.02} \right) \;, \quad \Omega h^2 = 3.65 \times 10^{-3} \left(\frac{T_0}{2.73K} \right)^3 \eta_{10} \;. \tag{3.104}$$

Thus, even if the thermal equilibrium is maintained the significant synthesis of nuclei can occur only at $T \leq 0.3MeV$. Then we can expect significant production $X_A \sim 1$ of nuclear species A at temperature $T \leq T_A$. However it turns out that the rate of the nuclear reaction is not high enough to maintain thermal equilibrium between various species. In order to study non equilibrium abundances in an expanding Universe is based on rate equations. Let us now review its general concepts.

3.12.1 Rate equations

Consider a reaction in which two particles 1 and 2 interact to form two other particles 3 and 4. For example, let us consider reaction $n + \nu_e = p + e$ that converts neutrons into protons in the forward direction and proton into neutrinos in the reverse direction. Another example is the reaction $p + e = H + \gamma$ where the forward reaction describes recombination of electron and proton forming a neutral hydrogen atom with the emission of photon. In general we are interested in how the number density n_1 of particle species 1 changes due to the reaction of the form $1 + 2 \Leftrightarrow 3 + 4$. Remember that even in case where there is no reaction the number density changes as $n_1 \propto a^{-3}$ due to the expansion of the Universe. In other words the quantity that changes due to the reaction is $n_1 a^3$. Further, the forward reaction will be clearly proportional to the product of the number densities $n_1 n_2$ while the reverse reaction will be proportional to $n_3 n_4$. Hence we can write the equation for the rate of the change of particle species n_1 in the form

$$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = \mu (A n_3 n_4 - n_1 n_2) \tag{3.105}$$

The left hand side is the relevant rate of change over and above that due to the expansion of the Universe. On the right hand side the two proportionality constants have been written as μ and $A\mu$ that generally are functions of time. Usually $\mu \simeq \sigma v$ where σ is the cross section for the relevant process and v is relative velocity. The left hand side has to vanish for system in thermal equilibrium with $n_i = n_i^{eq}$ where the superscript eq denotes the equilibrium densities of the different species labeled with $i = 1 \ldots 4$. If we insert in the above equation the condition $n_i = n_i^{eq}$ we can express A as

$$A n_3^{eq} n_4^{eq} - n_1^{eq} n_2^{eq} = 0 \Rightarrow A = \frac{n_1^{eq} n_2^{eq}}{n_3^{eq} n_4^{eq}} \tag{3.106}$$

and than the rate equation becomes

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = \mu n_1^{eq} n_2^{eq} \left(\frac{n_3 n_4}{n_3^{eq} n_4^{eq}} - \frac{n_1 n_2}{n_1^{eq} n_2^{eq}}\right) . \tag{3.107}$$

On the left hand side we can write $\frac{d}{dt} = aH\frac{d}{da}$ that shows that the relevant scale for this processes is H^{-1}. Clearly when $\frac{H}{\mu n_i} \ll 1$ the right hand side becomes ineffective because the factor $\frac{\mu}{H}$ factor. Then we see that the number of particles of species 1 does not change. In other words when the expansion rate of the Universe is large compared to the reaction rate ($\frac{\mu}{H} \ll 1$) the given reaction is ineffective in changing the number of particles. However this result does not mean that the reactions have reached thermal equilibrium and $n_i = n_i^{eq}$. In fact, the opposite situation occurs: The reactions are not fast enough to drive the number densities towards equilibrium densities and the number densities "freeze out" at non-equilibrium values. Of course the right hand side in (3.107) will also vanish when $n_i = n_i^{eq}$ that is the extreme limit of thermal equilibrium.

Using this general formalism we will now apply it to the process of nucleosynthesis which requires protons and neutrons that combine together to form bound nuclei of heavier elements like deuterium, helium... The abundance of these elements are going to be determined by the relative abundance of neutrons and protons in the Universe. For that reason we should start the discussion with the problem of the thermal equilibrium between protons and the neutrons in the early Universe. As long as the inter-conversion between n and p through the weak interaction processes

$$\nu + n \leftrightarrow p + e \ , \bar{e} + n \leftrightarrow p + \bar{\nu} \tag{3.108}$$

or their decay

$$n \leftrightarrow p + e + \bar{\nu} \tag{3.109}$$

is rapid with respect to the expansion rate of the Universe thermal equilibrium can be maintained. Then the equilibrium static physics implies that the equilibrium n/p ration is equal to

$$\left(\frac{n_n}{n_p}\right) = \frac{X_n}{X_p} = \exp(-Q/T) \ , \tag{3.110}$$

where $Q = m_n - m_p = 1.293 MeV$. For $T \gg Q$ the factor in the exponent is approaching zero and we obtain $X_n \approx X_p$. However when T drops below about $1.3 MeV$ the neutron fraction will drop exponentially on condition that the thermal equilibrium is still maintained. However to check weather the thermal equilibrium is maintained we have to compare the expansion rate with the reaction rate. The expansion rate is

$$H = \sqrt{\frac{8\pi G\rho}{3}} \ , \tag{3.111}$$

where

$$\rho = \frac{\pi^2}{30} g T^4 \ , \tag{3.112}$$

where $g \approx 10.75$ represents the relativistic degrees of freedom present at these temperatures. At $T = Q$ this gives $H \approx 1.1 s^{-1}$. The reaction rate needs to be computed from weak interaction theory. The neutron to proton conversion rate is approximated by

$$\lambda_{np} \approx 0.29 s^{-1} \left(\frac{T}{Q}\right)^5 \left[\left(\frac{Q}{T}\right)^2 + 6\left(\frac{Q}{T}\right) + 12\right] \ . \tag{3.113}$$

At $Q = T$ this gives $\lambda \approx 5 s^{-1}$ that is more rapid than the expansion rate. But as T drops below Q this decreases rapidly and the reaction ceases to be fast enough to maintain thermal equilibrium. Then we have to work out the neutron abundance using the equation (3.107).

If we denote $n_1 = n_n, n_3 = n_p$ and $n_2, n_4 = n_l$ where the subscript l stands for leptons then the equation (3.107) becomes

$$\frac{1}{a^3}\frac{d(n_n a^3)}{dt} = \mu n_l^{eq} \left(\frac{n_p n_n^{eq}}{n_p^{eq}} - n_n\right) \ . \tag{3.114}$$

To proceed we use the fact that μn_l^{eq} is equal to the rate of the neutron to proton conversion λ_{np}. We also use the relation

$$\frac{n_n^{eq}}{n_p^{eq}} = \exp(-Q/T) \tag{3.115}$$

Let us now introduce the fractional abundance

$$X_n = \frac{n_n}{(n_n + n_p)} \tag{3.116}$$

Then the equation (3.114) takes the form

$$\frac{dX_n}{dt} = \lambda_{np}((1 - X_n)e^{-Q/T} - X_n) \ , \tag{3.117}$$

where we have used

$$X_n + X_p = 1 \ , X_p = \frac{n_p}{n_n + n_p} \tag{3.118}$$

and also the fact

$$\frac{1}{a^3}\frac{d(n_n a^3)}{dt} = \frac{a^3(n_n+n_p)}{a^3}\frac{dX_n}{dt} \tag{3.119}$$

since $(n_n + n_p)a^3$ is constant. This equation can be integrated numerically and determine how the neutron abundance changes with time. The neutron fraction falls out of equilibrium when temperature drop below $1MeV$ and it freezes to about 0.15 at temperature below $0.5MeV$. As the temperature decreases further the neutron decays with a half life of $\tau_n \approx 886.7sec$ becomes important and starts to reduce the neutron number density. Then the only way how the neutrons can survive is through the synthesis of light elements. As the temperature falls further to $T = T_{He} \approx 0.28MeV$ significant amount of He could have been produced if the nuclear reaction rates were high enough. These reactions are all based on D, He and H and do not occur rapidly enough because the mass fraction of D, He and H are still quite small $[10^{-12}, 10^{-19}, 5\times 10^{-19}]$ at $T \simeq 0.3MeV$. The equilibrium deuterium abundance passes through unity at temperature of about $0.07MeV$ which is when nucleosynthesis can really begin.

The production of still heavier elements-even those like C, O which have higher binding energies than He is suppressed in the early Universe.

3.13 Decoupling of matter and radiation

In the early hot phase the radiation will be in thermal equilibrium with matter. As the Universe cools below $k_B T \simeq (\epsilon_a/10)$ is the binding energy of atoms the electrons and ions will combine to form neutral atoms and radiation will decouple from matter. This occurs at $T \simeq 3\times 10^3 K$. As the Universe expands further these photons will continue to exist without any further interaction. We shall now discuss some details related to the formation of neutral atoms and decoupling of photons.

The relevant reaction is

$$e + p = H + \gamma \ . \tag{3.120}$$

If the rate of this reaction is faster than the expansion rate then one can calculate the neutral fraction as follows. Introducing the fractional ionization X_i for each of the particle species and using the facts that $n_p = n_e$ and $n_p + n_H = n_B$. We also have $X_p = X_e$ and $X_H = \frac{n_H}{n_B} = 1 - \frac{n_p}{n_B} = 1 - X_e$. The equation that governs the time evolution of X_e that expresses the equilibrium situation now takes the form

$$\frac{1-X_e}{X_e^2} \approx 3.84\eta\left(\frac{T}{m_e}\right)^{3/2}\exp(B/T) \ , \tag{3.121}$$

where $\eta = 2.68\times 10^{-8}(\Omega_B h^2)$ is the baryon-to-photon ratio.We define T_e as the temperature at which 90 percent of the electrons have combined with protons. This implies $n_p = 0.1 n_B$ and hence $X_e = X_p = 0.1$. This leads to the condition

$$(\Omega_B h^2)^{-1}\tau^{3/2}\exp[-13.6\tau^{-1}] = 3.13\times 10^{-18} \ , \tag{3.122}$$

where $\tau = (T/1eV)$. The solution of this equation can be given by iterative procedure. For $\Omega_B h^2 = 1, 0.1, 0.01$ we then obtain $T_{atom} = 0.324eV, 0.307eV, 0.292eV$.

These results were based on the equilibrium densities. Then it is important to check that the rate of the reaction $p + e \leftrightarrow H + \gamma$ is fast enough to maintain equilibrium. It turns out however that this is not fully satisfied and hence we have to again use the rate equation. The rate equation (3.107) for $n_1 = n_e, n_2 = n_p, n_3 = n_H$ and $n_4 = n_\gamma$ and for $X_e = \frac{n_e}{n_e+n_H}$ takes the form

$$\frac{dX_e}{dt} = \alpha \left(\frac{\beta}{\alpha}(1 - X_e) - n_b X_e^2 \right) , \tag{3.123}$$

where the recombination rate α is the rate is given by

$$\alpha = 9.78 r_0^2 c \left(\frac{B}{T}\right)^{1/2} \ln\left(\frac{B}{T}\right) , \tag{3.124}$$

where $r_0 = \frac{e^2}{m_e^2 c^2}$ is classical electron radius. In (3.123) the ration β/α is given as

$$\frac{\beta}{\alpha} = \left(\frac{m_e T}{2\pi}\right)^{3/2} \exp[-B/T] \tag{3.125}$$

Using this result we obtain that the value of T_{atom} does not change significantly.

3.14 Structure formation and linear perturbation theory

The structure formation is based on the key idea that if there exist small fluctuations in the energy density in the early Universe, then gravitational instability then leads in a well understood manner leading to structures like galaxies today. The most popular model for generating these fluctuations is based on the idea that if the very early Universe went through the inflation phase then the quantum fluctuations of the field driving the inflation can lead to energy density fluctuations.

Let us illustrate this idea on the example of the massless scalar field ϕ minimally coupled to gravity. The action of the scalar field is

$$S_\phi = -\frac{1}{2}\int d^4x \sqrt{-g} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{3.126}$$

In spatial flat FRW background this action has the form

$$S_\phi = -\frac{1}{2}\int dx dt a^3(t)[-(\partial_t\phi)^2 + \frac{1}{a^2}(\partial_i\phi)^2] \tag{3.127}$$

so that the equation of motion takes the form

$$\partial_t(a^3\partial_t\phi) - a\partial_i\partial^i\phi = 0 \tag{3.128}$$

or equivalently

$$\ddot{\phi} + 3H(t)\dot{\phi} - \frac{1}{a^2}\partial_i\partial^i\phi = 0 , \tag{3.129}$$

where $\dot{x} = \partial_t x$, $\ddot{x} = \partial_t^2 x$. Thanks to the homogeneity and isotropy of space it is natural to work in the momentum representation where we search for the solutions in the form

$$e^{i\mathbf{x}\mathbf{k}}\phi_{\mathbf{k}}(t) \ . \tag{3.130}$$

If we insert (3.130) into (3.129) we obtain ordinary differential equation for $\phi_{\mathbf{k}}$ in the form

$$\ddot{\phi}_{\mathbf{k}} + 3H(t)\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\phi = 0 \ . \tag{3.131}$$

Note that $\mathbf{k}$ is a *coordinate momentum*. The physical momentum at time t is

$$\mathbf{p} = \frac{\mathbf{k}}{a} \tag{3.132}$$

and it depends on time.

Looking on (3.131) we see that the second term in it acts as a friction term. Then we can consider two regimes with the qualitatively different behavior of the modes $\phi_{\mathbf{k}}$: **Subhorizion modes:**

These modes are characterized condition

$$p = \frac{k}{a} \gg H \ . \tag{3.133}$$

Modes obeying this property are subhorizon modes since their physical length $\lambda \sim p^{-1}$ is much shorter than the Hubble distance H^{-1} that is a horizon size in matter and radion dominated Universe. More precisely, for modes obeying the condition (3.133) we can neglect the friction term in (3.131) and hence we get

$$\ddot{\phi} + \omega_{\mathbf{k}}^2(t)\phi = 0 \ , \omega_{\mathbf{k}}(t) = \frac{k}{a} \tag{3.134}$$

This equation has the general solution

$$\phi_{\mathbf{k}} = \frac{1}{a}e^{\pm i\int_{t_0}^t dt'\omega_{\mathbf{k}}(t')} \tag{3.135}$$

since

$$\begin{aligned}
\dot{\phi}_{\mathbf{k}} &= -H\phi_{\mathbf{k}} + i\omega_{\mathbf{k}}\phi_{\mathbf{k}} \approx i\omega_{\mathbf{k}}\phi_{\mathbf{k}} \ , \\
\ddot{\phi}_{\mathbf{k}} &= i\dot{\omega}_{\mathbf{k}}\phi_{\mathbf{k}} - \omega_{\mathbf{k}}^2\phi_{\mathbf{k}} = -iH\omega_{\mathbf{k}}\phi_{\mathbf{k}} - \omega_{\mathbf{k}}^2\phi_{\mathbf{k}} \approx -\omega_{\mathbf{k}}^2\phi_{\mathbf{k}} \ .
\end{aligned} \tag{3.136}$$

This solution (modulo slowly varying prefactor) describes oscillations with the frequently experiencing redshift (The frequency is lowered with time).

Superhorizon modes:

These modes are characterized by condition

$$p = \frac{k}{a} \ll H \ . \tag{3.137}$$

In this case the last term in (3.131) are negligible and the solutions are

$$\begin{aligned} \text{constant mode}: \quad & \phi_{\mathbf{k}} = const \ , \\ \text{growing mode}: \quad & \phi_{\mathbf{k}}(t) = K \int_{t_0}^{t} \frac{dt'}{a^3(t')} \ . \end{aligned} \tag{3.138}$$

It is clear that the constant mode is solution of (3.131). The growing mode is solution as well since

$$\dot{\phi}_{\mathbf{k}} = \frac{K}{a^3} \ , \ddot{\phi}_{\mathbf{k}} = -3H\phi_{\mathbf{k}} \ . \tag{3.139}$$

The gravitational waves obey precisely the same equations as (3.131) so that they have exactly the same behavior, in particular, for given k one of the superhorizon modes blows up at small t. It follows that the whole picture of the FRW Universe with small perturbations is thus self-consistent only if this modes vanishes at finite times.

Now recall that for radiation dominated and matter dominated Universe $H \sim t^{-1}$ while the scale factor behaves as $a \sim t^{1/2}$ for radion dominated Universe and $a \sim t^{2/3}$ for matter dominated Universe. Then the ratio of physical momentum to H behaves as

$$\frac{p(t)}{H(t)} \propto t^{1/2} \tag{3.140}$$

for radiation dominated Universe and

$$\frac{p(t)}{H(t)} \propto t^{1/3} \tag{3.141}$$

for matter dominated Universe. These results mean that all modes start as superhorizon and then enter the horizon. In the scalar mode example the requirement that the growing mode vanishes determines the initial date for each $\mathbf{k}$ up to overall amplitude. Then we have

$$\phi_{\mathbf{k}} = c_{\mathbf{k}} \ , \frac{k}{a} \ll H \ , \tag{3.142}$$

and

$$\phi_{\mathbf{k}} = c_{\mathbf{k}} \cos\left(\int_0^t dt' \omega_{\mathbf{k}}(t') \right) \ , \frac{k}{a} \gg H \ . \tag{3.143}$$

For density perturbations the oscillating behavior means that at late enough times there are sound waves in the primordial plasma with the wave-lengths that are shorter than the horizon size at each moment of time. Briefly speaking the fate of the primordial density perturbations is as follows. They stay constant until they enter the horizon at radiation or matter dominate stage. After that they start to oscillate and make the sound waves. The amplitudes of these waves grow during the matter dominated stage due to the gravitational instability. The regions with higher density tend to gravitationally attract matter and become even more overdense. The dense regions collapse and form gravitationally bound structures.

Let us now discuss in more details how the simple description given above is related to the more realistic situation. As long as the fluctuations are small one can study their evolution by linear perturbation theory. The basic idea of linear perturbation theory is well defined and simple. We write the metric as

$$g_{\mu\nu} = g_{\mu\nu}^{FRW} + h_{\mu\nu} \ , \tag{3.144}$$

where $g_{\mu\nu}^{FRW}$ is background FRW metric and $h_{\mu\nu}$ is small perturbations that propagate on the background characterized with $g_{\mu\nu}^{FRW}$. In the same way we perturb the source energy momentum tensor by

$$T_{\mu\nu} = T_{\mu\nu}^{FRW} + \delta T_{\mu\nu} \ , \tag{3.145}$$

where again $T_{\mu\nu}^{FRW}$ is the stress energy tensor for the background matter that solves the FRW equations and $\delta T_{\mu\nu}$ are perturbations. If we linearize the Einstein's equations one can relate the perturbed quantities by a relation of the form

$$\mathcal{L}(g_{\mu\nu}^{FRW})h_{\mu\nu} = \delta T_{\mu\nu} \ , \tag{3.146}$$

where $\mathcal{L}$ is second order linear differential operator depending on the background metric $g_{\mu\nu}^{FRW}$. As wa argued above due to the fact that the background is maximally symmetric one can separate out time and space and we can write down the equation for any given mode labeled with the wave vector $\mathbf{k}$ as

$$\mathcal{L}(a(t), \mathbf{k})h_{\mu\nu}(t, \mathbf{k}) = \delta T_{\mu\nu}(t, \mathbf{k}) \ . \tag{3.147}$$

Then carefull analysis performed in case of metric perturbations implies that the linearized equations of motion for gravity perturbations take the forms given in the toy example of the massless scalar fields studied above. More precisely, it can be shown,after some simplifications and presumption, that are all well justified, that perturbed metric can be written in the form

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)\delta_{ab}dx^a dx^b] \ . \tag{3.148}$$

In other words we obtain one perturbed scalar degree of freedom Φ. Then it can be shown that the dynamics of the mode Φ is governed by the equations that has the same form as (3.131).

4. Inflation cosmology

4.1 Problems of the standard Big-Bang model

The standard Big-Bang model suffers from number of problems. Before we enter in their discussion we review some properties of the Friedmann models at the early stage of the Universe.

The question is what can we say about the Hubble parameter $H = \frac{\dot{a}}{a}$, the density ρ and the quantity k?

At the earliest stages of the evolution of the Universe H and ρ could be arbitrarily large. On the other hand it is believed that for $\rho \geq M_P^4$ effects of quantum gravity are significant and the quantum fluctuations of metric exceed the classical value of $g_{\mu\nu}$. The standard cosmology where the metric is treated in the classical manner restricts to the region of phenomena where

$$\rho \leq M_P^4 \ , \quad T \leq M_P \sim 10^{19} GeV, \quad H < M_P \ . \tag{4.1}$$

We also have to stress that in the expanding Universe thermodynamics equilibrium cannot be established immediately but only when the temperature T is sufficiently low. The behavior of the non-equilibrium Universe at densities of order of the Planck density is very important problem.

Now we come to the list of problems of the standard hot Universe theory

4.2 Problems of the standard scenario

The singularity problem

The Friedmann equations imply that the density of matter in the Universe goes to infinity as $t \to 0$ and the corresponding solutions cannot be formally continued to the domain $t < 0$.

One of the most exciting questions of cosmology is whether anything existed *before* $t = 0$. If there is nothing before $t < 0$ the question is: where did the Universe come from?

Studies of the general structure of space-time near a singularity suggest that it is highly unlikely that this problem could be solved with the framework of the classical gravitation theory. One hope that these questions could be answered in the context of string theory. We will review some string theory inspired models in next sections. However these models are faced with many important and conceptional problems so that the problem of the birth of the Universe is the most challenging un answered question in physics.

Flatness Problem

The flatness problem concerns with the observation that the real density of the Universe, ρ, is known to be very close to the critical density ρ_c. Recall, that in the previous section we have studied the Friedmann equation

$$H^2 = \frac{1}{3M_P^2}\rho - \frac{k}{a^2} \ , \tag{4.2}$$

where now $M_P \equiv \frac{1}{\sqrt{8\pi G}} \sim 2 \cdot 10^{18} GeV$ is the four dimensional Planck mass. Recall also that $H = \frac{\dot{a}}{a}$ where $a(t)$ is the scale factor with the spacetime metric on the form

$$ds^2 = -dt^2 + a^2 d\Sigma \ , \tag{4.3}$$

where $d\Sigma$ is comoving volume element of space with $k = 0, +1, -1$ corresponding to flat, positively curved and negatively curved spaces respectively. As we known we can rewrite the Friedmann equation in the form

$$\Omega - 1 = \frac{k}{a^2 H^2} \ , \tag{4.4}$$

where Ω means the sum of particular Ω's. Note that for ordinary type of matter, $\frac{1}{a^2 H^2}$ will increase with time. To see this we use the continuum equation given by

$$\dot{\rho} + 3H(\rho + p) = 0 \ . \tag{4.5}$$

If we assume an equation of state of the form

$$p = w\rho \ , \tag{4.6}$$

for $w = \text{const}$ then the continuity equation can be written as

$$\frac{d\rho}{da}\frac{da}{dt} + 3\frac{\dot{a}}{a}(1+w)\rho = \frac{d\rho}{da} + 3(1+w)\frac{\rho}{a} = 0 \ , \tag{4.7}$$

that implies

$$\rho \sim a^{-3(1+w)} \ . \tag{4.8}$$

If we start with $\Omega \sim 1$ we obtain that $k \sim 0$. Then the Friedman equation is

$$H^2 \sim \rho \Rightarrow \frac{\dot{a}}{a} \sim a^{-3(1+w)/2} \tag{4.9}$$

that implies

$$daa^{(1+3w)/2} = t \Rightarrow a \sim t^{\frac{2}{3(1+w)}} \ . \tag{4.10}$$

As a consequence we get that

$$\frac{1}{a^2 H^2} \sim t^{2-\frac{4}{3(1+w)}} \ . \tag{4.11}$$

This expression grows with time for any $w > -1/3$-examples include pressureless dust with $w = 0$ and radiation with $w = 1/3$. Looking on the form of the Friedman equation (4.4) we see that, unless the Universe is exactly flat ($k = 0$) and, as a consequence $\Omega = 1$, Ω will rapidly evolve away from $\Omega = 1$. In order to have a value of Ω close to 1 today, one would therefore expect to need a value of Ω even closer to 1 in the early Universe. This is the famous *Flatness problem*. That is, how can Ω be so close to one?

We can argue alternatively as follows. Looking on the form of Friedmann equation we see that the curvature contribution is

$$|\Omega_{curv}| \equiv \frac{\rho_{curv}}{\rho_c} = \frac{3M_P}{a^2H^2} \ , \tag{4.12}$$

where we have defined the curvature contribution to the Friedmann equation as

$$|\rho_{curv}|\frac{3M_P}{a^2} \ . \tag{4.13}$$

The present value of the equation (4.12) is

$$|\Omega_{curv}| < 0.02 \ . \tag{4.14}$$

Since $|\rho|_{curv}$ scales as $1/a^2$ while the radiation matter and radiation scales as $1/a^3$ and $1/a^4$ respectively. This implies that the curvature contribution to the Friedman equations was even smaller in the past, for example

$$\begin{aligned} \text{nucleosynthesis}: \quad & |\Omega_{curv}| < 10^{-16} \ , \\ \text{electroweak epoch} \ , \quad & |\Omega_{curv}| < 10^{-26} \ . \end{aligned} \tag{4.15}$$

In other words the spatial curvature of the Universe was tiny at the beginning. The question is, why the initial conditions were so flat? This flatness problem cannot be solved within Hot Big Bang theory.

The total entropy and total mass problem

The question is why the total entropy S and total mass M of matter in the observable part of the Universe with R_p is so large. The total entropy S of the present Universe can be estimate as follows. The size of the observable part of the Universe is

$$l_{H,0} \sim 2H_0^{-1} \sim 10^{26} \ m$$

The entropy inside a sphere of the size $l_{H,0}$ is roughly of the order of the number of photons

$$S \sim N_\gamma \sim n_\gamma l_{H,0}^3 \ . \tag{4.16}$$

Using also the fact that

$$n_\gamma \sim T_\gamma^3 \sim 2.7 \ K$$

where T_γ is the temperature of the primordial background radiation. Then we finally obtain

$$S = 10^{88} \ . \tag{4.17}$$

On the other hand the estimate of the total mass in the observable Universe is

$$M \sim l_{H,0}^3 \rho_c \sim 10^{55} g \ . \tag{4.18}$$

In the Hot Big Bang theory the expansion of the Universe is almost adiabatic so this huge entropy should be built in as an initial condition. Certainly this initial condition is very special. Moreover, the condition of naturality, which is the statement that all dimensionless quantities should be of order 1 implies that such a initial conditions with huge entropy are rather un-natural.

Horizon problem

We known that the region of the Universe look very similar even though, assuming normal radiation dominated expansion of the early Universe, thay can not have been in causal contact. In fact, the horizon problem steams from the existence of particle horizons in FRW cosmologies. Horizons exist because there is only a finite amount of time since the Big Bang singularity and thus only a finite distance that photons can travel within the age of the Universe. Consider a photon moving along a radial trajectory in a flat Universe. In a flat, Universe, we can normalize the sale factor to be $a_0 = 1$. A radial null path obeys

$$0 = ds^2 = -dt^2 + a^2 dr^2 \tag{4.19}$$

so the comoving (coordinate) distance traveled by such a photon between times t_1 and t_2 is

$$\triangle r = \int_{t_1}^{t_2} \frac{dt}{a(t)} \ . \tag{4.20}$$

To get a physical distance as it would be measured by an observer at any time t simply multiply by $a(t)$. For simplicity, we are in matter dominated Universe for which

$$a = \left(\frac{t}{t_0}\right)^{2/3} \ . \tag{4.21}$$

The Hubble parameter is therefore given by

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} = a^{-2/3} H_0 \ , \tag{4.22}$$

where H_0 is Hubble parameter of today Universe. Then the photon travels a comoving distance

$$\triangle r = 2H_0^{-1}(\sqrt{a_2} - \sqrt{a_1}) \tag{4.23}$$

The comoving horizon size when $a = a_*$ is the distance a photon travels since the Big Bang

$$r_h(a_*) = 2H_0^{-1}\sqrt{a_*} \ . \tag{4.24}$$

The physical horizon size, as measured on the spatial hypersurface at a_* is therefore simply

$$d_h(a_*) = a_* r_h(a_*) = 2H_0^{-1} a_*^{3/2} = 2H_0^{-1} \frac{H_0}{H_*} = 2H_*^{-1} \ . \tag{4.25}$$

The horizon problem is simply the fact that CMB is isotropic to high degree of precision even though widely separated points on the last scattering surface are completely outside each other's horizons. When we look at the CMB we see the Universe at a scale factor $a_{CMB} \approx 1/200$. The comoving distance between a point on the CMB and an observer on Earth is

$$\triangle r = 2H_0^{-1}(1 - \sqrt{a_{CMB}}) \approx 2H_0^{-1} \ . \tag{4.26}$$

However, the comoving horizon distance for such a point is

$$r_h(a_{CMB}) = 2H_0^{-1}\sqrt{a_{CMB}} = 6 \times 10^{-2} H_0^{-1} \ . \tag{4.27}$$

Hence if we observe two widely separated parts of the CMB they will have non-overlapping horizons; different patches of the CMB sky were causally disconnected at recombination. On the other hand they are observed to be at the same temperature at high precision. This is the core of the famous horizon problem.

Problem of the large-scale homogeneity and isotropy of the Universe

As we argued in introduction all cosmological models are based on the presumption of absolutely homogeneous and isotropic Universe. Of course Universe is not absolutely homogeneous and isotropic at now at least on small scale and hence there is no reason to believe that it was homogeneous at its beginning. The most natural assumption is that the initial conditions at points that are sufficiently far from one another were chaotic and uncorrelated. On the other hand it was shown by Collins and Hawking that class of the initial conditions for which the Universe tends asymptotically (at large t) fo Friedmann Universe is one of measure zero among all possible conditions. In other words according to this classical analysis Friedmann model is very unprobable. This is the problem of large scale homogeneity and isotropy.

The galaxy formation problem

We know that Universe contains many inhomogeneities as stars, galaxies and so on. In order to explain the origin of galaxies one have to presume an existence of initial inhomogeneities whose spectrum is usually taken to be almost scale invariant. For a long time the origin of such density inhomogeneities remained obscure.

The baryon asymmetry problem

This is the problem why the Universe is added almost entirely of matter with almost no antimatter and why on the other hand the number of baryons is much less than number of photons $\frac{n_B}{n_\gamma} \sim 10^{-9}$.

The domain wall problem

It is natural to presume that the symmetry breaking occurs independently in all causally unconnected regions of Universe. Then at all these regions that comprise Universe at the time of symmetry-breaking phase transition, both field $\phi = +\mu/\sqrt{\lambda}$ and the field $\phi = -\mu/\sqrt{\lambda}$. Domains filled by the field $\phi = +\mu/\sqrt{\lambda}$ are separated from those with the field $\phi = -\mu/\sqrt{\lambda}$ by domain walls. It can be shown that the energy density of these walls is so high so that their existence is inconsistent with cosmological consequences. Since the theories based on the spontaneously breaking of gauge symmetry are very appealing and since in these theories domain walls arise in natural way we meet *Domain wall problem*. In other words how to deal with such theories in cosmology.

The primordial monopole problems

This problem is closely related to the domain wall problems. Many theories based on symmetry-braking mechanism can produce another nontrivial structures that are nontrivial configurations of the scalar and gauge fields and that are stable. However it can be shown that these objects are very massive. Moreover it can be also shown that the monopole density at present would be comparable with the baryon density. Thanks to the enormous massivity these objects we obtain that the Universe filled of monopoles is 10^{15} higher than the critical density. This implies that Universe filled with such matter would have collapsed long ago. The explanation of the mechanism how to deal with monopoles is one of the most important problems in cosmology.

Unwanted Relics

We have argued that for correct description of the early Universe the models of particle physics should be present. However these models contain monopoles and other topological defects. However the energy density of these objects can be very big and hence the monopole abundance in GUT is serious problem for cosmology if GUT have anything to do with reality.

4.3 Inflation as a solution

4.3.1 The General Idea of Inflation

The horizon problem is an extremely serious problem for the standard cosmology. Cosmological inflation is mechanism that can solve this problem.

The main idea is that the Universe undergoes a period of accelerated expansion defined as a period when $\ddot{a} > 0$ at early times. The effect of this acceleration is to quickly expand a small region of space to huge size. At this process the spatial curvature of the Universe is reduced and consequently we make the Universe extremely close to flat. In addition, the horizon size is greatly increased so that distant points on the CMB actually are in causal contact and unwanted relics are diluted, solving the monopole problem. Finally, quantum fluctuations imply that inflation cannot smooth out the Universe with perfect precision, so there is a spectrum of remnant density perturbations.

The general idea of inflation is that before Hot Big Bang (but after Planck era) the Universe was in vacuum-like state and then it went through the era of the exponential expansion

$$a(t) = \text{const} \cdot e^{\int H_{infl} dt} , \tag{4.28}$$

where H_{infl} is almost constant in time. Due to the exponential expansion a small patch of the Universe expands to great size. Let us presume that the duration of inflation t_{infl} exceeds 140 Hubble times

$$t_{infl} > \frac{140}{H_{infl}} . \tag{4.29}$$

Let us also presume that the size of the patch is initially at the order Planck size $l_P = \frac{1}{M_P} \sim 10^{-33} cm$. Then at the time t_{inf} the size exceeds the present horizon size $l_{H,0} \sim 10^{28} cm$. It is also clear the Universe flattens out, any initial inhomogeneities are diluted out. In the end of inflation, the Universe becomes spatially flat,homogeneous and isotropic at exponentially large spatial scales. This solves the horizon and flatness problems.

A natural way to ensure that the Universe expands exponentially is to assume that the matter at inflationary stage is in the vacuum-like state characterized with the energy density ρ_{infl} that is almost constant in time. At some point this energy density should transform into conventional energy density of hot plasma. This transformation is called reheating and after reheating the Hot Big Bang era begins. During reheating, huge entropy is released and this solves the entropy problems.

4.4 Many models of inflation

Before we come to the more detailed study of the question how the inflation works we give summary of some models of the inflation theory. The common property of these model is that the matter with suitable equation of state is in the form of the scalar field(s).

The initial model of inflation ("old inflation model") was based on idea that the scalar field ϕ was initially in a false vacuum with large potential energy. To end of inflation, a quantum tunneling from the false vacuum to the true vacuum was performed. However this model has the problem that it leads to an initially microscopical bubble of the true vacuum which cannot grow to contain our present observed Universe. Hence the attention shifted to models in which the scalar field ϕ slowly rolls during the inflation.

Models of scalar field-driven inflation can be divided into three groups:

- **Small-field inflation**
- **Large-field inflation**
- **Hybrid inflation**

Small field inflationary models are based on ideas from spontaneous symmetry breaking in particle physics. For example, let us consider the scalar field with the potential in the form

$$V(\phi) = \frac{1}{4}(\phi^2 - \sigma^2)^2 \ , \tag{4.30}$$

where we interpret σ as the symmetry breaking scale and λ as a dimensionless coupling constant. The main idea of the small-field models (”new inflation”) was that the scalar field starts to roll close to its symmetric point $\phi = 0$. At sufficient high temperature $\phi = 0$ is a stable ground state of the one-loop finite temperature effective potential $V_T(\phi)$. When the temperature drops below to some value that is smaller than T_c, $\phi = 0$ becomes unstable local minimum of $V_T(\phi)$ and ϕ can roll towards a ground state of the zero temperature potential (4.30) with

$$\phi_{gr} = \pm\sigma \ . \tag{4.31}$$

The problem of this model is that the slow-roll conditions [10]

$$\left(\frac{V'}{V}\right)^2 M_P^2 \ll 1 \ , \frac{V''}{V} M_P^2 \ll 1 \tag{4.32}$$

that for the potential (4.30) take the form

$$\frac{\phi^2}{(\phi^2 - \sigma^2)^2} \ll \frac{1}{M_P^2} \ , \frac{3\phi^2 - \sigma^2}{(\phi^2 - \sigma^2)^2} \ll \frac{1}{M_P^2} \tag{4.33}$$

and that have to be valid for inflation to works imply that

$$\sigma \sim M_P \ . \tag{4.34}$$

However this is in contradiction with the fact that we have to presume that σ is some symmetry breaking scale of the standard quantum field theory while M_P is the scale of the quantum gravity regime where the approximation of the quantum field

[10]Precise definition of these conditions will be given in next section

theory in curved space time cannot be valid. The potential (4.30) can be changed to satisfy the slow-roll conditions however this procedure needs several fine-tuning of the shape of the potential. A further problem of the slow-roll model is that the initial field velocity must be constrained to be small which is again fine-tuned initial condition.

As the alternative to the small-field inflationary models are *large-field inflation models* that are also known as *chaotic inflation*. The simplest example is provided by a massive scalar field with the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 \ . \tag{4.35}$$

In the chaotic inflation scenario it is presumed that the scalar field rolls towards the origin from large values of $|\phi|$. The slow roll conditions for the potential (**??**) takes the form [11]

$$|\phi| \gg M_P \ . \tag{4.36}$$

Values of $|\phi|$ comparable or larger than M_P are also required in other realizations of large-field inflations. The question is whether such a model can consistently be embedded in a realistic particle physics model, as for example supergravity. In many these models $V(\phi)$ receives supergravity-induced correction terms that destroys the flatness of the potential for $|\phi| > M_P$. The value $m \sim 10^{13} GeV$ is required in order to obtain the observed amplitude of density fluctuations.

With two scalar fields it is possible to construct a class of models which combine some of the nice features of large-field inflation models which is large set of the initial conditions that lead to inflation with the small-field inflation where the inflation takes place at sub-Planckian field values. These models are known as *Hybrid inflation*. For example, let us consider two scalar fields ϕ and ξ with the potential

$$V(\phi, \xi) = \frac{1}{4}\lambda_\xi(\xi^2 - \sigma^2)^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{2}g^2\phi^2\xi^2 \ . \tag{4.37}$$

In the absence of the thermal equilibrium it is natural to assume that $|\phi|$ begins at large values. For large ϕ the term

$$\frac{1}{2}g^2\phi^2\xi^2$$

that serves as an effective mass term for ξ is positive and hence ξ has stable minimum at $\xi = 0$. The parameters in (4.37) are chosen such that ϕ is slowly rolling for values of $|\phi|$ somewhat smaller than M_P but the parameters are chosen in such a way that the potential energy for these fields values is dominated by the first term in

[11] Note that the dimensional analysis that implies that V has dimension $[V] = 4$ in mass unit implies that $[\phi] = 1$.

(4.37). The field ϕ is slowly rolling whereas the potential energy is determined by the contribution from ξ. Once ϕ drops to the value

$$|\phi|_c = \frac{\sqrt{\lambda_\xi}}{g}\sigma \ . \tag{4.38}$$

For this value the effective potential for ξ takes the form

$$V(\phi_c, \xi) = \frac{\lambda_\xi}{4}(\phi^2 - 2\sigma^2)^2 \tag{4.39}$$

that has three extrema

$$\xi_0 = 0 \ , \quad V(0) = \lambda_\xi \sigma^4 \quad \xi_\pm = \pm\sqrt{2}\sigma \ , V(\phi_\pm) = 0 \tag{4.40}$$

that clearly shows that the configuration with $\xi = 0$ is unstable and decays to the one of the states $\xi_\pm = \pm\sqrt{2}\sigma$. Since in this case the ground state is not unique we have a possibility of the formation of topological defects at the end of the inflations.

After the slow-roll conditions break down the period of inflation ends and the inflation begins to oscillate around its ground state. Since the inflation field ϕ couples to other matter fields the energy of the Universe, that at the end of the period of inflation is stored completely in ϕ is transferred to the matter fields of the particle physics Standard model. The description of this process is very complicated,

4.5 How does the inflation work

The key property of the laws of physics that makes inflation possible is the existence of states of negative pressure. To recognize the effect negative pressure let us again consider Friedmann equation

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3}(\rho + 3p)a \ , \\ H^2 = \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ , \\ \dot{\rho} &= -3H(\rho + p) \ . \end{aligned} \tag{4.41}$$

Once again, the metric is given by Robertson-Walker form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2\right] \ , \tag{4.42}$$

where $k = 0, 1, -1$. From the first equation in (4.41) we see that positive pressure (ρ is always positive) contributes to the deceleration of the Universe while the negative pressure can cause acceleration. In other words, negative pressure produces a repulsive form of gravity.

The characteristic property of the inflation is that the physical wavelengths grow *faster* than the size of the Hubble radius

$$d_H = \frac{a(t)}{\dot{a}(t)} = \frac{1}{H}$$

as follows from the fact

$$\frac{\dot{\lambda}_{phys}}{\lambda_{phys}} = \frac{1}{a(t)\lambda_0}\frac{d(a(t)\lambda_0)}{dt} = \frac{\dot{a}}{a} = H = \frac{\dot{d}_H}{d_H} + d_H\frac{\ddot{a}}{a} \ . \tag{4.43}$$

This equation shows that during inflation when $\frac{\ddot{a}}{a} > 0$ the physical wavelengths become larger than the Hubble radius. However when the physical wavelength becomes larger than Hubble radius it is causally disconnected from physical processes. The inflationary era is followed by the radiation dominated and matter dominated stagers where the Hubble radius grows faster than the scale factor and the wavelengths that were outside now re-enter Hubble radius. This is the basic mechanism how the inflation explains the generation of temperature fluctuations and also the origin of the emergence of large scale formation: Briefly, quantum fluctuations generated early in the inflationary stage exit the Hubble radius during inflation and then eventually re-enter during the matter dominated era.

Remarkably, we can easily find form of the matter that produces negative pressure.

4.6 Slowly-Rolling Scalar Fields

In order the inflation to solve the problems of the standard cosmology it must be active at extremely early times. Thus we would like to study the earliest times in the Universe amenable to classical description. It is expected that this is around the Planck time t_P. For that reason we will retain values of Planck mass in the equation of this section. As we will see there are many models of inflation. In this section we will restrict ourselves to the study of the model of *chaotic inflation*.

Consider matter in the form of the scalar field ϕ that is described with the action

$$S_{matter} = -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right] \ . \tag{4.44}$$

In field theory the stress energy tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{4.45}$$

that for the action of the form $S = -\int d^4x\sqrt{-g}\mathcal{L}$ takes the form

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L} + 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}, \tag{4.46}$$

where we have used

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \ . \tag{4.47}$$

More precisely, for the action (4.44) the stress energy tensor takes the form

$$T_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}(\nabla_\alpha\phi)(\nabla_\beta\phi) + V(\phi)\right] \ , \tag{4.48}$$

where for the scalar field ϕ we have $\nabla_\alpha\phi = \partial_\alpha\phi$. Let us now restrict to the homogenous case in which all quantities depend only on cosmological time t and we also set $k = 0$. A homogenous real scalar field behaves as a perfect fluid with

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi) \ . \tag{4.49}$$

The other components of the stress energy tensor take the form

$$T_{ij} = -g_{ij}(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V) + \partial_i\phi\partial_j\phi \ . \tag{4.50}$$

If we define pressure as

$$p = \frac{1}{3}\sum_{i=1}^{3} T_{ii} \tag{4.51}$$

we get

$$p = \frac{\dot{\phi}^2}{2} - V(\phi) \ . \tag{4.52}$$

Thus any state which is dominated by the potential energy of a scalar field will have negative pressure.

Note also that the equation of motion for the scalar field are given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \ , \tag{4.53}$$

that can be thought of as a usual equation of motion for a scalar field in Minkowski space but with a friction term due to the expansion of the Universe. The Friedmann equation with such a field as a sole energy source is

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \ . \tag{4.54}$$

The accelerated expanssion occurs if the Universe is dominated by an energy component that approximates a cosmological constant. In that case the associated expansion rate will be exponential. From (4.49) we see that for $\dot{\phi}^2 \ll V(\phi)$ the potential energy of the scalar field is the dominant contribution to both the energy density and pressure ant the resulting equation of state is $p = -\rho$ that has the same form as the state equation for cosmological constant.

More technically, the *slow-roll approximation* for inflation involves neglecting the $\ddot{\phi}$ term in (4.53) and neglecting the kinetic energy compared of ϕ compared to the potential energy. In this case the scalar field equation of motion and the Friedmann equation become

$$\dot{\phi} = -\frac{V'}{3H} ,$$
$$H^2 = \frac{8\pi G}{3} V(\phi) . \tag{4.55}$$

The slow low conditions are conveniently characterized with so named *slow roll parameters*

$$\epsilon = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 , \eta = M_P^2 \frac{V''}{V} , \tag{4.56}$$

where

$$8\pi G = M_p^{-2} . \tag{4.57}$$

It is easy to see that the slow-roll conditions yield inflation. Recall that inflation is defined by

$$\frac{\ddot{a}}{a} > 0 \tag{4.58}$$

that using the fact that

$$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} \Rightarrow \frac{\ddot{a}}{a} = \dot{H} + \left(\frac{\dot{a}}{a}\right)^2$$

or alternatively

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 . \tag{4.59}$$

Then the inflation occurs when

$$\frac{\dot{H}}{H^2} > -1 . \tag{4.60}$$

But in slow roll

$$2\dot{H}H = \frac{8\pi G}{3} V'\dot{\phi} = -\frac{8\pi G}{9}\frac{V'^2}{H} \tag{4.61}$$

and hence

$$\frac{\dot{H}}{H^2} = -\frac{4\pi G}{9}\frac{V'^2}{H^4} = -\frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = -\epsilon \tag{4.62}$$

which will be small. Smallness of the second parameter η ensures that inflation will continue for a sufficient period.

It is useful to have a general expression that describes how much inflation occurs once it has begun. Such a quantity is the *number of e-folds* defined by

$$N(t) \equiv \ln\left(\frac{a(t_{end})}{a(t)}\right) . \tag{4.63}$$

Usually we are interested in how many e-folds occur between a given field value ϕ and the field value at the end of inflation ϕ_{end} where $\epsilon(\phi_{end}) = 1$. To do this we express $N(t)$ as

$$\begin{aligned} N(t) &= \ln\left(\frac{a(t_{end})}{a(t)}\right) = \int_{a(t)}^{a(t_{end})} \frac{da'}{a'} = \\ &= \int_t^{t_{end}} \frac{\dot{a}}{a} dt' = \int_t^{t_{end}} H dt' = \int_\phi^{\phi_{end}} H \frac{d\tilde{\phi}}{\dot{\tilde{\phi}}} = \\ &= -3 \int_\phi^{\phi_{end}} H^2 \frac{d\tilde{\phi}}{V'} = -\frac{1}{M_p^2} \int_\phi^{\phi_{end}} \frac{V}{V'} d\tilde{\phi} \ . \end{aligned} \tag{4.64}$$

The problem of the initial conditions for inflation is very subtle. In case of chaotic inflation in which we assume that the early Universe emerges from the Planck epoch with the scalar field taking different values in different part of the Universe with typically Planckian energies.

Let us now consider some examples of the potential that could lead to inflation. We start with the simple monomial

$$V = \lambda M_P^{4-\alpha} \phi^\alpha \ . \tag{4.65}$$

For potential above we obtain following slow roll parameters

$$\epsilon = \frac{\alpha^2 M_P^2}{2\phi^2} \ , \quad \eta = \alpha(\alpha - 1)\frac{M_P^2}{\phi^2} \ . \tag{4.66}$$

Inflation starts at a large value of ϕ and the inflaton then rolls slowly towards the minimum with increasing ϵ and η. Inflation ends when the slow roll conditions are saturated,

$$\phi \sim \lambda M_P \ . \tag{4.67}$$

The number of e-foldings we obtain before this happens is given by

$$\begin{aligned} N &= \ln\frac{a(t_e)}{a(t_i)} = \left(H dt = \frac{da}{a} \Rightarrow \int H dt = \ln(a_f) - \ln(a_i)\right) \int_{t_i}^{t_e} H dt = \\ &= \int_{\phi_i}^{\phi_e} H \frac{d\phi}{\dot{\phi}} = -\int_{\phi_i}^{\phi_e} \frac{3H^2}{V'} d\phi = -\frac{1}{M_P^2} \int_{\phi_i}^{\phi_e} \frac{V}{V'} d\phi = -\frac{1}{M_P^2 \alpha} \int_{\phi_i}^{\phi_e} \phi d\phi = \\ &= \frac{\phi_i^2}{2M_P^2 \alpha} - \frac{1}{4} \approx \frac{1}{2\alpha M_P^2} \phi_i^2 \end{aligned} \tag{4.68}$$

that implies

$$\phi_i = \sqrt{2\alpha N} M_P \gg M_P \ . \tag{4.69}$$

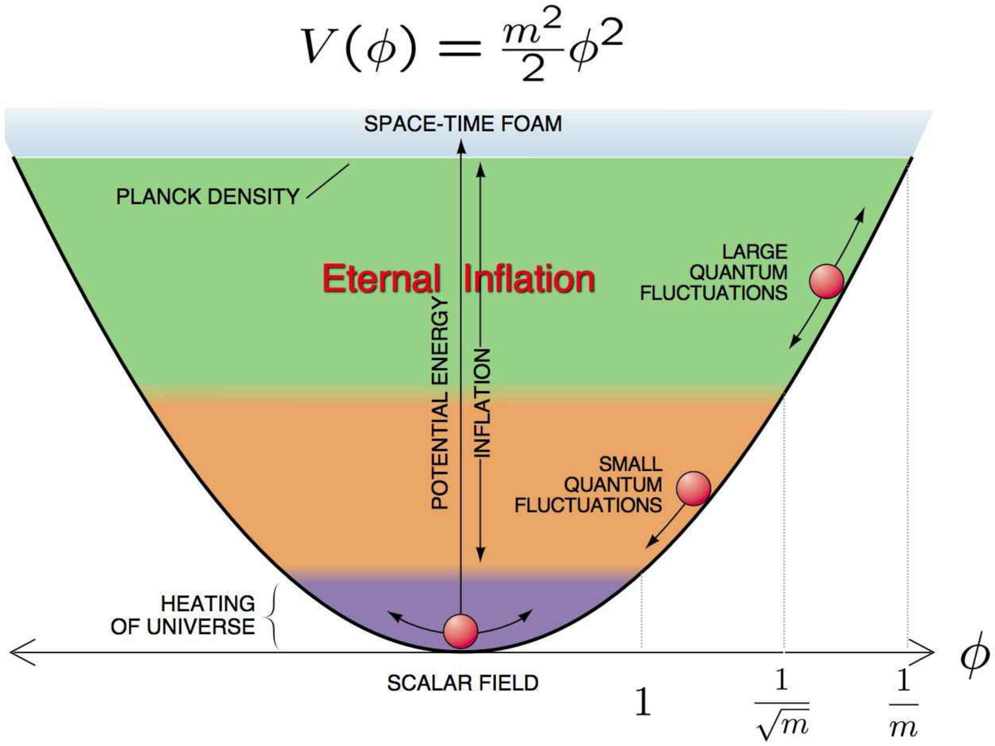

Figure 1: As an example that illustrates the main idea of inflation is motion of the scalar field in the theory with $V(\phi) = \frac{m^2}{2}\phi^2$. Several different regimes are possible, depending on the value of the field ϕ. If the potential energy density of the field is greater than the Planck density $M_p^4 = 1$, $\phi \sim m^{-1}$, quantum fluctuations of space-time are so strong that one cannot describe it in usual terms. Such a state is called space-time foam. At a somewhat smaller energy density (for $m \sim V(\phi) \sim 1$, $m^{-1/2} \sim \phi \sim m^{-1}$) quantum fluctuations of space-time are small, but quantum fluctuations of the scalar field ϕ may be large. Jumps of the scalar field due to quantum fluctuations lead to a process of eternal self-reproduction of inflationary universe which we are going to discuss later. At even smaller values of $V(\phi)$ (for $m^2 \sim V(\phi) \sim m$, $1 \sim \phi \sim m^{-1/2}$) fluctuations of the field ϕ are small; it slowly moves down as a ball in a viscous liquid. Inflation occurs for $1 \sim \phi \sim m^{-1}$. Finally, near the minimum of $V(\phi)$ (for $\phi \sim 1$) the scalar field rapidly oscillates, creates pairs of elementary particles, and the universe becomes hot.

Using this initial value ϕ_i we can determine the values of slow roll parameters at t_i

$$\epsilon_i \sim \frac{\alpha}{4N} \, , \eta \sim \frac{\alpha - 1}{N} \ . \tag{4.70}$$

Another example of the inflation potential is

$$V = V_0 e^{-\sqrt{\frac{2}{p}}\frac{\phi}{M_P}} \tag{4.71}$$

with the slow roll parameters

$$\epsilon = \frac{1}{p} \ , \eta = \frac{2}{p} \ . \tag{4.72}$$

Recall that for this potential we can combine the equation of motion to get

$$\dot{\phi} = -\frac{M_P}{\sqrt{3}}\frac{V'}{\sqrt{V}} = \sqrt{\frac{2}{3p}}\sqrt{V} \tag{4.73}$$

that has the solution

$$V \sim \frac{3M_4^2 p^2}{t^2} \tag{4.74}$$

and hence

$$H^2 \sim \frac{p^2}{t^2} \Rightarrow \ln a \sim p \ln t \Rightarrow a \sim t^p \ . \tag{4.75}$$

To gain more insight in the idea of inflation note that in most inflation models the energy density ρ is approximately constant leading to exponential expanssion of the scale factor. In fact, using $p = -\rho$ in the Friedmann equation we get

$$\ddot{a} = \frac{8\pi G}{3}\rho a \tag{4.76}$$

that in the approximation of $\rho = const$ can be solved with the ansatz $a = e^{\lambda t}$ that inserted in the equation above implies

$$\lambda^2 - \frac{8\pi G}{3}\rho_f = 0 \Rightarrow \lambda = \sqrt{\frac{8\pi G}{3}\rho_f} \ , \tag{4.77}$$

where ρ_f is constant energy density.

In the original model of inflation the state that drove the inflation involved a scalar field in a local (but no global) minimum of its potential energy.The scalar field state employed in the original version of inflation is called a *false vacuum* since the state temporally acts as if it were the state of lowest possible energy density. Classically this state is stable that there is no possibility how the scalar field crosses a potential energy barrier that separates it from the states of lower energy. However quantum mechanically this state would decay through tunneling. Initially it was hoped that this tunneling could successfully ends an inflation but it was soon found that the randomness of the bubble formation when the false vacuum decayed would produced large inhomogeneities.

This problem was solved in the *new inflation scenario* proposed by Linde. In this theory the inflation is driven by an scalar field with the potential in the form in the form

$$V = -\frac{A}{2}\phi^2 + \frac{B}{4}\phi^4 \tag{4.78}$$

that has minima at $\phi = 0, V(0) = 0$ that is a false vacuum and also minima at $\phi_\pm = \pm\sqrt{\frac{A}{B}}$ with $V(\phi_\pm) = -\frac{A^2}{4B}$. This scalar field is called *inflaton*. If this theory the inflation is driven by the scalar field on the plateau of the potential energy diagram (region around the point $\phi = 0$). If this plateau is flat enough, such a state can be stable enough for successful inflation. Soon after the introduction of the new inflation scenario it was shown that the inflaton potential need not have either a local minimum or a gentle plateau: This new scenario is known as a *chaotic inflation*.

4.7 Solving the problems of standard cosmology

To demonstrate the fact that inflation can solve the problems of the standard cosmology let us again consider the potential with the simplest form

$$V(\phi) = \frac{1}{2}m^2\phi^2 \ . \tag{4.79}$$

With this potential the Friedmann equation takes the form

$$\dot{\phi} = -\frac{m^2\phi}{3H} \ , H = \frac{m}{\sqrt{6}M_P}\phi \tag{4.80}$$

and we find

$$\phi = \phi_0 - \sqrt{\frac{2}{3}}\frac{m}{M_P}t \tag{4.81}$$

and

$$a = C\exp[\frac{m}{\sqrt{6}M_P}(\phi_0 t - \frac{\sqrt{2}M_P}{2\sqrt{3}}t^2)] = a_0\exp[\frac{1}{4M_P^2}(\phi_0^2 - \phi^2)] \ . \tag{4.82}$$

The period of time during the solution above is valid ends at $t \sim \triangle t$ at which

$$a(\triangle t) \sim a(0)\exp(\frac{1}{\epsilon^2}) \ . \tag{4.83}$$

If we take a typical value for m for which $\epsilon < 10^{-4}$ we obtain

$$a(\triangle t) \sim a(0) \times 10^{2.7\times 10^8} \ . \tag{4.84}$$

This has remarkable consequence. A proper distance L_P at $t = 0$ will inflate to a size $10^{10^8} cm$ after a time $\triangle t \sim 5 \times 10^{-36}\ s$. As we know the size of observable Universe today is $H_0^{-1} \sim 10^{28}\ cm$. Therefore, only a small fraction of the original Planck length comprises today's entire observable Universe.

General arguments

Inflation is not really a theory, but instead it is a paradigm, or class of theories. Each specific model of inflation makes definitive predictions but the class of the models as a whole can be tested only by looking for generic features that are common for all models. Nevertheless, there are number of features of the Universe that seem to be characterize consequences of inflation. The basic arguments for inflation are as follows:

- *The Universe is big*

  We know that Universe is very large; the visible part of the Universe contains about 10^{90} particles. Most of scientists believe that the creation of Universe can be explained in scientific terms. Thus we think about the theory that could explain how the Universe got so be so big. Such a theory has to explain the number of particles, 10^{90} or more. Simple way to get such a huge number, with small number as an input, is for the calculation to involve an exponential. The exponential expansion of inflation can explain this huge number. Moreover, inflationary cosmology suggests that, even though the observed Universe is incredible large, it is only a small fraction of the entire Universe.

- *The Hubble Expanssion*

  In standard FRW cosmology the Hubble expanssion is part of the postulates that define the initial conditions. But the inflation offers the possibility of explaining how the Hubble expansion began.

- *Homogeneity and Isotropy*

  As we have shown before the degree of uniformity of Universe is starling. The intensity of the cosmic microwave background radiation is the same in all directions. The cosmic background radiation was released 400000 years after big bang after the Universe cooled enough so that the opaque plasma neutralized into a transparent gas. The cosmic background radiation photons have mostly been traveling on straight lines since then so they provide an image of what the Universe looked like at 40000 years after big bang. The observed uniformity of radiation therefore implies that the observed Universe had become uniform in temperature by that time. In standard FRW cosmology a simple calculation shows that the uniformity could be established so quickly if signals could propagate at about 100 times the speed of light a proposition clearly contradicting the known laws of physics.

  In inflationary cosmology the uniformity is easily explained. It is created initially on microscopic scales by thermal thermal equilibrium processes and then inflation takes over and stretches the regions of uniformity to become large enough to encompass the observed Universe and more.

- *Flatness problem*

  The problem concerns the value of the ration

  $$\Omega_{tot} \equiv \frac{\rho_{tot}}{\rho_0} \ , \qquad (4.85)$$

  where ρ_{tot} is total mass density of the Universe and where $\rho_0 = \frac{3H^2}{8\pi G}$ is the critical density that would make the Universe spatially flat (In ρ_{tot} the vacuum energy, it is nonzero, is included.)

There is now general agreement that Ω_{tot} lies in the range

$$0.1 \leq \Omega_0 \leq 2 \ , \tag{4.86}$$

but it was very hard to pinpoint the value with more precision. Despite this large range the value of Ω at early times is highly constrained, since $\Omega = 1$ is an unstable equilibrium point of the standard model evolutions. Thus, if Ω was exactly equal to one, it would remain exactly one forever. On the other hand if Ω differs slightly from one in the early Universe, that difference-whether positive or negative, would be amplified with time. More generally, it can be shown that $\Omega - 1$ grows as

$$\Omega - 1 \begin{cases} t & \text{(during the reaiation} - \text{dominated era)} \\ t^{2/3} & \text{(during the matter} - \text{dominated era)} \end{cases} \tag{4.87}$$

It was shown that at $t = 1s$ when the processes of big bang nucleosynthesis were just beginning, Ω must be equal to one to an accuracy of one part of 10^{15}. Classical cosmology cannot explain this fact. In the context of modern particle physics cosmology, where we try to push all thinks all the way back to Planck scale $10^{-43} sec$ the problem becomes even more severe.

While this extraordinary flatness of the early Universe has o explanation in classical FRW cosmology, it is a natural prediction for inflation cosmology. During the inflationary period, we have following relation

$$\Omega - 1 \approx e^{-2H_{inf}t} \ , \tag{4.88}$$

where H_{inf} is Hubble parameter during inflation. Thus, as long as there is a sufficient period of inflation, Ω can start at almost any value and it will be driven to unity by the exponential expansion. Moreover, recent observation favored value of Ω_0 to be equal to $\Omega_0 = 1.02 \pm 0.02$ according with recent WMAP results that is in beautiful agreement with inflation.

- *Absence of magnetic monopoles*

  All grand unified theories predict that there should be, in the spectrum of possible particles,extremely massive particles carrying a net magnetic charge. It was shown in the context of the standard cosmology that magnetic monopoles would be produced so strongly so that they would overweigh everything else in the Universe by a factor of about 10^{12}. Such a large mass density would cause that the Universe would come to its big crunch in about 30.000 years. Inflation is simplest known mechanism to eliminate monopoles from the visible Universe even though they are still in the spectrum of possible particles. The monopoles are eliminated simply due to the fact that inflation diluted them to a completely negligible level.

- *Anisotropy of the cosmic microwave background radiation*

  The process of inflation smooths the Universe completely. On the other hand the density fluctuations are generated as inflation ends by the quantum fluctuations of the inflaton field. The general properties of these fluctuations are that are adiabatic, Gaussian, and nearly scale-invariant.

4.8 Reheating and Preheating

The great strength of inflation is its ability to redshift away all unwanted relics, such as topological defects. However during this process radiation and dust-like matter are similarly redshifted away to nothing so that at the end of inflation the Universe contains nothing but the inflationary scalar field condensate. The question is how does the matter arise and how is the Universe reheated?

The problem of reheating is very complicated and complex. In fact, the theory of reheating of the Universe after inflation is the most important application of the quantum theory of particle creation since almost all matter constituting the Universe was created during this process.

Now we sketch the standard picture.

Inflation ends when the slow-roll conditions are violated and the field begins to fall towards the minimum of the potential. Initially all energy density is in the inflation however now this energy is damped by two possible terms. Firstly, the expanssion of the Universe naturally damps the energy density. Secondly, the inflation may decay into other particles, such as radiation or massive particles, both fermionic or bosonic. To describe this process one introduce a phenomenological decay term Γ_ϕ into the scalar field equation. For example, if we consider the fermions only, then the rough expression for how the energy density evolves is

$$\dot{\rho}_\phi + (3H + \Gamma_\phi)\rho_\phi = 0 \ . \tag{4.89}$$

It can be shown that the inflaton undergoes damped oscilations and decays into radiation that equilibrates rapidly at a temperature known as the *reheat temperature* T_{RH}.

More precisely, early theory of reheating of Universe after inflation were based on the idea that the homogeneous inflation field can be represented as a collection of the particles of the field ϕ. Put differently, we expect that inflation field has the same form as the ordinary quantum field in the flat spacetime. Then we can model reheating as a decay of each particle separately and this process can be studied in the standard perturbative description of particle decay.Typically, it takes thousands of oscillations of the inflaton field until it decays into usual elementary particles by this mechanism.

In case of bosons the situation is more complicated since now inflaton oscilations may give rise to parametric resonance that is characterised by an extremely rapid decay that results into distributions of products that are far from equilibrium and only much later settles down to an equilibrium distribution at energy T_{RH}. Such a decay due to the parametric resonance is known as *preheating*. The parametric resonance is an example of the coherent field effect that leads to the homogeneous field decay much faster than would be predicted by perturbative effects. These coherent effects produce high energy, nonthermal fluctuations that could have significance for understanding developments at the early Universe, as for example baryogenesis.

4.9 Quantum fluctuations

The key problem is how to test an inflation. The answer is the structure formation. As we have seen an important reason to involve an inflation is to make the Universe smooth and flat. However as we observe every day there is a large amount of structure in Universe. This structure can be traced back to subtle variations in the matter distribution during the time when the cosmic microwave background was released. The naive application of inflation in fact excludes such non-uniformity. It is a nice example of the application of the quantum field theory in curved background that explains the emergence of non-uniformity.

The main point is that inflation magnifies microscopic quantum fluctuation to cosmic size and hence provides seeds for structure formations. It is very interesting that then the details of physics at the highest energy scales is therefore reflected in the distribution of galaxies and other structures on large scales. More precisely, the fluctuations start at their smallest scales and grow larger (in wavelength) as the Universe expands. Eventually they become larger than the horizon and free. Intuitively, the different parts of wave can no longer communicate with each other since light can not keep up with the expanssion of Universe. This is a consequence of the fact that the scale factor grows faster than the horizon which is a defining property of an accelerating and inflating Universe. At a later time, when inflation stops, the scale factor will start to grow slower than the horizon and the fluctuations will eventually come back within the causal horizon.The fluctuations will then appear as acoustic waves in the plasma and hence they will affect the CMB.

Let us now study this problem in more details. We assume that metric as well as the inflaton can be split into a classical background piece and a piece due to fluctuations according to

$$\begin{aligned} g_{\mu\nu} &= g^{(0)}_{\mu\nu} + h_{\mu\nu}(\tau, \mathbf{x}) \ , \\ \phi &= \phi^{(0)} + \delta\phi(\tau, \mathbf{x}) \ , \end{aligned} \tag{4.90}$$

where for convenience we have introduced *conformal time* τ such that the metric is given by

$$ds^2 = a(\tau)^2(d\tau^2 - d\mathbf{x}^2) \ . \tag{4.91}$$

Since the background metric is homogenous it is convenient to Fourier transform the fluctuation mode $\delta\phi$ as

$$\delta\phi(\tau, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \ . \tag{4.92}$$

Since we can presume that fluctuation are small in magnitude we can neglect the potential term for the fluctuation mode $\delta\phi$ so that its equation of motion takes to form

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu \delta\phi \right] = 0 \tag{4.93}$$

that using the (4.91) takes the form

$$\frac{1}{a^2} \delta\phi'' + \frac{2a'}{a} \delta\phi' - \frac{1}{a^2} \partial_i \partial^i \delta\phi = 0 \ , \tag{4.94}$$

where $(\dots)' = \frac{d(\dots)}{d\tau}$. Finally, using (4.92) we obtain differential equation for mode $\delta\phi_{\mathbf{k}}$

$$\delta\phi''_{\mathbf{k}} + 2\frac{a'}{a} \delta\phi'_{\mathbf{k}} + k^2 \delta\phi_{\mathbf{k}} = 0 \ . \tag{4.95}$$

If we introduce the rescaled mode $\mu_{\mathbf{k}} = a\delta\phi_{\mathbf{k}}$ so that

$$\delta\phi'_{\mathbf{k}} = \frac{\mu'_{\mathbf{k}}}{a} - \frac{\mu_{\mathbf{k}} a'}{a^2} \ , \delta\phi''_{\mathbf{k}} = \frac{\mu''_{\mathbf{k}}}{a^2} - 2\frac{\mu'_{\mathbf{k}} a'}{a^2} - \frac{\mu_{\mathbf{k}} a''}{a^2} + 2\frac{\mu_{\mathbf{k}} (a')^2}{a^3} \tag{4.96}$$

the equation (4.95) can be transformed into

$$\mu''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a} \right) \mu_{\mathbf{k}} = 0 \ . \tag{4.97}$$

It can be shown that the metric fluctuations can be reduced to two polarizations obeying an equation identical to the one for the scalar fluctuations. In what follows we will consider the scalar fluctuations only.

To proceed let us presume that the conformal factor depend on conformal time as

$$a \sim \tau^{1/2-\nu} \ , \tag{4.98}$$

where ν is a constant. An important example is $a \sim e^{Ht}$ with $H = \text{const.}$ where the change of coordinates gives

$$\frac{d\tau}{dt} = \frac{1}{a(t)} = e^{-Ht} \Rightarrow e^{-Ht} = -H\tau \Rightarrow a(\tau) = -\frac{1}{H\tau} \ . \tag{4.99}$$

Comparing with (4.98) we find that $-1 = 1/2 - \nu \Rightarrow \nu = 3/2$. Note also that the physical range of τ is $-\infty < \tau < 0$. Using now (4.98) the equation for fluctuation (4.97) takes the form

$$\mu''_{\mathbf{k}} + \left(k^2 - \frac{1}{\tau^2} \left(\nu^2 - \frac{1}{4} \right) \right) \mu_{\mathbf{k}} = 0 \ . \tag{4.100}$$

It is nice that the equation given above has solution known as a Hankel function. The general solution is given by

$$f_{\mathbf{k}}(\tau) = \frac{\sqrt{-\tau\pi}}{2}\left(C_1(k)H_\nu^{(1)}(-k\tau) + C_2(k)H_\nu^{(2)}(-k\tau)\right) \ , \tag{4.101}$$

where $C_1(k)$ and $C_2(k)$ are to be determined by initial conditions.

When we quantize this system we need to introduce oscillators $a_{\mathbf{k}}(\tau)$ and $a^\dagger_{-\mathbf{k}}(\tau)$ such that

$$\begin{aligned} \mu_{\mathbf{k}} &= \frac{1}{\sqrt{2k}}\left(a_{\mathbf{k}}(\tau) + a^\dagger_{-\mathbf{k}}(\tau)\right) \ , \\ \pi_{\mathbf{k}} = \mu'_{\mathbf{k}}(\tau) + \frac{1}{\tau}\mu_{\mathbf{k}}(\tau) &= -i\sqrt{\frac{k}{2}}\left(a_{\mathbf{k}}(\tau) - a^\dagger_{-\mathbf{k}}(\tau)\right) \ , \end{aligned} \tag{4.102}$$

obey standard commutation relation. It is important to stress that these operators are time dependent and can be expressed in terms of oscillators at a specific moment in time using the Bogolubov transformations

$$\begin{aligned} a_{\mathbf{k}}(\tau) &= u_{\mathbf{k}} a_{\mathbf{k}}(\tau_0) + v_{\mathbf{k}}(\tau) a^\dagger_{-\mathbf{k}}(\tau_0) \ , \\ a^\dagger_{-\mathbf{k}}(\tau) &= u^*_{\mathbf{k}}(\tau) a^\dagger_{-\mathbf{k}}(\tau_0) + v^*_{\mathbf{k}}(\tau) a_{\mathbf{k}}(\tau_0) \ , \end{aligned} \tag{4.103}$$

where

$$|u_{\mathbf{k}}(\tau)|^2 - |v_{\mathbf{k}}(\tau)|^2 = 1 \tag{4.104}$$

Then we can write the quantum field $\mu_{\mathbf{k}}$ as

$$\mu_{\mathbf{k}}(\tau) = f_{\mathbf{k}}(\tau) a_{\mathbf{k}}(\tau_0) + f^*_{\mathbf{k}}(\tau) a_{-\mathbf{k}}(\tau_0) \ , \tag{4.105}$$

where

$$f_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2k}}(u_{\mathbf{k}}(\tau) + v^*_{\mathbf{k}}(\tau)) \tag{4.106}$$

is given in (4.101).

Now we come the key question that is *what are the initial conditions?* The ussual choice is to consider the infinite past and choose a state annihilated by the annihilation operator

$$a_{\mathbf{k}}(\tau_0)\left|0,\tau_0\right\rangle = 0 \ , \tag{4.107}$$

for $\tau_0 \to -\infty$. However there is great debate about this choice in the past and is commonly known as a *Problem of transplanckian physics.* However we will not discuss this issue in this section and we will continue according to common practise. From (4.102) we get that

$$\pi_{\mathbf{k}}(\tau_0)\left|0,\tau_0\right\rangle = -1\sqrt{\frac{k}{2}} a^\dagger_{-\mathbf{k}}\left|0,\tau_0\right\rangle = -ik\mu_{\mathbf{k}}(\tau_0)\left|0,\tau_0\right\rangle \ . \tag{4.108}$$

Since the Henkel functions behave as for $\tau_0 \to -\infty$

$$\begin{aligned} H_\nu^{(1)}(-k\tau) &\sim \sqrt{-\frac{2}{k\tau\pi}} e^{-ik\tau} \ , \\ H_\nu^{(2)}(-k\tau) &\sim H_\nu^{(1)*}(-k\tau) \ , \end{aligned} \tag{4.109}$$

we find that the vacuum choice corresponds to $C_2(k) = 0$ and $|C_1(k)| = 1$.

In summary we have determined the quantum fluctuation and now we would like to see how they act on CMB. To do this we compute the size of the fluctuation according to

$$P(k) = \frac{4\pi k^3}{(2\pi)^3} \left\langle |\delta\phi_{\mathbf{k}}|^2 \right\rangle = \frac{k^3}{2\pi^2} \frac{1}{a^2} \left\langle |\mu_{\mathbf{k}}|^2 \right\rangle = \frac{k^3}{2\pi^2} \frac{1}{a^2} |f_{\mathbf{k}}|^2 = \frac{k^3}{2\pi^2} \frac{1}{a^2} \frac{|-\pi\tau|}{4} |H_\nu^{(1)}(-k\tau)|^2 \tag{4.110}$$

where $\langle(\ldots)\rangle$ mean the vacuum expectation value with respect to the sate $|0, \tau_0\rangle$. Note that we are working in Heisenberg representation where the quantum mechanical operators evolve with time while states not.

Now we should calculate (4.110) at late times, namely $\tau \to 0$. In this limit the Hankel function behaves as

$$H_\nu^{(1)}(-k\tau) \sim \sqrt{\frac{2}{\pi}} (-k\tau)^{-\nu} \tag{4.111}$$

and hence (4.110) for $\tau \to 0$ takes the form

$$P \sim \frac{1}{4\pi^2} \frac{1}{a^2} (-\tau)^{1-2\nu} k^{3-2\nu} \sim \frac{1}{4\pi^2} H^2 k^{3-2\nu} \ . \tag{4.112}$$

For $\nu = 3/2$ and for slow roll when H for $\tau \to 0$ is almost constant we can set the scale of the fluctuations. In fact, we find the well known scale invariant spectrum for $\nu = 3/2$

$$P = \frac{1}{4\pi^2} H^2 \ . \tag{4.113}$$

It can be shown that this is more or less the whole story in case of the gravitational, or tensor, perturbations. The scalar fluctuations obey similar equation

$$P_s \sim \left(\frac{H}{\dot{\phi}} \right)^2 \frac{1}{4\pi^2} H^2 \ . \tag{4.114}$$

Ussualy we express the deviation from the scale invariance by introducing spectral indices according to

$$\begin{aligned} n_s - 1 &= \frac{d \ln P_s}{d \ln k} = 3 - 2\nu_s \ , \\ n_T &= \frac{d \ln P_T}{d \ln k} = 3 - 2\nu_T \ , \end{aligned} \tag{4.115}$$

where ν_s refers to the scalar perturbations and ν_T refers to the gravitational, or tensor perturbations. While not clear from our simplified analysis, the $\nu's$ need not be the sam in the two cases. Observations show that n_s is very close to 1 consistent with the basic idea of inflation. It is extreme important to find any slight deviation from the scale invariant vale which could give important information about the inflationary potential.

In fact, the flatness of the spectrum of density fluctuations, together with flatness of the Universe $\Omega = 1$ constitute the two most robust predictions of inflationary cosmology. On the other hand there is an important difference between the prediction of flatness of the Universe and the flatness of the spectrum of perturbations of metric. It is difficult (though possible) to construct an inflationary model deviating from the prediction $\Omega = 1$. On the other hand the situation with the flatness of the spectrum is opposite: It is very difficult (though possible) to construct a model with an exactly flat spectrum of perturbations of metric. In this sense, existence of a small deviation of the spectrum of inflationary perturbations from the flat spectrum (i.e. breaking of the scale invariance of the spectrum) represents an additional robust prediction of inflation.

4.10 Eternal Inflation

The eternal inflation scenario is based on the discovery of the process of self-reproduction of inflationary Universe.In fact, this process exists in old inflationary theory and in the new one but its significance was appreciated after discovery of eternal inflation in the simplest versions of the chaotic inflation scenario.

In the case of the new inflation, the exponential expansion occurs as the scalar field rolls from the false vacuum state at the peak of the potential energy towards to the true vacuum. Remarkably, it was shown very briefly after introduction of this model that the new inflation scenario is generically eternal. The key point is that, even though classically the field would roll off the hill, quantum mechanically there is always an amplitude for it to remain at the top.

The time scale for the decay of the false vacuum is controlled by

$$m^2 = - \left. \frac{\partial^2 V}{\partial^2 \phi} \right|_{\phi=0} , \tag{4.116}$$

which is the negative mass-squared of the scalar field when it is at the top of the hill on the potential. This is a free parameter of each model but m has to be small compared to Hubble constant or lese the model does not lead to enough inflation.

In other words, for parameters choosen so that the inflation works, the exponential decay of false vacuum is slower than an exponential expanssion. Even if the false vacuum is decaying, the expansion outruns the decay and the total volume of false vacuum actually increases with time rather than decreases. Thus inflation does not end at all places at once,instead it ends at localized patches, in a succession that continues at infinitum. Each patches essentially a whole Universe so that it can be said that inflation produces not just one Universe but an infinite number of Universes.

In the context of the chaotic Universe models the situation is slightly subtle even if it was shown by A. Linde that these models are eternal as well.We know that inflation occurs as the scalar field rolls down a hill of the potential energy diagram. As the field rolls down the hill quantum fluctuations will be superimposed on top of the classical motion. The best way to think about this is to ask what happens during one time interval of duration $\triangle t = H^1$ (Hubble time) in a region of one Hubble volume H^3. Suppose that ϕ_0 is the average value of ϕ in this region at the start of the interval. By definition of a Hubble time the rate of the expanssion is given by

$$a(t + \triangle t)/a(t) = e^{H\triangle t} = e \ . \tag{4.117}$$

This means that the change of volume is

$$V(t + \triangle t)/V(t) = a^3(t + \triangle t)H^{-3}/(a^3(t)H^{-3}) = e^3 \tag{4.118}$$

Since $e^3 \approx 20$ we see that volume will expand by a factor 20. Since correlations are extended typically over one Huble length if follows that in the end of the Hubble time the initial Hubble size region grows and breaks up into 20 independent Hubble sized regions.

During the time interval $\triangle t$ the classical field ϕ is rolling down the hill. On the other hand the classical change in the field $\triangle\phi_{cl}$ during the time interval $\triangle t$ is going to be modified by quantum fluctuations $\triangle\phi_{qu}$ which can drive the field upwards or downward relative to classical trajectory. For any one of the 20 regions at the end of the Hubble time we can describe the change of the field as

$$\triangle\phi = \triangle\phi_{cl} + \triangle\phi_{qu} \ . \tag{4.119}$$

In the crude approximation the fluctuation is treated as a free quantum field. This fact implies that $\triangle\phi_{qu}$ the quantum fluctuation averaged over one of the 20 Hubble volumes at the end, will have a Gaussian probability distribution, with a with of order $H/2\pi$. Then there is then a probability that the sum of the two terms on the right hand side will be positive-that the scalar field will fluctuate up instead down. As long as the probability is bigger than 1 in 20 then the number of inflating regions with $\phi > \phi_{cl}$ will be larger at the end of the interval than at the beginning. This process will then go on forever so inflation will never end.

We see that the condition for an existence of eternal inflation is that the probability for the scalar field to go up must be bigger than $1/e^3 \approx 1/20$. It can be shown that criterion implies the relation

$$\frac{H^2}{\dot{\phi}_{cl}} > 3.8 \tag{4.120}$$

The probability that $\triangle\phi$ is positive tends to increase as one considers larger and larger values of ϕ so that sooner or later one reaches the point when the inflation becomes eternal. In fact for that reason we think that inflation is almost always eternal.

The eternal inflation follows from the observation that in many models large quantum fluctuations that are produced during inflation may locally increase the value of the energy density in some parts of the Universe. These reasons then expand at a greater rate than their parent domains and quantum fluctuations in them lead to production of new inflationary domains which expand even faster. This leads to an eternal process of self-reproduction of the Universe.

In order to understand the process of self-reproduction we should remember that the processes separated by distances l greater than H^{-1} proceed independently one another. This is a consequence of the fact that during an exponential expanssion the distance between any two objects separated by more than H^{-1} is growing with speed exceeding the speed of light. Then an observer in the inflationary Universe can see only the processes occurring inside the horizon of radius H^{-1}. In this sense any inflationary domain of initial radius exceeding H^{-1} can be considered as a separate mini-Universe.

In order to study the behavior of such a mini-Universe we should take into account the quantum fluctuations. Let us consider an inflationary domain of initial radius H^{-1} containing sufficient homogeneous field with initial value $\phi \gg M_p^2$. From the basic equation of the inflation model

$$H = \frac{m\phi}{\sqrt{6}} \ , \dot{\phi} = -m\sqrt{\frac{2}{3}} \tag{4.121}$$

we can deduce that during time interval $\triangle t = H^{-1}$ the field inside the domain will be reduced by $\triangle\phi$ that follows from the second equation above

$$\frac{\triangle\phi}{\triangle t} = -m\sqrt{\frac{2}{3}} \Rightarrow \triangle\phi = -m\sqrt{\frac{2}{3}}H^{-1} = -\frac{2}{\phi} \ , \tag{4.122}$$

where in the second step we have used the first equation in (4.121). On the other hand it can be shown that the quantum fluctuation of the field ϕ is

$$|\delta\phi(x)| \approx \frac{H}{2\pi} = \frac{m\phi}{2\pi\sqrt{6}} \ . \tag{4.123}$$

Then we see that the magnitude of quantum fluctuation is larger than $\triangle\phi$ for

$$\frac{m\phi^*}{2\pi\sqrt{6}} \approx \frac{2}{\phi^*} \Rightarrow \phi^* \sim \frac{5}{\sqrt{m}} \tag{4.124}$$

Then for $\phi \ll \phi^*$ the decrease of the field ϕ due to the classical motion is much greater than the average amplitude of the quantum fluctuations $\delta\phi$ generated during the same time. On the other hand for $\phi \gg \phi^*$ one has $\delta\phi(x) \gg \triangle\phi$. Since the typical wave length of the fluctuation mode is $\sim H^{-1}$ it turns out that the whole domain after the time $\triangle t = H^{-1}$ divides into following number of domain with almost homogenous field

$$a(\triangle t)H^{-1}/H^{-1} = e^{3HH^{-1}} \sim 20 \qquad (4.125)$$

where the first expression express the physical size of the domain divided wave length. In summary, we get 20 separated domains of size H^{-1}, each containing almost homogenous field $\phi - \triangle\phi + \delta\phi$. In almost half of these domains the field ϕ grows by $|\delta\phi(x)| - \triangle\phi \approx H/2\pi$ rather than decreases. This means that the total volume of the Universe containing *growing* field ϕ increases 10 times. During the next time interval $\triangle t = H^{-1}$ this process repeats. Thus, after the two time intervals H^{-1} the total volume of the Universe containing the growing scalar field increases 100 times. In other words the Universe enters eternal process of self-reproduction.

One should however be careful with interpretation of this result. There is still an ongoing debate of whether eternal inflation is eternal only in the future or also in the past. To see this precisely where is the problem let us consider any particular time-like geodetic line at the stage of inflation. For any given observer following this geodetic the duration t_i of the stage of inflation on this geodesic will be finite. On the other hand eternal inflation implies that if one takes all such geodesics and calculate the time t_i for each of them, then there will be no upper bound for t_i. In other words for each time T there will be such geodesic which experience inflation for the time $t_i > T$.

Similarly, if we study any particular geodesic in the past time direction, one can prove that it has finite length. In other words, the inflation n any particular point in the Universe should have a beginning at some time τ_i. However there is no reason to expect that there is an upper bound for all τ_i on all geodesics. If this upper bound does not exist, then eternal inflation is eternal not only in the future but also in the past.

Put differently, there is a beginning for each part of the Universe and there will be an end for inflation at any particular point. But there will be no end for the evolution of Universe as a whole in the eternal inflation scenario and at present we do not have any reason to believe that there was a single beginning of the evolution of the whole Universe at some moment $t = 0$ which was traditionally associated with Big Bang.

If this scenario is correct, then physics alone cannot provide a complete explanation for all properties of our part of the Universe.

4.11 Eternal Inflation: Implications

Even if the other Universes that are created during the eternal inflation are too remote to imagine observing directly we will see that an eternal inflation has real consequences in terms of the way we extract predictions from theoretical models.

Firstly, the eternal inflation implies that all hypothesis about initial conditions for the Universe, such as the Hartle and Hawking no boundary proposal, the tunneling proposals by Vilekin or Linde become totally divorced from observation. This follows from the presumption of the eternal inflation with its infinite production of pocket Universes. Then one can expect that the statistical properties of inflating region should approach a steady state which is independent on initial condition. Unfortunatelly there are great problems with the study of this steady state, for example, the properties of this state seems to depend crucially on the super-Planckian physics which we do not understand at present. It is however possible that string theory could be helpful with this study. More precisely, the same quantum fluctuations that make eternal inflation possible tend to drive the scalar field further and further up to potential energy curve so that some attempts that wanted to quantity the steady state require the imposition of some kind of a boundary condition at large ϕ.

Even if the Universe forgets the details of its genesis the question, how the Universe began still remain interesting. To see this note that eternally inflating Universes continue forever once they start they are apparently not eternal into the past. [12]

The second consequence of the eternal inflation is that the probability of the onset of inflation becomes totally irrelevant provided that the probability is not identically zero. In fact, this observation is slightly in the clash with our previous claim that chaotic inflation gives better result that the new inflation scenario. Even if the initial conditions necessary for the new inflation scenario cannot be justified on the basis of the thermal equilibrium as was proposed in original papers, in the context of the eternal inflation it is sufficient to conclude that the probability for the required initial conditions is nonzero.

The third consequence of the eternal inflation is the possibility that it offers to rescue the predictive power of theoretical physics. Here we mean the status of M-theory. Even if this theory by itself has uniqueness it appears that the vacuum is far from unique. Since the predictions will depend on the properties of the vacuum, the predictive power of M-theory could be limited. Eternal inflation however provides a possible mechanism to remedy this problem since it might help to constrain the vacuum state of the real Universe and hopefully significantly enhance the predictive power of M-theory. We must however stress that this is pure speculation whose validity is not justified but one can hope that recent works in the context of the string theory landscape could bring new light on this conjecture.

[12] This remark implies that the word “eternal” is not technically correct, we should rather speak about “semi-eternal” or “future-eternal” Universe.

4.12 Does Inflation Need a Beginning

We know that according to the inflation scenario is eternal in the future. Than a natural question arrives: Is it possible that the inflation is eternal into the past? There is a nice theorem by Borde, Guth and Vlenkin (2003) that proves that the answer to this question is no. There is of course no conclusion that an eternally inflating model must have a unique beginning and no conclusion that there is an upper bound on the length of all backwards-going geodesics from a given point. In other words this theorem shows that some new physics would be needed do describe the past boundary of the inflating region.

4.13 Inflation and Observations

It is very nice that inflation can make prediction which can be tested by cosmological observations. The inflationary prediction for nearly flat spectrum of density perturbation is in agreement with both you measurements of the CMB anisotropy and observations of structures in the Universe.

Let us also give another example where the inflation cosmology gives very nice explanation of the observation date.

Today,we have three-dimensional map of the distribution of galaxies in space that contain more than one hundred thousand galaxies.They clearly indicate that the luminous matter in the Universe is neither uniformly nor randomly distributed. We see clusters of galaxies,superclusters, filaments and voids that are regions of space empty of galaxies. The distribution can be quantified in terms of the luminosity power spectrum.

As we have also seen another observation window in cosmology is the cosmic microwave background radiation. This radiation is characterised by a surprising isotropy, in other words it looks the same from all different directions on the sky. However this radiation has also fractional level of a bit less than 10^{-4} of anisotropies. These anisotropies can be characterised in terms of their angular power spectrum. The sky map (that is clearly two-dimensional of topology of sphere) of anisotropies is expanded in spherical harmonics Y_{lm}

$$\frac{\triangle T}{T}(\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\phi,\theta) \ , \tag{4.126}$$

where θ, ϕ are the usual angles on the surface of two-sphere. It can be shown that the angular power spectrum of CMB has characteristic pattern of anisotropies. The challenge of cosmology is to explain both the overall isotropy of CMB and the specific patter of anisotropies.

In order to explain these observation structures we have to look to the very early Universe. The reason is that the Standard Big Bang cosmology that describes the cosmological evolution at late times where the notion "late times" means the times that includes period of nucleosynthesis and later implies that the length scales that are currently observed were outside the Hubble radius in the early times and no causal structure formation scenario is possible.

It is great success of inflationary cosmology that can explains all problems we listed above and also provides a causal mechanism for the origin of inhomogeneities in the Universe.

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