$y'' + py' + qy = f(x)$

$y'' + py' + qy = 0$

λ_1, λ_2

y_0

$y = y_0 + y_p$

a) $f(x) = P_m(x)$ $\quad x,\ x+2,\ 2x-3,\ x^7,\ x^2+x-1,\ 1$

$y_p = x^k Q_m(x)$ $\quad k=0$ ✓ $\quad k=1$ ✓ je nula kořenem rovnice ?

b) $f(x) = P_m(x)\, e^{\alpha x}$ je α kořenem ch. rovnice ?

$y_p = x^k Q_m(x) e^{\alpha x}$, k ... je násobnost kořene α

1) $y'' - y = x$

$f(x) = x$

$y'' - y = 0$ ✓

$\lambda^2 - 1 = 0$ ✓

$(\lambda-1)(\lambda+1) = 0$

$\lambda_1 = 1,\ \lambda_2 = -1$ ✓

$y_0 = C_1 e^x + C_2 e^{-x}$ ✓

$y_p = Ax + B$ ✓

$y_p' = A$

$y_p'' = 0$

$0 - (Ax + B) = x$

$-Ax - B = x$

$-A = 1 \quad -B = 0$

$A = -1 \quad B = 0$

$y_p = -x$ ✓

$y = C_1 e^x + C_2 e^{-x} - x$ ✓

2) $y'' - y' = x$

$f(x) = x$

$y'' - y' = 0$

$\lambda^2 - \lambda = 0$

$\lambda(\lambda-1) = 0$

$\lambda_1 = 0,\ \lambda_2 = 1$

$y_0 = C_1 + C_2 e^x$

$y_p = x \cdot (Ax + B)$

$y_p = Ax^2 + Bx$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$2A - (2Ax + B) = x$

$-2Ax + 2A - B = x$

$-2A = 1 \quad 2A - B = 0$

$A = -\frac{1}{2} \quad 2\cdot(-\frac{1}{2}) - B = 0$

$B = -1$

$y_p = -\frac{1}{2}x^2 - x$

$y = C_1 + C_2 e^x - \frac{1}{2}x^2 - x$

3) $y'' - 3y' + 2y = 2x+5$, $y(0) = 4$, $y'(0) = 0$

$y'' - 3y' + 2y = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-1)(\lambda-2) = 0$

$\lambda_1 = 1, \ \lambda_2 = 2$

$y_0 = C_1 e^x + C_2 e^{2x}$

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$0 - 3A + 2(Ax+B) = 2x+5$

$2Ax + (-3A+2B) = 2x+5$

$2A = 2 \rightarrow A = 1$

$-3 + 2B = 5 \rightarrow B = 4$

$\boxed{y_p = x+4}$

$y = y_0 + y_p = \boxed{C_1 e^x + C_2 e^{2x} + x + 4}$ $y(0)=4$, $y'(0)=0$

$y' = C_1 e^x + 2C_2 e^{2x} + 1$

$4 = C_1 e^{0x} + C_2 e^{2\cdot 0x} + 0 + 4$

$4 = C_1 + C_2 + 4$

$C_1 + C_2 = 0$

$C_1 = -C_2$

$0 = C_1 e^{0x} + 2C_2 e^{0x} + 1$

$0 = C_1 + 2C_2 + 1$

$0 = -C_2 + 2C_2 + 1$

$C_2 = -1$

$C_1 = 1$

$\boxed{y = e^x - e^{2x} + x + 4}$

5) $y'' + 5y' + 6y = 1$

$y'' + 5y' + 6y = 0$

$\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+2)(\lambda+3) = 0$

$\lambda_1 = -2, \ \lambda_2 = -3$

$\boxed{y_0 = C_1 e^{-2x} + C_2 e^{-3x}}$

$y_p = A$

$y_p' = 0$

$y_p'' = 0$

$0 + 5\cdot 0 + 6A = 1$

$A = \frac{1}{6}$

$\boxed{y_p = \frac{1}{6}}$

$y_{b2}' = 0, \ y_{b2}'' = 0$

$0 + 0 + 6\cdot\frac{1}{6} = 1$

$\boxed{y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{6}}$

5) $y'' - 3y' + 2y = 12e^{3x}$ $\quad f(x) = 12e^{3x} \quad \alpha = 3$

$y'' - 3y' + 2y = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$y_0 = C_1 e^x + C_2 e^{2x}$

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$9Ae^{3x} - 3(3Ae^{3x}) + 2(Ae^{3x}) = 12e^{3x}$

$2Ae^{3x} = 12e^{3x}$

$A = 6$

$y_p = 6e^{3x}$

$\boxed{y = C_1 e^x + C_2 e^{2x} + 6e^{3x}}$

6) $y'' - y = e^{-x}$ $\quad f(x) = e^{-x} \quad \alpha = -1$

$\lambda^2 - 1 = 0$

$\lambda_1 = 1, \lambda_2 = -1$

$\boxed{y_0 = C_1 e^x + C_2 e^{-x}}$

$y_p = x \cdot Ae^{-x}$

$y_p' = Ae^{-x} + Ax \cdot (-e^{-x})$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x}$

$-Axe^{-x}$

$-Ae^{-x} - Ae^{-x} + Axe^{-x} - xAe^{-x} = e^{-x}$

$-A - A + Ax - Ax = 1$

$-2A = 1$

$A = -\frac{1}{2}$

$\boxed{y_p = -\frac{1}{2}xe^{-x}}$

$\boxed{y = C_1 e^x + C_2 e^{-x} - \frac{1}{2}xe^{-x}}$

7) $y'' - 2y' + y = e^x$ $\quad f(x) = e^x \quad \alpha = 1$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda_{1,2} = 1$

$\boxed{y_0 = C_1 e^x + C_2 xe^x}$

$y_p = x^2 Ae^x$

$y_p' = 2Axe^x + Ax^2 e^x$

$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2 e^x$

$2Ae^x + 4Axe^x + Ax^2 e^x - 4Axe^x - 2Ax^2 e^x + Ax^2 e^x = e^x$

$2Ae^x = e^x$

$2A = 1$

$A = \frac{1}{2}$

$\boxed{y_p = \frac{1}{2}x^2 e^x}$

$\boxed{y = C_1 e^x + C_2 xe^x + \frac{1}{2}x^2 e^x}$

8) $y' - 1y = x$

$\lambda - 1 = 0$

$\lambda = 1$

$y_0 = Ce^x$

$y_p = Ax + B$

$y_p' = A$

$A - Ax - B = x$

$A - B = 0 \quad -A = 1$

$A = -1$

$B = -1$

$y_p = -x - 1$

$$\boxed{y = Ce^x - x - 1}$$

$y' - y = 0$

$\frac{dy}{dx} = y$

$\int \frac{dy}{y} = \int dx$

$y_0 = Ce^x$

$y = C(x) \cdot e^x$

$y' = C'e^x + Ce^x$

$C'e^x + Ce^x - Ce^x = x$

$C'e^x = x$

$C'(x) = xe^{-x}$

$C(x) = \int xe^{-x}dx = \left| \begin{matrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{matrix} \right|$

$= -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$

$y = (-xe^{-x} - e^{-x} + C)e^x = \underline{Ce^x - x - 1}$