Exercise sheet 3

1. Is the class of ordinals in bijection with the class of cardinals? Prove or confute it.
2. Compute the cardinal numbers $\beth_0^{\beth_2^{\beth_1}}$, and $\beth_\omega \cdot \aleph_\omega$.
3. Compute the ordinal numbers $\omega_2^{\omega_3}$ and $(\omega \cdot \omega)^\omega \cdot \omega_1$.
4. Using Cantor normal forms, decide whether the ordinals $\omega^{\omega_1} \cdot \omega_1$ and $\omega_1^{\omega_1}$ are equal.
5. Define the function f from ordinals to cardinals as follows:

   - $f(0) = 0$;
   - $f(\alpha + 1) = \aleph_{f(\alpha)}$;
   - if α is a limit ordinal, $f(\alpha) = \sup_{\beta<\alpha} f(\beta)$.

   Show that a cardinal κ is weakly inaccessible if and only if $f(\kappa) = \kappa$.
6. Recall the definition of transitive closure of a set X as:

   - $X_0 = X$;
   - $X_{n+1} = \bigcup X_n$;
   - $TC(X) = \bigcup_{n\in\mathbb{N}} X_n$.

   Show that this really is the transitive closure of X, i.e. if Y is a transitive set such that $X \subseteq Y$ then $TC(X) \subseteq Y$.
   Conclude that X is transitive if and only if $X = TC(X)$.
7. Assume that there is a huge cardinal. Can we conclude that Vopěnka's principle is provable in ZFC set theory?