A global property of plane curves: rotation index

Definition. A subset $C \subseteq E_2$ is called **immersed curve of the class** C^r, $r \geq 1$ if there exists a regular motion $f : I \to E_2$ tdy C^r such that $C = f(I)$ for some open interval $I \subseteq \mathbb{R}$.

In immersed curve $C \subseteq E_2$ is called **immersed curve of the class** C^r if there exists a parametrization $f : [a,b] \to E_2$, $a, b \in \mathbb{R}$ such that $f([a,b]) = C$, $f(a) = f(b)$ and $f|_{(a,b)} \to E_2$ is a regular motion of the class C^r such that $f_+^{(i)}(a) = f_-^{(i)}(b)$, $i \leq r$.

If moreover maps $f|_{[a,b)}$ and $f|_{(a,b]}$ are injective then C is called **simple closed immersed curve**.

For simplicity, we shall just talk about *closed* and *simple closed* curves (which will be implicitly assumed to be immeresed). In this setting we shall introduce a new definition of the curvature for which we shall consider E_2 as *oriented* Euclidean space:

Definition. At each point of the curve $f(t)$ we define *oriented Frenet frame* $(f(t); e_1(t), \bar{e}_2(t))$ as follows: $e_1(t) = \frac{f'(t)}{||f'(t)||}$ and $(e_1(t), \bar{e}_2(t))$ is a positive orthonormal basis. Assuming $f(s)$ is the arc-length parametrization, we call the number $\bar{\kappa}(s) \in \mathbb{R}$ satisfying $e_1'(s) = \bar{\kappa}(s)\bar{e}_2(s)$ *oriented curvature* at the point $f(s)$.

Frenet formulae are similar to the non-oriented version: $e_1'(s) = \bar{\kappa}(s)\bar{e}_2(s)$ and $e_2'(s) = -\bar{\kappa}(s)\bar{e}_1(s)$. Note the oriented curvature could be negative (think about examples).

Proposition. Let $f : [a,b] \to E_2$ be closed curve $C$of the class C^r. Then there is a function $\theta : [a,b] \to \mathbb{R}$ of the class C^r such that $e_1(t) = (\cos\theta(t), \sin\theta(t))$ which satisfies $\theta'(t) = \bar{\kappa}(t)||f'(t)||$. Moreover, the difference $\theta(b) - \theta(a)$ is independent on the choice of θ.

Proof. Existence of θ is obvious: choose $\theta(a)$ such that $e_1(a) = \big(\cos\theta(a), \sin\theta(a)\big)$ and then extend θ continuously to the interval $[a,b]$. More precisely, using the arc-length parametrization we have $\cos\theta(s) = (e_1(s), \varepsilon_1)$ and $\sin\theta(s) = -(\bar{e}_2(s), \varepsilon_1)$ where ε_1 is the first vector of the standard basis. Then $\theta(s)$ is of tha class C^r. (Do we need both previous relations? Why?) By differentiating we obtain $\theta'(s) = \bar{\kappa}(s)$. Reparametrization $s = s(t)$, $\frac{ds}{dt} > 0$ the yields $\theta'(t) = \bar{\kappa}(t)||f'(t)||$.

To show independence of the difference $\theta(b) - \theta(a)$ on the choice of θ, assume $\varphi(t)$ is another function satisfying the proposition. Then $\theta(t) - \varphi(t) = 2k(t)\pi$ for some continuous function $k(t) \in \mathbb{Z}$. Thus $k(t)$ is a constant. □

Thus the difference $\theta(b) - \theta(a)$ is obtained using a parametrization $f : [a,b] \to E_2$ of an immersed closed curve (but is independent on a reparametrization). Consider e.g. various parametrizations of the circle $(\cos t, \sin t)$ where either $t \in [0, 2\pi]$ or $t \in [0, 4\pi]$ etc.

Definition. The number $n_C := \frac{1}{2\pi}\big[\theta(b) - \theta(a)\big]$ is called *rotation index* of the closed curve uzavřené křivky C from the proposition.

Example. The curve $f(t) = (\cos 2\pi t, \sin 2\pi t)$ for $t \in [0, m]$, $m \in \mathbb{N}$ has the rotation index m. (This curve is of course a circle.) What are examples of closed curves with negative rotation index?

Theorem. *It holds* $n_C = \frac{1}{2\pi}\int_a^b \bar{\kappa}(t)||f'(t)||dt$.

Moreover, n_C *is independent on a reparametrization preserving the orientation. Reparametrizations changing orientation change the sign of* n_C.

Proof. The first part of the proof follows from the relation $\theta'(t) = \bar{\kappa}(t)||f'(t)||$. (In)dependence on the reparametrization $t = t(\tau)$ follows from the form of the integral on the right hand side after substituion $t = t(\tau)$. □

Recall convex subset $T \subseteq \mathbb{R}^2$ satisfies pak $\overline{x_1x_2} \subseteq T$ for each $x_1, x_2 \in T$. Here $\overline{x_1x_2}$ denotes the segment with endpoints x_1 and x_2.

Lemma. Let $T \subseteq \mathbb{R}$ be a convex set and $e : T \to S^1$ be a function of the class C^r. Then there exists a function $\theta : T \to \mathbb{R}$ of the class C^r satisfying $e(x) = (\cos\theta(x), \sin\theta(x))$, $x \in T$. Moreover, if $\theta(x)$ and $\varphi(x)$ are two such functions then they differ by $2k\pi$ for some $k \in \mathbb{Z}$.

Here we denote by S^1 a circle. Note this technical lemma (stated without proof) is a two-dimensional proof of the proposition

The following theorem is the main result of this section:

Theorem (Hopf's Umlaufsatz)**.** *If* $f : [a,b] \to E_2$ *is a simple closed curve* C *then* $n_C = \pm 1$.

The opposite implication does not hold. Why?

Proof. We can assume $a = 0$ and that f is arc-length parametrization. Put $\Delta = \{(s,t) \mid 0 \leq s \leq b\} \subseteq \mathbb{R}^2$ and define the function $h : \Delta \to S^1$ as follows:

$$h(s,t) = \begin{cases} e_1(s) & s = t \\ -e_1(0) & (s,t) = (0,b) \\ \frac{f(t)-f(s)}{||f(t)-f(s)||} & \text{otherwise.} \end{cases}$$

The set Δ is convex and function $h(s,t)$ is continuous. Further we can assume that $f(0) = (0,0)$ and that this is the "lowest" point on the curve, i.e. that this point has minimal y-coordinate. Then $e_1(0)$ is (up to the sign) first vector of the standard basis, i.e. $e_1(0) = \pm\varepsilon_1$. Further we shall assume $e_1(0) = \varepsilon_1$ (which might change the orientation of C).

It follows from the lemma that $h(s,t) = \big(\cos\tilde{\theta}(s,t), \sin\tilde{\theta}(s,t)\big)$ for continuous function $\tilde{\theta} : \Delta \to \mathbb{R}$. Using $\theta(s)$ from the proposition, then the theorem sayes that

$$n_C = \frac{1}{2\pi}\int_a^b \bar{\kappa}(s)ds = \frac{1}{2\pi}\big(\theta(b) - \theta(0)\big) = \frac{1}{2\pi}\big(\tilde{\theta}(b,b) - \tilde{\theta}(0,0)\big) =$$
$$= \frac{1}{2\pi}\big[\big(\tilde{\theta}(0,b) - \tilde{\theta}(0,0)\big) + \big(\tilde{\theta}(b,b) - \tilde{\theta}(0,b)\big)\big].$$

Here $N_1 := \tilde{\theta}(0,b) - \tilde{\theta}(0,0)$ is the angle which measures the change of the radious vector. Hence $N_1 = \pi$ since the curve lies in the upper half-plane. Similarly, the angle $N_2 = \tilde{\theta}(b,b) - \tilde{\theta}(0,b)$ measures the change of the vector opposite to the radius vector, i.e. $N_2 = \pi$. Thus $n_C = 1$. □