Thm. 1.42 Suppose G is a Lie group and $H \subseteq G$ a closed subgroup. Then the homogeneous space G/H admits a unique structure of a smooth mfd. s.t. $\pi : G \longrightarrow G/H$ is a submersion (smooth + $T_g\pi$ is surj. $\forall g \in G$).

In particular, $\dim(G/H) = \dim(G) - \dim(H)$.

Moreover, $\ell : G \times G/H \longrightarrow G/H$, $\ell(g', gH) = g'g H$, is a smooth left-action of G on G/H.

Proof. By Thm. 1.28, H is a Lie subgroup of G.
We write $\mathfrak{h} \subseteq \mathfrak{g}$ for the Lie subalgebra corresp. to H.

Let us construct a smooth atlas for G/H.

Choose a vector space compliment $\mathfrak{k}$ of $\mathfrak{h}$ in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}.$$

Consider
$$\begin{aligned} \varphi : \mathfrak{k} \oplus \mathfrak{h} &\longrightarrow G \\ (X, Y) &\longmapsto \exp(X)\exp(Y). \end{aligned}$$

By the proof of Thm. 1.29, we know that $\exists$ open neighbhds W of 0 in $\mathfrak{k}$ with $\underline{\exp(W) \cap H = \{e\}}$ and an open neighbhd. of 0 in $\mathfrak{h}$ s.t. $\varphi|_{W \times V} : W \times V \longrightarrow U$)

is a diffeomorphism onto an open neighbhd. U' of e in G.

By possibly shrinking W, we may also assume that for $X_1, X_2 \in W$
$\underline{\exp(X_1)^{-1}\exp(X_2)} \in U'$ (by continuity).

Now consider

$$F : \mathfrak{a} \times H \longrightarrow G$$

$$F(X,h) := \exp(X)h$$

<u>Claim</u>: $F\big|_{W\times H} : W\times H \longrightarrow U =: F(W\times H)$ is a diffeom.

onto an open neighbhd. U of e in G.

<u>Injectivity</u>: Assume $\exp(X_1)h_1 = \exp(X_2)h_2$ $\quad X_1, X_2 \in W$, $h_1, h_2 \in H$

$$\Rightarrow \underline{\underline{h_2^{-1}h_1 = \underline{\exp(X_1)^{-1}\cdot\exp(X_2)}}} \in H \cap U' = \exp(V)$$

$$\Rightarrow \exists\, Y \in V \text{ s.t. } \exp(Y) = \exp(X_1)^{-1}\cdot\exp(X_2)$$

$$\Rightarrow \varphi(X_1, Y) = \varphi(X_2, 0)$$

$$\Rightarrow Y = 0 \text{ and } X_1 = X_2$$

$$\Rightarrow h_2 = h_1$$

i.e $F\big|_{W\times H}$ is injective.

Moreover, $F(X, \underline{\exp(Y)}) = \varphi(X, Y)$

Locally around $W \times \{e\}$ we can write F as $\underline{\varphi \circ (\underline{id, \exp^{-1}})}$.

$\Longrightarrow \underline{T_{(X,e)}} F$ is a linear isomorphism $\forall X \in W$.

By $F \circ (id, \rho^h) = \rho^h \circ F$, we conclude that

$\underline{T_{(X,h)}} F$ is an isomorphism $\forall X \in W, \forall h \in H$.

$\Longrightarrow F$ is a local diffeom. around any point in $W \times H$

Injectivity + being a local diffeo $\Longrightarrow U := F(W \times H)$ is open and $F: W \times H \to U$ diffeom. onto U.

Construct now the atlas for G/H :

$\pi : G \to G/H$, $\pi(U) \subseteq G/H$ is open $\Big(\pi^{-1}(\pi(U))$
$\quad p$
$= U$

$\Rightarrow \pi(U)$ is open$\Big)$

Set

$$\psi : W \to \pi(U)$$
$$\psi(x) = \pi\big(\exp(x)\big)$$

<u>Claim</u> ψ is a homeomorphism.

Injectivity: $X_1, X_2 \in W$ with $\psi(X_1) = \psi(X_2)$

i.e. $\exp(X_1)H = \exp(X_2)H$.

$\Rightarrow X_1 = X_2$ since F is injective on $W \times H$ $\Rightarrow \psi$ is injective surjectivity of ψ follows directly from that of F.

$\Rightarrow$ ψ is a bijection and evidently it is continuous as a comp. of continuous maps.

$W' \subset W$ open subset, then $\pi^{-1}(\psi(W')) = F(\underline{W' \times H})$ is open in G $\Longrightarrow$ $\psi(W') \subseteq G/H$ is open.

For $g \in G$, set $U_g := \pi(\lambda_g(U)) = \{ g \exp(X) H : X \in W \}$
$\subseteq G/H$.

and let $u_g : U_g \to W$ be defined by $u_g := \psi^{-1} \circ \ell_{g^{-1}}$

$\{(U_g, u_g)\}_{g\in G}$ is a smooth atlas for G/H.

$$\underline{(u_{g'} \circ u_g^{-1})}(x) = u_{g'}(g\exp(x)H) = \psi^{-1}((g')^{-1} g \exp(x) H)$$

$$= pr_1(F^{-1}(\underbrace{(g')^{-1} g \exp(x)}_{\in U}))$$

$$\Longrightarrow u_{g'} \circ u_g^{-1} = pr_1 \circ F^{-1} \circ \lambda_{(g'^{-1}g)} \circ \exp \text{ is smooth.}$$

For this C^∞-structure on G/H, $\pi : G \to G/H$ is a smooth submersion: $\underbrace{\psi \circ \pi \circ \varphi}_{= pr_1} : W \times V \to W$ is a smooth submersion.

Such a submersion has the universal property that for any map $f : G/H \longrightarrow N$ to another smooth mfd. N

f is smooth $\Longleftrightarrow$ $f \circ \pi : G \to G/H \longrightarrow N$ is smooth.

The univ. property implies uniqueness of the C^∞-str. on G/H by applying it to the identity $(G/H, \mathcal{A}) \xrightarrow{\mathrm{id}} (G/H, \mathcal{B})$ for two smooth structures $\mathcal{A}$ and $\mathcal{B}$.

It remains to show that $\ell : G \times G/H \to G/H$ is smooth.

$$\ell \circ \underline{\mathrm{id} \times \pi} : G \times G \longrightarrow G \times G/H \xrightarrow{\ell} G/H \quad \text{is smooth}$$

$$\shortparallel$$

$$\underline{\pi \circ \mu} : G \times G \longrightarrow G \longrightarrow G/H$$

Since π is a surj. subm., so is $\mathrm{id} \times \pi$ and so the univ. property of submersions implies ℓ is smooth.

$\square$.

It is not difficult to see that:

Thm. 1.43 Suppose M is a smooth mfd. equipped with a smooth transitive left-action $\ell : G \times M \to M$ of a Lie group G.
Then for any $x \in M$, G_x is closed a closed subgroup of G and the natural bijection

$$\begin{array}{ccc} G/G_x & \xrightarrow{\sim} & Gx = \underline{M} \\ gG_x & \longmapsto & g \cdot x \end{array}$$

is a diffeomorphism.

Until the 19th century (before Riemannian geometry), people understood by „geometry" almost exclusively Euclidean geometry.

In order to incorporate non-Euclidean geometries (parallel postulate does not hold) F. Klein proposed in its Erlangen program a broader notion of the geometry.

Geometry in the Sense of Klein (Klein geometry) $\rightleftharpoons$

C^∞-mfd. M equipped with a smooth transitive left-action of a Lie group.

If we fix a base point $x \in M$ and set $G_x =: H$,
then $M \cong G/H$ is a homogeneous space with G acting on G/H by left multiplication.

The geometry specified by such an action is the study of figures / properties of figures that are left invariant under the group. So the geometric structure on M is specified indirectly by saying what its automorphisms (symmetries) are.

Examples

(1) High-school geometry / Euclidean geometry,

$M = \mathbb{R}^n$

$G = \mathrm{Euc}(n) = \{ \underset{\in \mathbb{R}^n}{x} \longmapsto Ax + b \ : \ A \in O(n),\ b \in \mathbb{R}^n \}$.

$\parallel$

$\mathrm{Isom}(\mathbb{R}^n, g_{euc})$.

equipped with standard inner product $\langle , \rangle$.

$G \times M \longrightarrow M$ transitive left action of G on M.

$((A,b), x) \longmapsto Ax + b$

$x = 0 \in \mathbb{R}^n$ $\quad G_0 = O(n)$ $\quad \mathbb{R}^n \simeq \mathrm{Euc}(n) / O(n)$.

② Affine geometry

$M = \mathbb{R}^n \qquad G = \mathrm{Aff}(n) = \{ x \mapsto \underline{Ax+b} \ : \ A \in GL(n,\mathbb{R}),\ b \in \mathbb{R}^n \}$

$A^n = \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \ : \ x_0 = 1 \} \subseteq \underline{\mathbb{R}^{n+1}}$ affine hyperplane.

Elements of $GL(n+1,\mathbb{R})$ that preserve A^n are of the form

$$\left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline b & A \end{array} \right) : b \in \mathbb{R}^n,\ A \in GL(n,\mathbb{R}) \right\} \subseteq GL(n+1,\mathbb{R})$$

$\cong$

$\mathrm{Aff}(n) \qquad \underline{\begin{pmatrix} 1 \\ x \end{pmatrix} \in A^n} \qquad \mathrm{Aff}(n) / \mathrm{Aff}(n)_0 \cong \mathrm{Aff}(n) / GL(n,\mathbb{R}) \cong \mathbb{R}^n.$

$\mathrm{Aff}(n) \times A^n \longrightarrow A^n$ transitive left action.

No measure of distance and length, but the concept of parallel lines, colinearity ... still sense.

③ $M = S^n \subseteq \underline{\mathbb{R}^{n+1}}$ equipped with round metric $\underline{g_{rd}}$.

$O(n+1) \simeq \mathrm{Isom}(S^n, g_{rd})$.

$O(n+1) \times S^n \longrightarrow S^n$ acts transitively on S^n.

$S^n \simeq O(n+1) \big/ O(n)$.

$G_{e_1} \simeq O(n)$

$e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$

Analogue of the parallel postulate for the analogue of „lines" on S^n, namely the geodesics of g_{rd},

does not hold !

Any two great circles meet at two points.

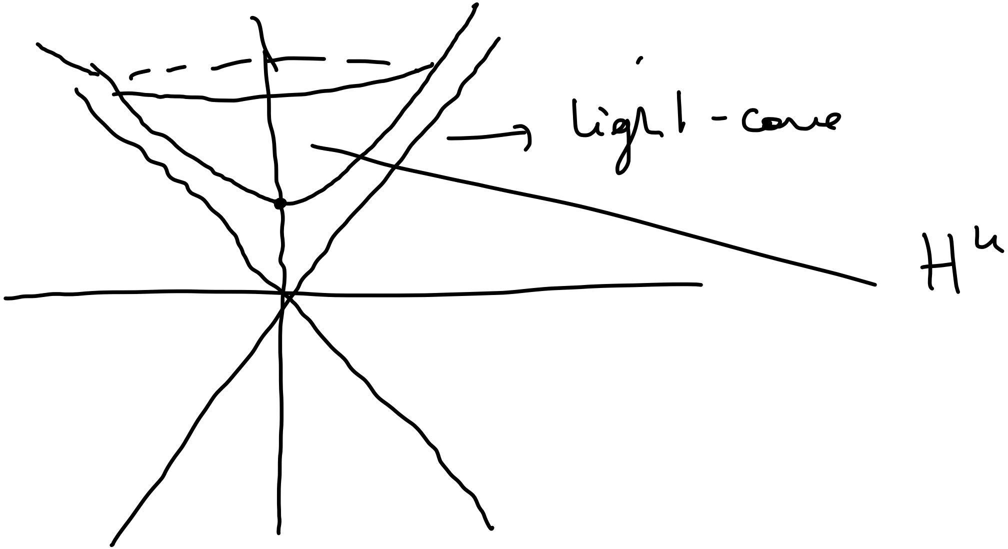
4. $\mathbb{R}^{n+1} = \mathbb{R}^{n,1}$ equipped with standard Lorentzian

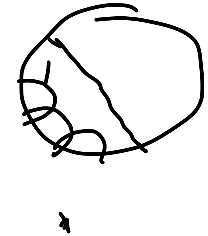
$x = (x_0, \dots, x_n)$ inner product $\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$

$$= x^t \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} y$$

$$H^n = \{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1 , x_0 > 0 \}$$

$\hookrightarrow$ n-dim hyperbolic space equipped with its standard metric g_{hyp}





„Parallel postulate" does not hold for geodesics
does not hold: ∃ infinitely many geodesics through
a point not intersecting a given one.

5 Classical projective geometry.

$\mathbb{R}P^n$ = n-dim. proj. space = 1-dim. subspace of $\mathbb{R}^{n+1}$.

$\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}P^n$

Projective lines of $\mathbb{R}P^n$ = images of 2-dim. subspaces of $\mathbb{R}^{n+1}$ under π

$\underline{GL(n+1, \mathbb{R})} \times \mathbb{R}P^n \to \mathbb{R}P^n$ transitive left-action.

$(A, [x]) \longmapsto [Ax]$

$Z(GL(n+1, \mathbb{R})) = (\mathbb{R} \setminus \{0\}) I_{n+1}$

$$\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \overset{[x]}{x} = \lambda x$$

$PGL(n+1, \mathbb{R}) := GL(n+1, \mathbb{R}) / Z(GL(n+1, \mathbb{R}))$

↑ projective linear group.

↑ normal subgroup.

$\Rightarrow$ left action $\underline{PGL(n+1, \mathbb{R})} \times \mathbb{R}P^n \xrightarrow{\sim} \mathbb{R}P^n$

$PGL(n+1, \mathbb{R}) \cong \{ f : \mathbb{R}P^n \to \mathbb{R}P^n : f$ maps projective lines to projective lines $\}$.

$PGL(n+1, \mathbb{R}) / \left[\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \right] \simeq \mathbb{R}P^n$.

"stabilizer of line $[e_1]$"